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Fractional discrete-continuous model of heat transfer process

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The paper proposes a new, state space, finite dimensional, fractional order model of a heat transfer in one dimensional body. The time derivative is described by Caputo operator. The second order central difference describes the derivative along the length. The analytical formulae of the model responses are proved. The stability, convergence, and positivity of the model are also discussed. Theoretical results are verified by experiments.

Key words: non integer order systems, heat transfer equation, finite difference, Caputo operator, positive systems

1. Introduction

Fractional Order (FO) models for different physical systems and phenomena have been presented by various Authors. For example FO chaotic systems are presented in [3], the FO model of a diffusion process is given for example in [7] and [34]. A number of interesting examples can be also found in [4, 6] and [30]. Positive fractional systems are presented for example in [8, 10].

Heat processes can be also described using fractional order approach. Examples are given among others in [1, 5, 11, 12]. New fractional operators: Caputo-Fabrizio or Atangana-Baleanu are here employed too. For example [33] presents the use of Caputo-Fabrizio operator in modeling of heat transfer, an application of operators with the non singular kernel to modeling of thermal processes was deeply analysed in paper [2]. The fractional model of one dimensional heat transfer process using Caputo-Fabrizio operator is presented in [19, 27]. The same

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process described with the use of Atangana-Baleanu operator is presented in [20]. The fully discrete, fractional order model using Grünwald-Letnikov operator to express both derivatives along time and length is proposed in the paper [22].

In this paper we propose and analyse a new, finite dimensional, state-space model of the one dimensional, experimental heat plant. The proposed model is analogical to the models of the RC ladder network discussed e.g. in [13, 14].

The paper is organized as follows. Preliminaries recall elementary ideas from fractional calculus as well as the heat transfer equation with fractional derivative along time and second order derivative along the length. Next the proposed fractional, discrete-continuous model is presented. The analytical formulae of the step and impulse responses are proved. The stability, positivity, and convergence are also discussed. Finally the proposed results are experimentally verified using experimental data.

2. Preliminaries

A presentation of elementary ideas is started with a definition of a non integer-order, integro-differential operator. It was given for example by [4, 9, 10, 30]:

Definition 1 *(The elementary non integer order operator) The non integer-order integro-differential operator is defined as follows:*

$$ {}_aD_t^{\alpha} f(t) = \begin{cases} \dfrac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}}, & \alpha > 0, \\ f(t), & \alpha = 0, \\ \displaystyle\int_a^t f(\tau)(\mathrm{d}\tau)^{\alpha}, & \alpha < 0, \end{cases} \tag{1} $$

where a and t denote time limits for operator calculation, $\alpha \in \mathbb{R}$ denotes the non integer order of the operation.

Next an idea of Gamma Euler function will be remembered (see for example [10]):

Definition 2 *(The Gamma function)*

$$ \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \mathrm{d}t. \tag{2} $$

An idea of Mittag-Leffler function needs to be given next. It is a non-integer order generalization of exponential function $e^{\lambda t}$ and it plays crucial role in the solution of fractional order (FO) state equation. The one parameter Mittag-Leffler function is defined as underneath:

Definition 3 *(The one parameter Mittag-Leffler function)*

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + 1)}. \tag{3}$$

and the two parameter Mittag-Leffler function is defined as:

Definition 4 *(The two parameters Mittag-Leffler function)*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + \beta)}. \tag{4}$$

For $\beta = 1$ the two parameter function (4) turns to one parameter function (3).

The integro-differential operator can be described by different definitions, given by Grünwald and Letnikov (GL definition), Riemann and Liouville (RL definition) and Caputo (C definition). In this paper the C definition is applied (see for example [4, 9, 10, 30]):

Definition 5 *(The Caputo definition of the FO operator)*

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(M-\alpha)} \int_0^t \frac{f^{(M)}(\tau)}{(t-\tau)^{\alpha+1-M}} \mathrm{d}\tau, \tag{5}$$

where $M - 1 < \alpha < M$ *denotes the non integer order of operation and* $\Gamma(..)$ *is the Gamma function* (2).

For the Caputo operator the Laplace transform can be defined (e.g. [10]):

Definition 6 *(The Laplace transform for Caputo operator)*

$$\begin{gathered}
\mathcal{L}({}_0^C D_t^\alpha f(t)) = s^\alpha F(s), \qquad \alpha < 0, \\
\mathcal{L}({}_0^C D_t^\alpha f(t)) = s^\alpha F(s) - \sum_{k=0}^{M-1} s^{\alpha-k-1} {}_0D_t^k f(0), \\
\alpha > 0, \quad M - 1 < \alpha \leqslant M \in \mathbb{Z}.
\end{gathered} \tag{6}$$

Consequently, the inverse Laplace transform for non integer order function is expressed as follows ([10]):

$$\begin{gathered}
\mathcal{L}^{-1}[s^\alpha F(s)] = {}_0D_t^\alpha f(t) + \sum_{k=0}^{M-1} \frac{t^{k-1}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+), \\
M - 1 < \alpha < M, \quad M \in \mathbb{Z}.
\end{gathered} \tag{7}$$

In (7) $\Gamma(..)$ is the Gamma function (2).

