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Dedicated to Professor Jan Stochel
on the occasion of his 70th birthday

POSITIVE SOLUTIONS FOR NONPARAMETRIC ANISOTROPIC SINGULAR SOLUTIONS

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Abstract. We consider an elliptic equation driven by a nonlinear, nonhomogeneous differential operator with nonstandard growth. The reaction has the combined effects of a singular term and of a “superlinear” perturbation. There is no parameter in the problem. Using variational tools and truncation and comparison techniques, we show the existence of at least two positive smooth solutions.

Keywords: variable Lebesgue and Sobolev spaces, anisotropic regularity, anisotropic maximum principle, truncations and comparisons, Hardy inequality.

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1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following anisotropic singular Dirichlet problem

$$\begin{cases} -\operatorname{div} a(z, Du) = \beta(z)u(z)^{-\eta(z)} + f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \, u > 0. \end{cases} \tag{1.1}$$

Here, $a : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ is measurable in $z \in \Omega$, continuous, monotone in $x \in \mathbb{R}^N$ (hence maximal monotone, too) and satisfies nonstandard growth conditions. These conditions are general enough to incorporate in our framework many differential operators of interest, such as the anisotropic p-Laplacian and the anisotropic (p, q)-Laplacian. In the reaction, we have the combined effects of a singular term $\beta(z)u(z)^{-\eta(z)}$ with $\beta \in L^\infty(\Omega)_+\backslash\{0\}$ and $\eta \in C(\bar\Omega)$ satisfying $0 < \eta(z) < 1$ for all $z \in \bar\Omega$ and a perturbation $f(z, x)$ which is a Carathéodory function (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$ the function $x \mapsto f(z, x)$ is continuous) and we assume that it has $(p_+ -1)$-superlinear growth as $x \to +\infty$. Here, $p \in C(\bar\Omega)$ is the function that controls the nonstandard growth of $a(\cdot)$ and $p_+ = \max_{\bar{\Omega}} p$. Near 0^+, the perturbation $f(z,\cdot)$ exhibits an oscillatory behavior.

We point out that there is no parameter $\lambda > 0$ in our problem. Usually, singular problems are parametric and the presence of the parameter $\lambda > 0$ is helpful because by varying $\lambda > 0$, we can achieve desirable geometric configurations necessary to apply the minimax theorems of the critical point theory. Problem (1.1) does not involve a parameter and this requires new arguments. The study of anisotropic singular problems is lagging behind and only recently there have been established existence and multiplicity results, all for parametric problems. We mention the works of Bai, Papageorgiou & Zeng [1], Byun & Ko [2], Guarnotta, Marano & Moussaoui [8], Papageorgiou, Rădulescu & Zhang [14], Zeng & Papageorgiou [17], and Saoudi & Ghanmi [15]. A survey of anisotropic problems (including double phase equations) can be found in Papageorgiou [10].

In the present paper, using variational tools from the critical point theory, together with truncation and comparison techniques, we show that problem (1.1) has at least two positive smooth solutions.

2. MATHEMATICAL BACKGROUND AND HYPOTHESIS

Anisotropic problems require the use of Lebesgue and Sobolev spaces with variable exponents. For more information about these spaces, we refer to the books of Cruz Uribe & Fiorenza [3] and of Diening, Harjulehto, Hästo & Ruzička [4].

By $L^0(\Omega)$ we denote the space of all measurable functions $u : \Omega \to \mathbb{R}$. We identify two such functions which differ only on a Lebesgue null subset of Ω. Let $r \in C(\bar{\Omega})$ and set

$$r_- = \min_{\bar{\Omega}} r \quad \text{and} \quad r_+ = \max_{\bar{\Omega}} r.$$

Consider the set $V_1 = \{r \in C(\bar{\Omega}) : 1 < r_-\}$. Then for $r \in V_1$, we define the "variable Lebesgue space" $L^{r(z)}(\Omega)$ by

$$L^{r(z)}(\Omega) = \left\{ u \in L^0(\Omega) : \int_\Omega |u|^{r(z)} \mathrm{d}z < \infty \right\}.$$

We equip $L^{r(z)}(\Omega)$ with the so-called "Luxemburg norm" $\|\cdot\|_{r(z)}$ defined by

$$\|u\|_{r(z)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|u|}{\lambda} \right)^{r(z)} \mathrm{d}z \le 1 \right\}.$$

Evidently, $\|\cdot\|_{r(z)}$ is the Minkowski functional of the convex, absorbing and balanced set

$$C = \left\{ u \in L^0(\Omega) : \varrho_p(u) = \int_\Omega |u|^{p(z)} \mathrm{d}z \le 1 \right\}.$$

The function $\varrho_p(\cdot)$ is known as the "modular function".

Normed in this way, $L^{p(z)}(\Omega)$ becomes a Banach space which is separable, reflexive (in fact, uniformly convex, since $x \mapsto |x|^{p(z)}$ is a uniformly convex function).