A fractional linear state equation takes the following form:

$$\begin{aligned} {}_0D_t^{\alpha}x(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{8}$$

where $\alpha \in (0, 1)$ is the fractional order of the state equation, $x(t) \in \mathbb{R}^N$, $u(t) \in \mathbb{R}^L$, $y(t) \in \mathbb{R}^P$ are the state, control and output vectors respectively, A, B, C are the state, control, and output matrices, respectively, $[0; t]$ is the considered time interval.

Finally, the fundamental stability condition of the time-continuous, FO system described by the state equation needs to be recalled. It is formulated by the Matignon theorem (see for example [3], p. 22, Theorem 1.3).

Theorem 1 *(The stability of FO system described by state equation) The FO system described by the state equation (8) is stable if the following condition is satisfied:*

$$|Arg(eig(A))| > \alpha\frac{\pi}{2}, \tag{9}$$

where $0 < \alpha < 2$ is the fractional order of the system, $eig(A)$ are the eigenvalues of the state matrix A.

3. The experimental system and its non integer order, state space model using Caputo operator

The simplified scheme of the considered heat plant is shown in Fig. 1. It is a thin copper rod heated by an electric heater located at one end of the rod. The temperature is measured using miniature RTD sensors Pt-100.

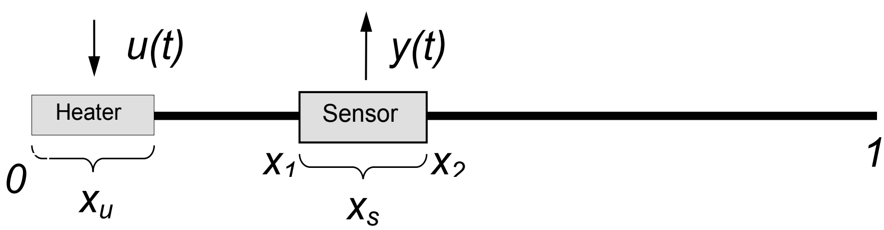

Figure 1: The simplified scheme of the experimental system

To simplify the further considerations assume that the length of the rod equals to 1.0. Consequently the heater is $0.0 < x_u < 1.0$ long, sensors are x_s long and they are attached in points: 0.29, 0.50 and 0.73 of rod length. More details about the construction of the whole system are given in the section "Experimental Results".

The basic mathematical model describing the heat transfer in the rod is the partial differential equation of the parabolic type. Due to fact that the both frontal surfaces of rod are much smaller than its side surface, we assume the homogeneous Neumann boundary conditions at the both ends. The heat exchange along the length needs to be considered. The control and observation are distributed because the size of heater and sensors should be included in model. Such a model with integer orders of both differentiations has been considered in papers [16–18]. The non integer order model with respect to time, employing the Caputo operator was given in [24], its properties were analyzed also in [28]. It takes the following form:

$$\begin{cases} {}_0^C D_t^{\alpha} Q(x,t) = a_w \dfrac{\partial^2 Q(x,t)}{\partial x^2} - R_a Q(x,t) + b(x)u(t), \\ \dfrac{\partial Q(0,t)}{dx} = 0, \qquad t \geqslant 0, \\ \dfrac{\partial Q(1,t)}{dx} = 0, \qquad t \geqslant 0, \\ Q(x,0) = Q_0, \qquad 0 \leqslant x \leqslant 1, \\ y(t) = k_0 \displaystyle\int_0^1 Q(x,t)c(x)\,\mathrm{d}x. \end{cases} \tag{10}$$

In (10) α is the non integer order of the system, $a_w > 0$, $R_a \geqslant 0$ denote coefficients of heat conduction and heat exchange, k_0 is a steady-state gain of the model, Q_0 is the initial spatial temperature distribution in the rod, $b(x)$ and $c(x)$ are heater and sensor functions. They take the following, simple form:

$$b(x) = \begin{cases} 1, & x \in [0, x_u], \\ 0, & x \notin [0, x_u]; \end{cases} \tag{11}$$

$$c(x) = \begin{cases} 1, & x \in [x_1, x_2], \\ 0, & x \notin [x_1, x_2]. \end{cases} \tag{12}$$

4. The proposed discrete-continuous FO model of the plant

Divide the rod into N short sections $\Delta x = \dfrac{1}{N}$ long. Consequently, the first and second derivative along length in Eq. (10) can be approximated by 1'st order forward difference and 2'nd order central difference, analogically, as it was done in [15, 31]:

$$\frac{\partial Q(x,t)}{\partial x} = \frac{Q(x+\Delta x,t) - Q(x,t)}{\Delta x} + o(\Delta x), \tag{13}$$

$$\frac{\partial^2 Q(x,t)}{\partial x^2} = \frac{Q(x+\Delta x,t) - 2Q(x,t) + Q(x-\Delta x,t)}{\Delta x^2} + o(\Delta x^2), \tag{14}$$

where $o(..)$ is the spatial truncation error, depending on the mesh spacing described by N (see [31], Eq. (14)). Next, introduce the following notation:

$$Q_n(t) = Q(n\Delta x, t), \quad n = 0, 1, \ldots, N. \tag{15}$$

Using the above convention the functions of heater and sensor (11) and (12) are expressed as:

$$b_n = \begin{cases} 1, & n\Delta x \in [0, x_u], \\ 0, & n\Delta x \notin [0, x_u], \end{cases} \quad n = 0, 1, \ldots, N; \tag{16}$$

$$c_n = \begin{cases} 1, & x \in [x_1, x_2], \\ 0, & n\Delta x \notin [x_1, x_2], \end{cases} \quad n = 0, 1, \ldots, N. \tag{17}$$

Using the approximations (13), (14) with notation (15) with neglecting $o(\Delta x)$, $o(\Delta x^2)$, and using (16), (17) to heat equations (10) we obtain:

$$\begin{cases} Q_0(t) = Q_1(t), \\ \ldots \\ {}_0^C D_t^{\alpha} Q_n(t) = a_w \dfrac{Q_{n+1}(t) - 2Q_n(t) + Q_{n-1}(t)}{\Delta x^2} - R_a Q_n(t) + b_n u(t), \\ \qquad\qquad n = 1, \ldots, N-1, \\ \ldots \\ Q_{N-1}(t) = Q_N(t), \\ Q_n(0) = 0, \quad n = 0, 1, \ldots, N, \\ y(t) = k_0 \Delta x \displaystyle\sum_{n=0}^{N} Q_n(t) c_n. \end{cases} \tag{18}$$

Equation (18) can be expressed as the following, fractional order, finite dimensional state equation:

$$\begin{cases} {}_0^C D_t^{\alpha} Q(t) = AQ(t) + Bu(t), \\ y(t) = CQ_n(t), \end{cases} \tag{19}$$

where $Q(t) = [Q_1(t) \ldots Q_n(t)]^T \in \mathbb{R}^N$ is the state vector, $u(t) \in \mathbb{R}$ is the control, $y(t) \in \mathbb{R}$ is the output. Next the state, control and output matrices A, B, and C

are defined as follows:

$$A = d\begin{bmatrix} -1-R & 1 & \dots & & 0 \\ 1 & -2-R, & 1\dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & & \dots & 1 & -1-R \end{bmatrix}_{N\times N}, \tag{20}$$

where:

$$\begin{aligned} R &= R_a\frac{\Delta x^2}{a_w}, \\ d &= \frac{a_w}{\Delta x^2}; \end{aligned} \tag{21}$$

$$B = [b_0, b_1, \dots, b_N]^T. \tag{22}$$

In (22) b_n is defined by (16).

$$C = [c_0, c_1, \dots, c_N]. \tag{23}$$

In (23) c_n is defined by (17).

4.1. The Jordan form of the model

The eigenvalues of the state matrix A are as follows ([13]):

$$\lambda_n = d\left[-2(1-\cos(\phi_n) - R\right], \quad n = 1,\dots,N \tag{24}$$

or equivalently:

$$\lambda_n = 2a_w N^2\left(\cos(\phi_n) - 1\right) - R_a, \quad n = 1,\dots,N, \tag{25}$$

where:

$$\phi_n = \frac{n\pi}{N+1}, \quad n = 1,\dots,N. \tag{26}$$

The matrix P transforming the system to the Jordan canonical form takes the following form (see [13], Eqs. (6)–(11)):

$$P = \sqrt{\frac{2}{N+1}}\begin{bmatrix} \sin\phi_1 & \sin 2\phi_1 & \dots & \sin N\phi_1 \\ \sin\phi_2 & \sin 2\phi_2 & \dots & \sin N\phi_2 \\ \dots & \dots & \dots & \dots \\ \sin\phi_N & \sin 2\phi_N & \dots & \sin N\phi_N \end{bmatrix}_{N\times N}. \tag{27}$$

It can be checked that $P^2 = I \iff P^{-1} = P$. Then the Jordan canonical form of the state equation (19) is as follows:

$$\begin{cases} {}_0^C D_t^{\alpha} Q^*(t) = A^* Q^*(t) + B^* u(t), \\ y(t) = C^* Q_n^*(t), \end{cases} \tag{28}$$

where:

$$
\begin{aligned}
A^* &= PAP = d\frac{2}{N+1}\operatorname{diag}\{\lambda_1,\ldots,\lambda_N\},\\
B^* &= PB = \sqrt{\frac{2}{N+1}}\left[b_1^*,\ldots,b_N^*\right]^T,\\
C^* &= CP = \sqrt{\frac{2}{N+1}}\left[C_1^*,C_2^*,C_3^*\right]^T,\\
C_j^* &= \left[c_{j1}^*,\ldots,c_{jN}^*\right].
\end{aligned}
\tag{29}
$$