Let $r' \in V_1$ be defined by $r'(z) = \frac{r(z)}{r(z)-1}$ for all $z \in \bar{\Omega}$ (that is, $\frac{1}{r(z)} + \frac{1}{r'(z)} = 1$ for all $z \in \bar{\Omega}$). Then $L^{r'(z)}(\Omega) = L^{r(z)}(\Omega)^*$ and we have the following version of the Hölder inequality

$$\int\limits_\Omega |uh| \mathrm{d}z \leq \left[\frac{1}{r_-} + \frac{1}{r'_-}\right] \|u\|_{r(z)} \|h\|_{r'(z)}$$

for all $u \in L^{r(z)}(\Omega)$, all $h \in L^{r'(z)}(\Omega)$. Suppose $r, s \in V_1$, and assume that $r(z) \leq s(z)$ for all $z \in \bar{\Omega}$, then

$$L^{s(z)}(\Omega) \hookrightarrow L^{r(z)}(\Omega) \text{ continuously.}$$

Using the variable Lebesgue spaces, we can define the corresponding "variable Sobolev spaces". So let $r \in V_1$. We define

$$W^{1,r(z)}(\Omega) = \{u \in L^{r(z)}(\Omega) : |\mathrm{D}u| \in L^{r(z)}(\Omega)\}.$$

Here, $\mathrm{D}u$ is the weak gradient of u. We equip $W^{1,r(z)}(\Omega)$ with the following norm

$$\|u\|_{1,r(z)} = \|u\|_{r(z)} + \|\mathrm{D}u\|_{r(z)} \text{ for all } u \in W^{1,r(z)}(\Omega),$$

where

$$\|\mathrm{D}u\|_{r(z)} = \||\mathrm{D}u|\|_{r(z)}.$$

With this norm, $W^{1,r(z)}(\Omega)$ becomes a Banach space which is separable, reflexive (in fact, uniformly convex).

Let

$$C^{0,1}(\bar{\Omega}) = \{r : \bar{\Omega} \to \mathbb{R} : r \text{ is Lipschitz continuous}\}$$

and let $r \in V_1 \cap C^{0,1}(\bar{\Omega})$. We define

$$W_0^{1,r(z)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,r(z)}}.$$

This is also a separable reflexive (in fact, uniformly convex) Banach space. Moreover, the Poincaré inequality holds, namely there is $C = C(\Omega) > 0$, such that

$$\|u\|_{r(z)} \leq C\|\mathrm{D}u\|_{r(z)} \text{ for all } u \in W_0^{1,r(z)}(\Omega).$$

Therefore on $W_0^{1,r(z)}(\Omega)$ we can use the equivalent norm $\|\cdot\|$ defined by

$$\|u\| = \|\mathrm{D}u\|_{r(z)} \text{ for all } u \in W_0^{1,r(z)}(\Omega).$$

If $r \in V_1$, then we define

$$r^*(z) = \begin{cases} \frac{Nr(z)}{N-r(z)}, & \text{if } r(z) < N \\ +\infty, & \text{if } N \leq r(z) \end{cases} \quad \text{for all } z \in \bar{\Omega}.$$

Suppose $p, r_- \in V_1$, with $r \in C^{0,1}(\bar{\Omega})$ and assume that $r_- < N$, $p(z) \le r(z)$ (resp. $p(z) < r(z)$) for all $z \in \bar{\Omega}$. Then we have

$$W_0^{1,r(z)}(\Omega) \hookrightarrow L^{p(z)}(\Omega) \text{ continuously (resp. } W_0^{1,r(z)}(\Omega) \hookrightarrow L^{p(z)}(\Omega) \text{ compactly).}$$

If $u \in L^0(\Omega)$, then we set

$$u^+(z) = \max\{u(z), 0\}, \quad u^-(z) = \max\{-u(z), 0\} \quad \text{for all } z \in \Omega.$$

Evidently, $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,r(z)}(\Omega)$, then $u^\pm \in W_0^{1,r(z)}(\Omega)$. If $u, v \in L^0(\Omega)$ and $u(z) \le v(z)$ for a.a. $z \in \Omega$, then we define

$$[u, v] = \{h \in W_0^{1,r(z)}(\Omega) : u(z) \le h(z) \le v(z) \text{ for a.a. } z \in \Omega\},$$

$$\operatorname{int}_{C_0^1(\bar{\Omega})}[u, v] = \text{ interior in } C_0^1(\bar{\Omega}) \text{ of } [u, v] \cap C_0^1(\bar{\Omega}).$$

Here,

$$C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega = 0}\}.$$

This is an ordered Banach space, with order (positive) cone

$$C_+ = \{u \in C_0^1(\bar{\Omega}) : 0 \le u(z) \text{ for all } z \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\right\},$$

where $\frac{\partial u}{\partial n} = (\mathrm{D}u, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

If X is a Banach space and $\varphi \in C^1(X)$, then

$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \text{ (the critical set of } \varphi).$$

We say that $\varphi(\cdot)$ satisfies the "C-condition", if it has the following property: "Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence."