With respect to (16), (17) and (29) elements of matrices B^* and C^* are as follows:

$$
b_n^* = \sum_{m=1}^{N_b} \sin m\phi_n, \qquad n = 1,\ldots,N. \tag{30}
$$

$$
c_{jm}^* = \sum_{n=N_{c1}}^{N_{c2}} \sin m\phi_n, \qquad m = 1,\ldots,N. \tag{31}
$$

Using results presented in [32], Eq. (5) elements (30) and (31) can be presented without sum:

$$
b_n^* = \frac{\sin\dfrac{N_b\phi_n}{2}\sin\dfrac{(N_b+1)\phi_n}{2}}{\sin\dfrac{\phi_n}{2}}, \tag{32}
$$

$$
c_{jm}^* = \frac{\sin\dfrac{N_{c2}\phi_m}{2}\sin\dfrac{(N_{c2}+1)\phi_m}{2} - \sin\dfrac{N_{c1}\phi_m}{2}\sin\dfrac{(N_{c1}-1)\phi_m}{2}}{\sin\dfrac{\phi_m}{2}}. \tag{33}
$$

The indices N_b, N_{c1} and N_{c2} describe elements of B and C matrices different from zero. They are determined by the construction of the plant:

$$
\begin{aligned}
N_b &= \operatorname{Int}(N x_u),\\
N_{c1} &= \operatorname{Int}(N x_1),\\
N_{c2} &= \operatorname{Int}(N x_2).
\end{aligned}
\tag{34}
$$

In (34) Int(..) denotes the nearest integer value.

4.2. The step and impulse responses

Assume the homogenous initial condition $Q_n(0) = 0, n = 1, \ldots, N$ and the control being the Heaviside function: $u(t) = 1(t)$. Then the step response is as follows:

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{s}C^*(s^\alpha I - A^*)^{-1}B^*\}. \tag{35}$$

Using results given in [24] we obtain:

$$y(t) = C^* Q^*(t) B^*, \tag{36}$$

where B^* and C^* are expressed by (29),
$Q^*(t) = \text{diag}\{Q_1^*, \ldots, Q_N^*\}$ and $Q_n(t)$ is as follows:

$$Q_n^*(t) = \frac{E_\alpha(\lambda_n t) - 1}{\lambda_n}, \quad n = 1, .., N. \tag{37}$$

In (37) $E_\alpha(..)$ denotes the one parameter Mittag-Leffler function (3).

The equation (36) can be applied to modeling of the considered heat system. This will be illustrated by the example.

The impulse response of the considered model can be computed as the following inverse Laplace transform:

$$g(t) = \mathcal{L}^{-1}\{C^*(s^\alpha I - A^*)^{-1}B^*\}. \tag{38}$$

The equation (38) takes the following form:

$$g(t) = C^* G^*(t) B^*, \tag{39}$$

where: $G^*(t) = \text{diag}\{G_1^*(t), \ldots, G_N^*(t)\}$. The n-th component $G_n^*(t)$ can be calculated using formula of the impulse response for the elementary FO transfer function given in [3], p. 10, Eq. (1.31):

$$G_n^*(t) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda_n t^{-\alpha}), \quad n = 1, .., N. \tag{40}$$

In (40) $E_{\alpha,\alpha}(..)$ is the two-parameter Mittag-Leffler function (4).

4.3. The stability

The stability of the proposed FO model will be discussed using Matignon Theorem (9). The spectrum of the model is expressed by (24), (25). The eigenvalues are purely real, single and separated. For $a_w > 0$ and $R_a \geqslant 0$ each eigenvalue lies between $-R_a$ and $-4a_w N^2 - R_a$. This denotes that:

$$|\text{Arg}(eig(A))| = \pi \quad \forall n = 0, \ldots, N. \tag{41}$$

This means that the proposed model is asymptotically stable for each $N > 0$ and each fractional order $0.0 < \alpha < 2.0$.

4.4. The convergence

The Rate of Convergence (ROC) in the considered case is defined as the rate of the steady-state response of the system for two consecutive, increasing dimensions of the state equation N. Denote the steady state response of the model by $y_{ss}(N)$. It is equal:

$$y_{ss}(N) = -C_N^*(A_N^*)^{-1}B_N^* . \tag{42}$$

In (42) lower index $_N$ denotes the size of model, A, B and C describe the system with respect to (29). Consequently the exactly defined ROC_{ex} is as follows:

$$ROC_{ex}(N) = \left|1 - \frac{y_{ss}(N+1)}{y_{ss}(N)}\right| . \tag{43}$$

The $ROC_{ex}(N)$ is a discrete function of model order N. For $N \to \infty$ it is expected to go to 0.0. The $ROC_{ex}(N)$ as a function of N is irregular (see Fig. 2), however its exponential envelope is visible. This allows to deduce that it can be estimated using discrete exponential function.

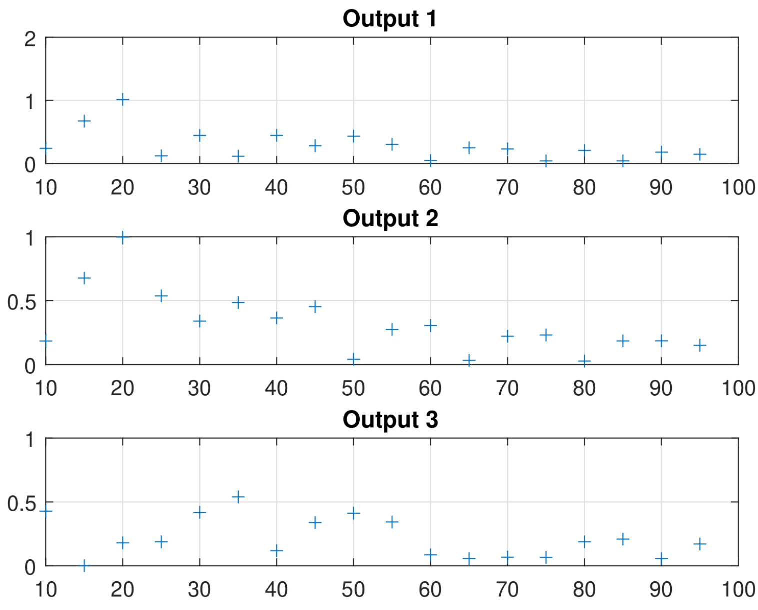

Figure 2: The exact values of $ROC_{ex}(N)$ for all outputs, $N = 10, \ldots, 100$

The exact analytical formula expressing (43) with the use of (25) and (29) is too complicated to compute the order N assuring the predefined value of ROC. That's why the exponential estimation of ROC is proposed and discussed below.

The trend of $ROC_{ex}(N)$ shown in Fig. 2 can be esimated by the discrete, exponentially damped sine function or the exponential function (see for example [29],

p. 2, Eq. (1.2) and p. 7, Eq. (1.8) respectively). The estimation using exponential function is as follows:

$$ROC_{est}(N) = ke^{-a\Delta x(N-N_{\min})}. \tag{44}$$

In (44) k and a are constants applied to fit the estimation to exact data $ROC_{ex}(N)$, $N_{\min}$ is the minimum value of the considered N. This approximation can be used to estimate the order N_p assuring the predefined value p of ROC. It is given by the proposition given underneath.

Proposition 1 *(The size of model N_p assuring the predefined value of ROC)*
Consider the model of one dimensional heat transfer (18)–(23)*. Assume that the required, predefined value is: $ROC_{est} = p$, $0 < p < 1$. The dimension N_p of the model, assuring keeping the predefined value p is equal:*

$$N_2 = \operatorname{Int}\left(\frac{\ln\left(\frac{p}{k}\right) + a\Delta x N_{\min}}{a\Delta x}\right). \tag{45}$$

In (45) Int(..) *denotes the nearest integer value, $N_{\min}$ is the minimum value of order N considered in calculations.*

Proof. The fixed value p is expressed as follows:

$$p = ke^{-a\Delta x(N-N_{\min})}. \tag{46}$$

By logarithmized both sides of (46) we obtain:

$$\ln\left(\frac{p}{k}\right) = a\Delta x\left(N - N_{\min}\right). \tag{47}$$

The nearest integer from solution of (47) relative to N gives directly (45) and the proof is completed. □

The above condition will be verified using experimental results in the next section.

4.5. The positivity

Remember that the necessary and sufficient condition of the internal positivity (see for example [10], p. 18) is that the state matrix A is required to be a Metzler matrix and matrices B, C must contain only nonnegative entries.

Next recall the considered base model before transformation to the Jordan form, described by the equation (8) with matrices A, B and C described by (20)-(23) and their elements described by (11) and (12) respectively. It can be noted that:

- the state matrix A is the Metzler matrix,

- all entries of B and C matrices, defined by (11) and (12) are equal 0 or 1. This means that they are nonnegative.

This means that the model before transformation to Jordan form is internally positive and consequently, externally positive.

The situation turns to be a little bit more complicated for Jordan canonical form due to the change of state space model. Matrix A^* is still being Metzler matrix, but matrices B^* and C^* (29) do not have only non negative entries. This implies that the internal positivity is lost. However, the external positivity is still being kept due to the fact that the impulse responses of the systems before and after transformation to the Jordan form are the same.