Let $1 < q < \infty$ and consider the q-Laplacian

$$\Delta_q u = \operatorname{div}(|\mathrm{D}u|^{q-2}\mathrm{D}u) \text{ for all } u \in W_0^{1,q}(\Omega).$$

By $\hat{\lambda}_1(q)$ we denote the first eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$. We know that $\hat{\lambda}_1(q) > 0$, it is simple, isolated and if $\hat{u}_1(q) \in W_0^{1,q}(\Omega)$ is the corresponding L^q-normalized (that is, $\|\hat{u}_1(q)\|_q = 1$), positive eigenfunction corresponding to $\hat{\lambda}_1(q)$, then $\hat{u}_1(q) \in \operatorname{int} C_+$ (see, for example, Gasinski & Papageorgiou [7]).

Let $\vartheta(z, t)$ be measurable in z, C^1 in $t \in (0, \infty)$ and

$$0 < \hat{C} \le \frac{\vartheta_t'(z, t)t}{t} \le c_0,$$

$$c_1 t^{p(z)-1} \le \vartheta(z,t) \le c_2(t^{\mu(z)-1} + t^{p(z)-1}) \text{ for a.a. } z \in \Omega, \text{ all } t \ge 0$$

with $c_1, c_2 > 0$, $p, \mu \in C^{0,1}(\bar{\Omega})$, $1 < \mu_- \le \mu_+ < p_- < N$.

Our hypotheses on the coefficient $\beta(\cdot)$ and the exponent $\eta(\cdot)$, are the following:

$$H_0 : \beta \in L^\infty(\Omega)\backslash\{0\}, \beta(z) \ge 0 \text{ for a.a. } z \in \Omega$$

and

$$\eta \in C(\bar{\Omega}) \text{ with } 0 < \eta(z) < 1 \text{ for all } z \in \bar{\Omega}.$$

The hypothesis on the map $a(\cdot)$ involved in the definition of the differential operator, are the following:

$$H_1 : a(z,y) = a_0(z,|y|)y \text{ for all } y \in \mathbb{R}^N, \text{ with } a_0 \in C(\bar{\Omega} \times \mathbb{R}_+) \cap C^1(\Omega \times (0,+\infty)),$$

$$a_0(z,t) > 0 \text{ for all a.a. } z \in \Omega, \text{ all } t > 0 \text{ and for a.a. } x \in \Omega.$$

(i) $t \mapsto a_0(z,t)t$ is strictly increasing, $\lim_{t\to 0^+} a_0(z,t)t = 0$ uniformly for a.a. $z \in \Omega$;
(ii) $|\nabla_y a(z,y)| \le c_3 \frac{\vartheta(z,|y|)}{|y|}$ for all $y \in \mathbb{R}^N\backslash\{0\}$, some $c_3 > 0$;
(iii) $\frac{\vartheta(z,|y|)y}{|y|}|\xi|^2 \le (\nabla_y a(z,y)\xi, \xi)_{\mathbb{R}^N}$ for all $y \in \mathbb{R}^N\backslash\{0\}$, all $\xi \in \mathbb{R}^N$;
(iv) if $G_0(z,t) = \int_0^t a_0(z,s)s\mathrm{d}s$, then there exists $q \in (1, p_-)$ such that $t \mapsto G_0(z, t^{\frac{1}{q}})$ is convex, $\limsup_{t\to 0^+} \frac{G_0(z,t)}{t^q} \le c^*$ uniformly for a.a. $z \in \Omega$;
(v) $\Sigma_{k=1}^N |\frac{\partial g(z,t)}{\partial z_k}| \le c_4(1 + |\ln\gamma|)g(z,t)$ with $g(z,t) = a_0(z,t)t, t \in [\gamma, 1], \gamma \in (0,1)$.

Remark 2.1. These hypotheses are dictated by the anisotropic regularity theory of Fan [5] (which generalizes the classical isotropic nonlinear regularity theory) and the anisotropic maximum principle (see Papageorgiou, Rădulescu & Zhang [14] and Zhang [18]). These hypotheses imply that $G_0(z,\cdot)$ is strictly convex and increasing on $\mathbb{R}_+$. We set $G(z,y) = G_0(z,|y|)$ for all $z \in \Omega$, all $y \in \mathbb{R}^N$. Then $G(z,\cdot)$ is convex, differentiable in $y \in \mathbb{R}^N$ and for a.a. $z \in \Omega$, $\nabla G(z,y) = a(z,y)$, that is, $G(z,\cdot)$ is the primitive of $a(z,\cdot)$. The convexity of $G(z,\cdot)$ and since $G(z,0) = 0$, imply that

$$G(z,y) \le (a(z,y),y)_{\mathbb{R}^N} \text{ for a.a. } z \in \Omega, \text{ all } y \in \mathbb{R}^N. \tag{2.1}$$

These hypotheses provide a broad framework in which we can fit many differential operators of interest. Hypotheses H_1 and (2.1), lead to the following growth conditions on $a(\cdot)$ and $G(\cdot)$.