This can be also proved using external positivity condition given in the paper [21]. The system having diagonal state operator is externally postive iff:

$$C^* B^* \geqslant 0. \tag{48}$$

With respect to (29) the condition (48) takes the following form:

$$C^* B^* = C P P^{-1} B = C B.$$

The B and C matrices have only nonnegative entries (see (16), (17), (22), (23)). Additionally if $N_b < N_{c1}$, then $CB = 0$. This implies that the condition (48) is met for each plant parameters and fractional order α.

The above considerations will be illustrated by the example.

5. Experimental results

The laboratory plant employed to experiments is shown in Fig. 3. The rod is 260 [mm] long. The control is given as the standard current 0–20 [mA] by the analog output of the PLC. This current is amplified to the range 0–1.5 [A] and given to the heater. The temperature is measured using RTD sensors Pt-100. In the considered case the size and location of sensors are as follows:

$$\begin{cases} x = 0.29: & x_1 = 0.26, \quad x_2 = 0.32, \\ x = 0.50: & x_1 = 0.47, \quad x_2 = 0.53, \\ x = 0.73: & x_1 = 0.70, \quad x_2 = 0.76. \end{cases}$$

The temperature is directly read by analog inputs of the PLC in Celsius degrees. Data acquisition is done by PLC cooperating with SCADA. The whole system is connected via PROFINET. The step response of the plant as a function of time and length is illustrated by Fig. 4. It was tested between 0 and $T_f = 300[s]$ with sample time $h = 1$ [s]. This relatively long sample time is sufficient.

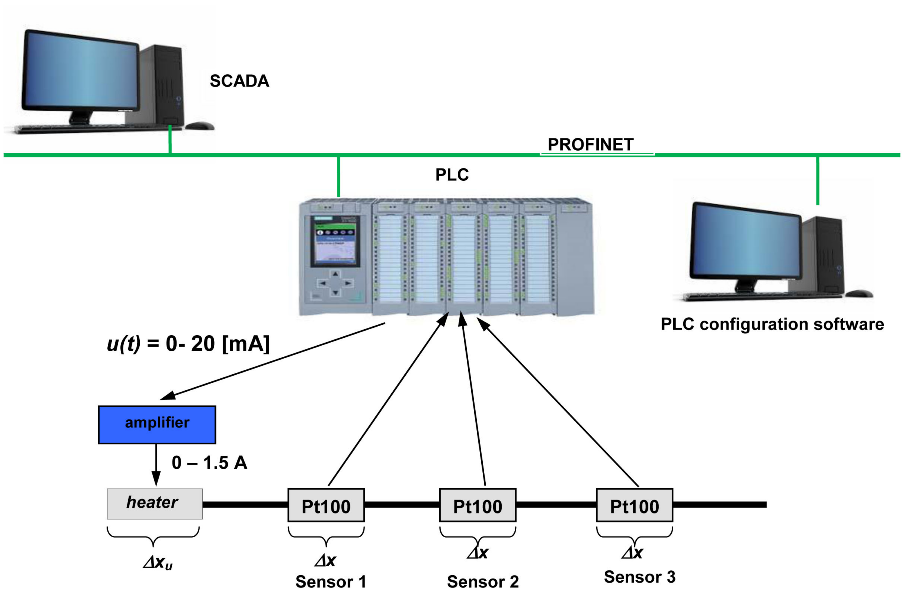

Figure 3: The construction of the experimental system

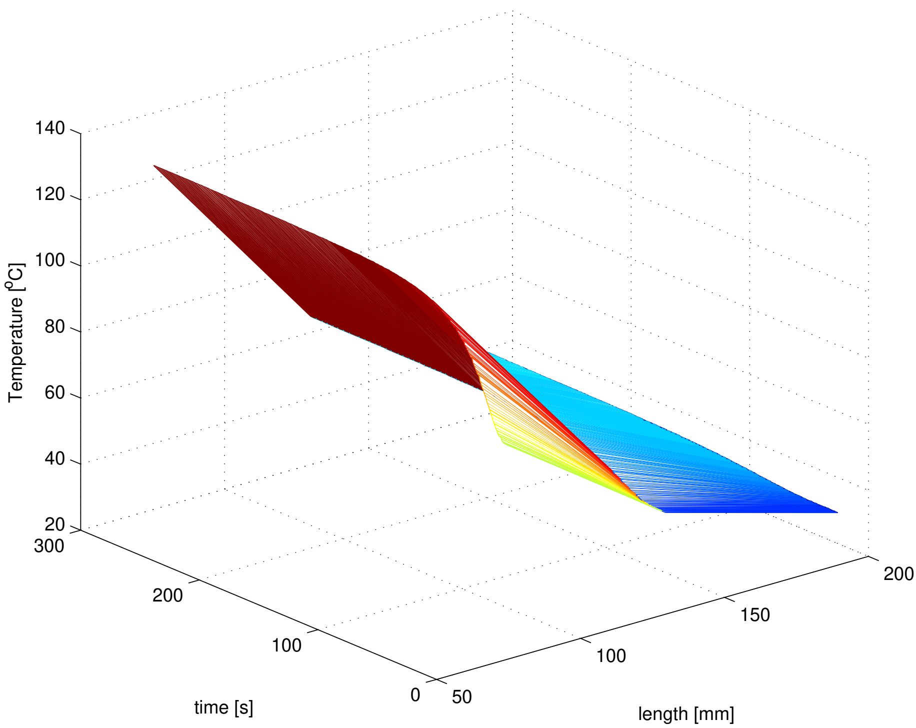

Figure 4: The step response of the plant as a function of time and length

The quality of the model we deal with was estimated using typical Mean Square Error (MSE) cost function:

$$MSE = \frac{1}{3K_s} \sum_{j=1}^{3} \sum_{k=1}^{K_s} \left(y_{e_j}^{+}(k) - y_j^{+}(k) \right)^2 . \tag{49}$$

In (49) K_s denotes the number of collected samples for one sensor, $y_{e_j}^{+}(k)$ and $y_j^{+}(k)$ are step responses of plant and model in k-th time step.

The parameters of the model: a_w, R_a and α were estimated via minimization of the cost function (49) using MATLAB function *fminsearch*. Results are given in Table 1 and illustrated by Fig. 5.

Table 1: Parameters of the model

N	α	a_w	R_a	MSE
22	0.9212	0.0003	0.0362	0.0369

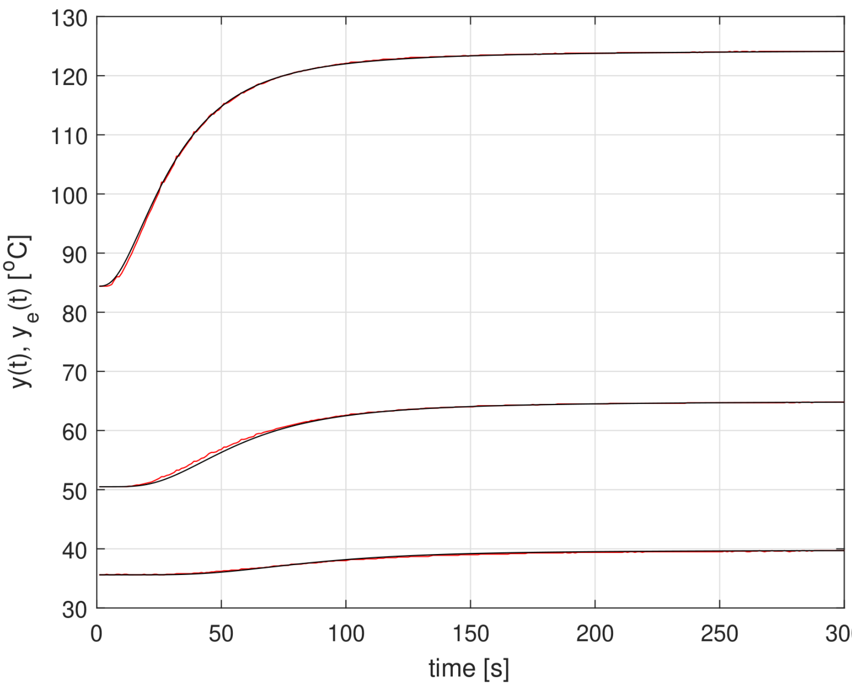

Figure 5: Comparison of the proposed model to experiment for $N = 22$ (red line-experiment, black line-model)

Next the convergence of the proposed model was examined using condition (45). The parameters of the model are given in 1, the parameters of approximation (44) assuring the fitting the estimate to real dependence are following: $k = 1.2$, $a = 0.005$. Assume that the expected value of ROC is equal $p = 0.2$. Using condition (45) we obtain $N_p = 82$. This result is illustrated by Fig. 6.