Lemma 2.2. *If hypothesis* H_1(i), (ii), (iii) *hold, then*

(a) $a(z,y)$ *is a Carathéodory map, which is monotone in* $y \in \mathbb{R}^N$ (*hence maximal monotone*);
(b) $|a(z,y)| \le c_5(|y|^{\mu(z)-1} + |y|^{p(z)-1})$ *for a.a.* $z \in \Omega$, *all* $y \in \mathbb{R}^N$, *some* $c_5 > 0$;
(c) $\frac{c_1|y|^{p(z)}}{p(z)-1} \le (a(z,y),y)_{\mathbb{R}^N}$ *for a.a.* $z \in \Omega$, *all* $y \in \mathbb{R}^N$;
(d) $\frac{c_1|y|^{p(z)}}{p(z)(p(z)-1)} \le G(z,y) \le c_6(1 + |y|^{p(z)})$ *for a.a.* $z \in \Omega$, *all* $y \in \mathbb{R}^N$, *some* $c_6 > 0$.

If $a(z,y) = |y|^{p(z)-2}y$, then we have the anisotropic p-Laplacian

$$\Delta_{p(z)}u = \operatorname{div}(|\mathrm{D}u|^{p(z)-2}\mathrm{D}u) \text{ for all } u \in W_0^{1,p(z)}(\Omega).$$

If $a(z,y) = |y|^{p(z)-2}y + |y|^{q(z)-2}y$ with $q(z) \le p(z)$ for all $z \in \bar{\Omega}$, then we have the anisotropic (p,q)-Laplacian

$$\Delta_{p(z)}u + \Delta_{q(z)}u \text{ for all } u \in W_0^{1,p(z)}(\Omega).$$

The hypotheses on the perturbation $f(z,x)$ are the following:

H_2 : $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \le \hat{a}(z)(1 + x^{r(z)})$ for a.a. $z \in \Omega$, all $x \ge 0$ with $\hat{a} \in L^\infty(\Omega)$ and $r \in C(\bar{\Omega})$ with $p(z) < r(z) < p^*{}_-$ for all $z \in \bar{\Omega}$;

(ii) if $F(z,x) = \int_0^x f(z,s)\mathrm{d}s$, then $\lim_{x\to+\infty} \frac{F(z,x)}{x^{p_+}} = +\infty$ uniformly for a.a. $z \in \Omega$ and there exists $\tau \in C(\bar{\Omega})$ such that

$$\tau(z) \in \left((r_+ - p_-)\frac{N}{p_-}, p^*{}_-\right),$$

$$0 < \beta_0 \le \liminf_{x\to+\infty} \frac{f(z,x)x - p_+F(z,x)}{x^{\tau(z)}} \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exists $\eta \in L^\infty(\Omega)$ such that

$$\hat{\lambda}_1(q)c^* \le \eta(z) \text{ for a.a. } z \in \Omega, \quad \hat{\lambda}_1(q)c^* \ne \eta,$$

$$\eta(z) \le \liminf_{x\to0^+} \frac{f(z,x)}{x^{q-1}} \text{ uniformly for a.a. } x \in \Omega,$$

$$(q \in (1,p_-) \text{ as postulated by hypothesis } H_1(\text{iv}));$$

(iv) there exists $\gamma > 0$ such that

$$\frac{\beta(z)}{\gamma^{\eta(z)}} + f(z,\gamma) \le -\tilde{c} < 0 \text{ for a.a. } z \in \Omega$$

and we can find $\hat{\xi}_\gamma > 0$ such that for a.a. $x \in \Omega$ the mapping

$$x \mapsto f(z,x) + \hat{\xi}_\gamma x^{p(z)-1}$$

is nondecreasing on $[0,\gamma]$.

3. AN AUXILIARY PROBLEM

The difficulty that we encounter when we study singular problems is due to the fact that the energy functional of the problem is not C^1 and so we cannot use the minimax results of the critical point theory. We need to find ways to bypass the singularity and deal with C^1-functionals.

For this reason in this section, we consider an auxiliary anisotropic Dirichlet problem, the solution of which will help us "neutralize" the singularity.

Note that hypotheses H_2 (i), (iii) imply that given $\epsilon > 0$, we can find $c_\epsilon > 0$ such that

$$f(z,x) \geq (\eta(z) - \epsilon)x^{q-1} - c_\epsilon x^{r(z)-1} \text{ for a.a. } x \in \Omega, \text{ all } x \geq 0. \tag{3.1}$$

Then we introduce the Carathéodory function $k_\epsilon(z,x)$ defined by

$$k_\epsilon(z,x) = \begin{cases} (\eta(z) - \epsilon)(x^+)^{q-1} - c_\epsilon (x^+)^{r(z)-1} & \text{if } x \leq \gamma, \\ (\eta(z) - \epsilon)\gamma^{q-1} - c_\epsilon \gamma^{r(z)-1} & \text{if } \gamma < x, \end{cases} \tag{3.2}$$

with $\gamma > 0$ as postulated by hypothesis H_2 *(iv)*.

We consider the following anisotropic Dirichlet problem

$$\begin{cases} -\operatorname{div} a(\mathrm{D}u(z)) = k_\epsilon(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, u \geq 0. \end{cases} \tag{3.3}$$

Proposition 3.1. If hypotheses H_0, H_1 hold, then for all $\epsilon > 0$ small problem (3.3) has a unique positive solution $\bar{u} \in \operatorname{int} C_+$.