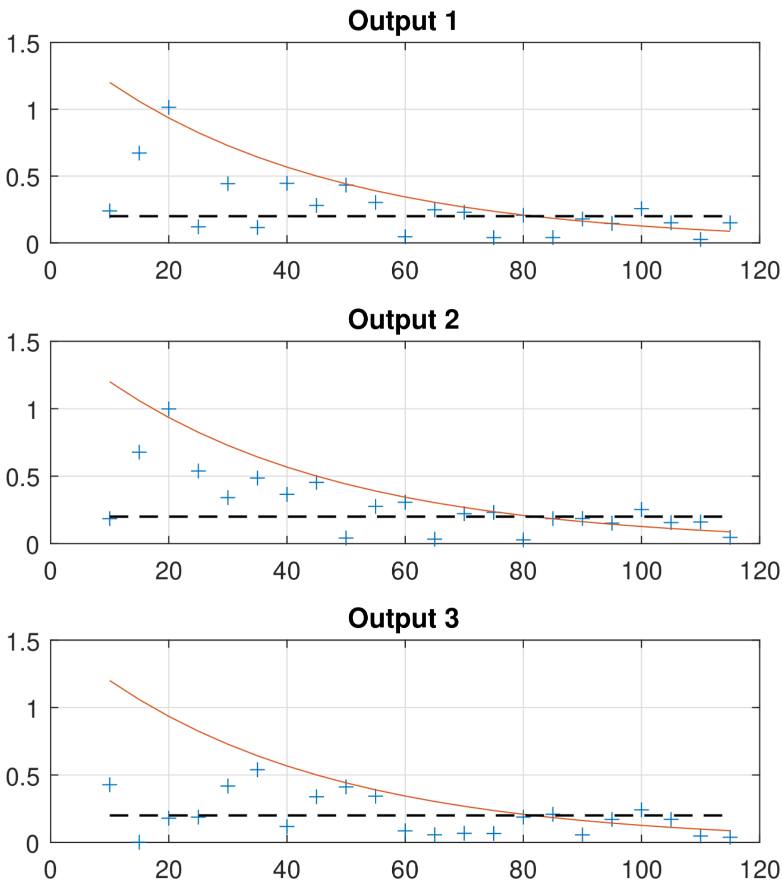

Figure 6: The exact ROC (43) (crosses), its estimate (44) (solid line) and threshold value of p (the dotted line) for all outputs, $N = 10, \ldots, 120$

Finally, the positivity of the model was tested. With respect to (48) we calculate CB. Using (22), (23) and model parameters from Table 1 we obtain:

$$BC = [0; 0; 0]^T.$$

This result is compliant to (48). It confirms the external positivity of the system. As an additional illustration, the impulse response of the system is shown in Fig. 7.

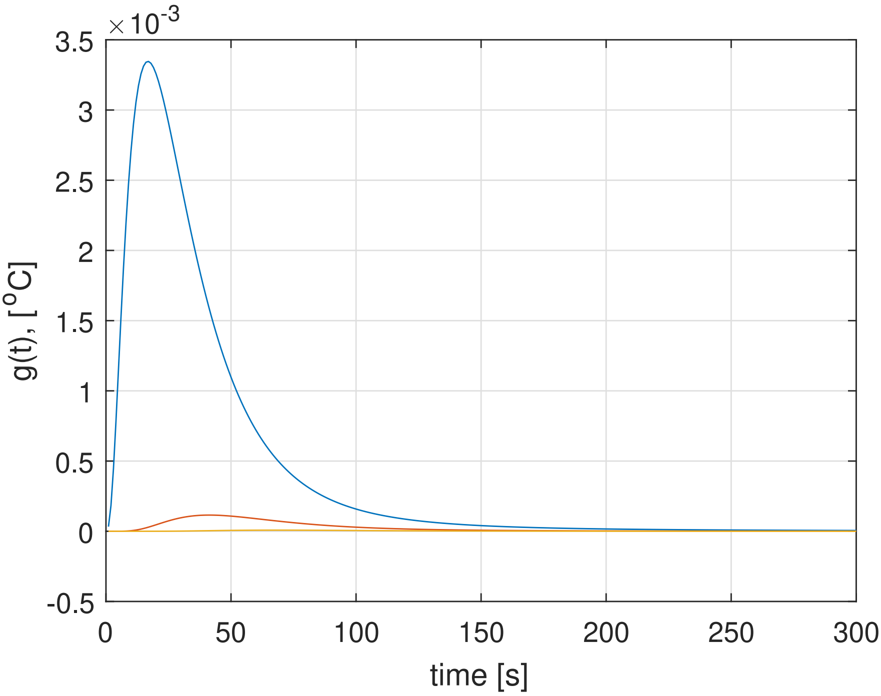

Figure 7: The impulse response (38) of the system for all sensors. (sensor 1: blue, sensor 2: orange, sensor 3: yellow)

6. Final conclusions

The first final conclusion from the paper is that the proposed discrete-continuous model well describes the heat transfer in one dimensional plant. The proposed model is asymptotically stable for each discretization step $\frac{1}{N}$ and each fractional order $0.0 < \alpha < 2.0$.

Next, the model is externally positive and its external positivity does not depend on its size and fractional order.

The convergence of the model can be estimated using a simple exponential estimator, proposed by authors. The use of this estimator allows to analytically compute the model size N assuring its predefined rate of convergence.

The comparison of the presented model to infinite dimensional models previously proposed in papers: [24, 28] is in the press.

It is important to note that the proposed one dimensional model can be emploeyd to analyse heat transfer in 3D homogenous bodies, e.g. in modeling of thermal insulation of a wall.

The future investigation of the presented problems will cover the comparing of the proposed model to models with diagonal state matrix, using different fractional operators and presented in papers [20, 23–27]. The model with triangular state matrix proposed in the paper [22] needs also to be compared to others. The models will be compared in the sense of accuracy, convergence and numerical complexity.

An another interesting issue recently analyzed by authors is the generalization of the proposed model to 2D case, to modeling dynamics of temperature fields measured with the use of thermal camera.

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