Proof. Let $K_\epsilon(z,x) = \int_0^x k_\epsilon(z,s)\mathrm{d}s$ and consider the C^1-functional $\psi_\epsilon : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\epsilon(u) = \int_\Omega G(z, \mathrm{D}u)\mathrm{d}z - \int_\Omega K_\epsilon(z,u)\mathrm{d}z \text{ for all } u \in W_0^{1,p(z)}(\Omega).$$

Using Lemma 2.2(d) and (3.2), we see that $\psi_\epsilon(\cdot)$ is coercive. Also using the anisotropic Sobolev embedding theorem, we infer that $\psi_\epsilon(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $\bar{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$\psi_\epsilon(\bar{u}) = \inf\{\psi_\epsilon(u) : u \in W_0^{1,p(z)}(\Omega)\}. \tag{3.4}$$

Hypothesis H_1(iv) implies that for the given $\epsilon > 0$ (see (3.2)), we can find $\delta_0 = \delta_0(\epsilon) \in (0, \gamma)$ such that

$$G(z,y) \leq \frac{c^* + \epsilon}{q}|y|^q \text{ for a.a. } z \in \Omega, \text{ all } |y| \leq \delta_0. \tag{3.5}$$

Recall that $\hat{u}_1(q) \in \operatorname{int} C_+$. So we can find $t \in (0,1)$ small such that

$$t\mathrm{D}\hat{u}_1(q)(z) \in [0, \delta_0] \text{ for all } z \in \bar{\Omega}. \tag{3.6}$$

Using (3.6) and (3.5), we obtain

$$\psi_\epsilon(t\hat{u}_1(q)) \le \frac{c^*+\epsilon}{q}t^q\hat{\lambda}_1(q) - \frac{t^q}{q}\int_\Omega [\eta(z)-\epsilon]\hat{u}_1(q)^q \mathrm{d}z + \frac{c_\epsilon t^{r_-}}{r_-}p_r(\hat{u}_1(q))$$
$$= \frac{t^q}{q}\left[\int_\Omega (c^*\hat{\lambda}_1(q)-\eta(z))\hat{u}_1(q)^q \mathrm{d}z + 2\epsilon\right] + \frac{c_\epsilon t^{r_-}}{r_-}p_r(\hat{u}_1(q))$$
(recall that $\|\hat{u}_1(q)\|_q = 1$ and that $t \in (0,1)$).

Since $\hat{u}_1(q) \in \operatorname{int} C_+$ and $c^*\hat{\lambda}_1(q) \le \eta(z)$ for a.a. $z \in \Omega$, with strict inequality on a set of positive measure, we see that

$$\varrho_0 = \int_\Omega (c^*\hat{\lambda}_1(q)-\eta(z))\hat{u}_1(q)^q \mathrm{d}z < 0.$$

So, for $\epsilon \in (0, -\frac{1}{2}\varrho_0)$, we have

$$\psi_\epsilon(t\hat{u}_1(q)) \le c_7 t^{r_-} - c_8 t^q \text{ for some } c_7, c_8 > 0, \text{ all } t > 0.$$

Since $q < r_-$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned}
&\psi_\epsilon(t\hat{u}_1(q)) < 0,\\
&\Rightarrow\ \psi_\epsilon(\bar{u}) < 0 = \psi_\epsilon(0) \text{ (see (3.4))},\\
&\Rightarrow\ \bar{u} \ne 0.
\end{aligned}$$

From (3.4), we have

$$\begin{aligned}
&\psi_\epsilon'(\bar{u}) = 0 \text{ in } W^{-1,p'(z)}(\Omega) = W_0^{1,p(z)}(\Omega)^*.\\
&\Rightarrow\ \langle V(\bar{u}), h\rangle = \int_\Omega k_\epsilon(z,\bar{u})h\mathrm{d}z \text{ for all } h \in W_0^{1,p(z)}(\Omega),
\end{aligned} \tag{3.7}$$

with $V : W_0^{1,p(z)}(\Omega) \to W^{-1,p'(z)}(\Omega)$ being the monotone nonlinear operator defined by

$$\langle V(u), h\rangle = \int_\Omega (a(z,\mathrm{D}u),\mathrm{D}h)_{\mathbb{R}^N}\mathrm{d}z \text{ for all } u, h \in W_0^{1,p(z)}(\Omega).$$

In (3.7) we use the text function $h = -\bar{u}^- \in W_0^{1,p(z)}(\Omega)$. Using Lemma 2.2(c), we have

$$\begin{aligned}
&\int_\Omega \frac{c_1}{p(z)-1}|\mathrm{D}\bar{u}^-|^{p(z)}\mathrm{d}z \le 0 \text{ see (3.2)},\\
&\Rightarrow\ \frac{c_1}{p_+-1}\varrho_p(\mathrm{D}\bar{u}) \le 0\\
&\Rightarrow\ \bar{u} \ge 0, \bar{u} \ne 0 \text{ (by Poincaré's inequality).}
\end{aligned}$$

Next, in (3.7) we use the test function $h = (\bar{u}-\gamma)^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}
&\langle V(\bar{u}), (\bar{u}-\gamma)^+\rangle \\
&= \int\limits_\Omega \left((\eta(z)-\epsilon)\gamma^{q-1} - c_\epsilon \gamma^{r(z)-1}\right)(\bar{u}-\gamma)^+ \mathrm{d}z \text{ (see (3.2))} \\
&\le \int\limits_\Omega f(z,\gamma)(\bar{u}-\gamma)^+ \mathrm{d}z \text{ (see (3.1))} \\
&\le 0 \text{ (see hypothesis } H_2(\mathrm{iv})) \\
&\Rightarrow \bar{u} \le \gamma.
\end{aligned}$$

So, we have proved that

$$\bar{u} \in [0,\gamma], \ \bar{u} \ne 0. \tag{3.8}$$

Then from (3.8), (3.2) and (3.7), we refer that $\bar{u} \in W_0^{1,p(z)}(\Omega)$ is a positive solution of the auxiliary problem (3.3). From Fan & Zhao [6], we know that $\bar{u} \in L^\infty(\Omega)$. Then the regularity theory of Fan [5] implies that $\bar{u} \in C_+ \backslash \{0\}$. Finally, the anisotropic maximum principle of Zhang [18] implies that $\bar{u} \in \operatorname{int} C_+$.

Next, we show the uniqueness of this positive solution. For this purpose we introduce the integral functional $j : L^1(\Omega) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) : \begin{cases} \int_\Omega G(z, \mathrm{D}u^{\frac{1}{q}})\mathrm{d}z & \text{if } u \ge 0, \ u^{\frac{1}{q}} \in W_0^{1,p(z)}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From Theorem 2.2 of Takač & Giacomoni [16], we have that $j(\cdot)$ is convex. Suppose that $\bar{v} \in W_0^{1,p(z)}(\Omega)$ is another positive solution of problem (3.3). Again we have $\bar{v} \in \operatorname{int} C_+$. Using Proposition 4.1.22 of Papageorgiou, Rădulescu & Repovs [12, p. 274], we have

$$\frac{\bar{u}}{\bar{v}} \in L^\infty(\Omega), \quad \frac{\bar{v}}{\bar{u}} \in L^\infty(\Omega). \tag{3.9}$$

Let $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (the effective domain of $j(\cdot)$) and consider $h = \bar{u}^q - \bar{v}^q \in W_0^{1,p(z)}(\Omega)$. On account of (3.9) for $t \in (0,1)$ small, we have

$$\bar{u}^q + th \in \operatorname{dom} j, \quad \bar{v}^q + th \in \operatorname{dom} j.$$

Then the convexity of $j(\cdot)$ implies that the directional derivative of $j(\cdot)$ at $\bar{u}^q$ and at $\bar{v}^q$ in the direction h exists and using Green's identity (see also Takač & Giacomoni [16, Theorem 2.5]), we have

$$j'(\bar{u}^q)(h) = \frac{1}{q}\int\limits_\Omega \frac{-\operatorname{div} a(z,\mathrm{D}\bar{u})}{\bar{u}^{q-1}} h \mathrm{d}z = \frac{1}{q}\int\limits_\Omega \frac{(\eta(z)-\epsilon)\bar{u}^{q-1} - c_\epsilon \bar{u}^{r-1}}{\bar{u}^{q-1}} h \mathrm{d}z,$$

$$j'(\bar{v}^q)(h) = \frac{1}{q}\int\limits_\Omega \frac{-\operatorname{div} a(z,\mathrm{D}\bar{v})}{\bar{v}^{q-1}} h \mathrm{d}z = \frac{1}{q}\int\limits_\Omega \frac{(\eta(z)-\epsilon)\bar{v}^{q-1} - c_\epsilon \bar{v}^{r-1}}{\bar{v}^{q-1}} h \mathrm{d}z.$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Hence

$$0 \le c_\epsilon \int_\Omega (\bar{v}^{r(z)-1} - \bar{u}^{r(z)-1})(\bar{u}^q - \bar{v}^q)\mathrm{d}z \le 0,$$

which gives $\bar{u} = \bar{v}$. This proves the uniqueness of the positive solution $\bar{u} \in \operatorname{int} C_+$ of problem (3.3). $\square$

4. POSITIVE SOLUTIONS

Now, we will use $\bar{u} \in \operatorname{int} C_+$ from Proposition 3.1, in order to generate positive solutions for problem (1.1). Recall that $\bar{u} \in \operatorname{int} C_+$ and $0 \le \bar{u} \le \gamma$ (see (3.8)).

Proposition 4.1. If hypothesis H_0, H_1, H_2 hold, then problem (1.1) has a positive solution $u_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[0, \gamma]$.

Proof. For every $h \in W_0^{1,p(z)}(\Omega)$, we have

$$\varrho_p\left(\frac{h}{\bar{u}^{\eta(\cdot)}}\right) \le \int_\Omega \bar{u}^{(1-\eta(z))p(x)}\left(\frac{|h|}{\bar{u}}\right)^{p(z)} \mathrm{d}z. \tag{4.1}$$

Let $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \bar{\Omega}$. Since $\bar{u} \in \operatorname{int} C_+$, we can find $c_9 > 0$ such that

$$\hat{d} \le c_9 \bar{u}. \tag{4.2}$$

Using (4.2) in (4.1), we obtain

$$\begin{aligned}
\varrho_p\left(\frac{h}{\bar{u}^{\eta(\cdot)}}\right) &\le c_{10}\varrho_p\left(\frac{h}{\hat{d}}\right) \text{ for some } c_{10} > 0 \\
&\le c_{11} \max\left\{\left\|\frac{h}{\hat{d}}\right\|_{p(z)}^{p_-}, \left\|\frac{h}{\hat{d}}\right\|_{p(z)}^{p_+}\right\} \text{ for some } c_{11} > 0 \\
&\le c_{12} \max\left\{\|\mathrm{D}h\|_{p(z)}^{p_-}, \|\mathrm{D}h\|_{p(z)}^{p_+}\right\} \text{ for some } c_{12} > 0.
\end{aligned}$$

Here we have used the anisotropic Hardy's inequality due to Harjulehto, Hästo & Koskenoja [9]. It follows that

$$\frac{h}{\bar{u}^{\eta(\cdot)}} \in L^{p(z)}(\Omega) \text{ for all } h \in W_0^{1,p(z)}(\Omega). \tag{4.3}$$

We introduce the Carathéodory function $\hat{f}(z, x)$ defined by

$$\hat{f}(z,x) = \begin{cases} \beta(z)\bar{u}(z)^{-\eta(z)} + f(z, \bar{u}(z)) & \text{if } x < \bar{u}(z), \\ \beta(z)x^{-\eta(z)} + f(z,x) & \text{if } \bar{u}(z) \le x \le \gamma, \\ \beta(z)\gamma^{-\eta(z)} + f(z,\gamma) & \text{if } \gamma < x. \end{cases} \tag{4.4}$$

We set $\hat{F}(z,x) = \int_0^x \hat{f}(z,s)\mathrm{d}s$ and consider the functional $\hat{\varphi} : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}(u) = \int_\Omega G(z, \mathrm{D}u)\mathrm{d}z - \int_\Omega \hat{F}(z,u)\mathrm{d}z \text{ for all } W_0^{1,p(z)}(\Omega).$$

We know that $\hat{\varphi} \in C^1(W_0^{1,p(z)}(\Omega))$ (see (4.3) and Proposition 3 of Papageorgiou & Smyrlis [11]). From Lemma 2.2 and (4.4), it is clear that $\hat{\varphi}(\cdot)$ is coercive. Also, using the anisotropic Sobolev embedding theorem, we see that $\hat{\varphi}(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W_0^{1,p(z)}(\Omega)$ such that

$$\begin{aligned} &\hat{\varphi}(u_0) = \inf\{\hat{\varphi}(u) : u \in W_0^{1,p(z)}(\Omega)\} \\ &\Rightarrow \hat{\varphi}'(u_0) = 0 \text{ in } W^{-1,p'(z)}(\Omega) = W_0^{1,p(z)}(\Omega)^* \\ &\Rightarrow \langle V(u_0), h\rangle = \int_\Omega \hat{f}(z,u_0)h\mathrm{d}z \text{ for all } h \in W_0^{1,p(z)}(\Omega). \end{aligned} \tag{4.5}$$

In (4.5) we use the test function $h = (\bar{u} - u_0)^+ \in W_0^{1,p(z)}(\Omega)$. Then

$$\begin{aligned} &\langle V(u_0), (\bar{u} - u_0)^+\rangle \\ &= \int_\Omega (\beta(z)\bar{u}^{-\eta(z)} + f(z,\bar{u}))(\bar{u} - u_0)^+\mathrm{d}z \text{ (see (4.4))} \\ &\geq \int_\Omega f(z,\bar{u})(\bar{u} - u_0)^+\mathrm{d}z \\ &\geq \int_\Omega \left((\eta(z) - \epsilon)\bar{u}^{q-1} - c_\epsilon \bar{u}^{r(z)-1}\right)(\bar{u} - u_0)^+\mathrm{d}z \text{ (see (3.1))} \\ &= \langle V(\bar{u}), (\bar{u} - u_0)^+\rangle \text{ (see Proposition 4.1)}, \\ &\Rightarrow \langle V(\bar{u}) - V(u_0), (\bar{u} - u_0)^+\rangle \leq 0, \\ &\Rightarrow \bar{u} \leq u_0. \end{aligned}$$

Next, in (4.5) we choose the test function $h = (u_0 - \gamma)^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned} &\langle V(u_0), (\bar{u} - \gamma)^+\rangle \\ &= \int_\Omega (\beta(z)\gamma^{-\eta(z)} + f(z,\gamma))(u_0 - \gamma)^+\mathrm{d}z \text{ (see (4.4))} \\ &\leq 0 \text{ (see hypothesis } H_2\text{(iv))} \\ &\Rightarrow u_0 \leq \gamma. \end{aligned}$$

We have proved that

$$u_0 \in [\bar{u}, \gamma], \tag{4.6}$$

so u_0 is a positive solution of (1.1) (see (4.4), (4.5)).

The anisotropic regularity theory implies that $u_0 \in \operatorname{int} C_+$.

Let $\hat{\xi}_\gamma > 0$ be as postulated by hypothesis H_2(iv). We have

$$\begin{aligned}
&-\operatorname{div} a(\mathrm{D}u_0) + \hat{\xi}_\gamma u_0^{p(z)-1} \\
&\geq -\operatorname{div} a(\mathrm{D}u_0) + \hat{\xi}_\gamma u_0^{p(z)-1} - \beta(z) u_0^{-\eta(z)} \\
&= f(z, u_0) + \hat{\xi}_\gamma u_0{}^{p(z)-1} \\
&\geq f(z, \bar{u}) + \hat{\xi}_\gamma \bar{u}^{p(z)-1} \text{ (see (4.6) (see hypothesis } H_2\text{(iv))} \\
&\geq (\eta(z) - \epsilon)\bar{u}^{q-1} - c_\epsilon \bar{u}^{r(z)-1} + \hat{\xi}_\gamma \bar{u}^{p(z)-1} \text{ (see (3.1))} \\
&= -\operatorname{div} a(\mathrm{D}\bar{u}) + \hat{\xi}_\gamma \bar{u}^{p(z)-1} \text{ (see Proposition 4.1)} \\
&\Rightarrow u_0 - \bar{u} \in \operatorname{int} C_+ \\
&\text{(see Proposition 2.4 of Papageorgious, Rădulescu \& Repovs [13]).}
\end{aligned} \tag{4.7}$$

Also, we have

$$\begin{aligned}
&-\operatorname{div} a(\mathrm{D}u_0) + \hat{\xi}_\gamma u_0^{p(z)-1} - \beta(z) u_0^{-\eta(z)} \\
&= f(z, u_0) + \hat{\xi}_\gamma u_0{}^{p(z)-1} \\
&\leq f(z, \gamma) + \hat{\xi}_\gamma^{p(z)-1} \text{ (see (4.6) and hypothesis } H_2\text{(iv))} \\
&\leq -\tilde{c} - \beta(z)\gamma^{-\eta(z)} + \hat{\xi}_\gamma \gamma^{p(z)-1} \text{ (see hypothesis } H_2\text{(iv))} \\
&\leq -\operatorname{div} a(\mathrm{D}\gamma) + \hat{\xi}_\gamma \gamma^{p(z)-1} - \beta(z)\gamma^{-\eta(z)}, \\
&\Rightarrow u_0(z) < \gamma \text{ for all } z \in \bar{\Omega} \\
&\quad \text{(see Proposition A4 of Papageorgiou, Rădulescu \& Zhang [14]).}
\end{aligned} \tag{4.8}$$

From (4.7) and (4.8), we conclude that

$$u_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[\bar{u}, \gamma].$$

The proof is now complete. □

We can produce a second positive solution distinct from u_0.

Proposition 4.2. If hypothesis H_0, H_1, H_2 hold, then problem (1.1) has a second positive solution

$$\hat{u} \in \operatorname{int} C_+, \quad \hat{u} \neq u_0.$$

Proof. We introduce the Carathéodory function $f_0(z, x)$ defined by

$$f_0(z, x) = \begin{cases} \beta(z)\bar{u}(z)^{-\eta(z)} + f(z, \bar{u}(z)) & \text{if } x \leq \bar{u}(z), \\ \beta(z) x^{-\eta(z)} + f(z, x) & \text{if } \bar{u}(z) < x. \end{cases} \tag{4.9}$$

We set $F_0(z, x) = \int_0^x f_0(z, s)\mathrm{d}s$ and consider the C^1-functional $\varphi_0 : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(z, \mathrm{D}u)\mathrm{d}z - \int_\Omega F_0(z, u)\mathrm{d}z \text{ for all } u \in W_0^{1,p(z)}(\Omega).$$

From (4.4) and (4.9) we see that

$$\varphi_0|_{[\bar{u},\gamma]} = \hat{\varphi}|_{[\bar{u},\gamma]}. \tag{4.10}$$

Recall that u_0 is a minimizer of $\hat{\varphi}(\cdot)$ (see the proof of Proposition 4.2). Also we know that $u_0 \in \mathrm{int}_{C_0^1\bar{\Omega}}[\bar{u},\gamma]$. Then from (4.10) we infer that

$$\begin{aligned} &u_0 \text{ is a local } C_0^1(\bar{\Omega})\text{-minimizer of } \varphi_0(\cdot),\\ &\Rightarrow u_0 \text{ is a local } W_0^{1,p(z)}(\Omega)\text{-minimizer of } \varphi_0(\cdot)\,(\text{see [14, Proposition A3]}). \end{aligned} \tag{4.11}$$

Using (4.9) we can check easily that

$$K_{\varphi_0} \subseteq \{u \in W_0^{1,p(z)}(\Omega) : \bar{u} \leq u(z) \text{ for a.a. } z \in \Omega\} \cap \mathrm{int}\, C_+. \tag{4.12}$$

From (4.12) and (4.9), we see that we may assume that

$$K_{\varphi_0} \text{ is finite} \tag{4.13}$$

or otherwise we already have an infinity of positive smooth solutions of (1.1) and so we are done. Then (4.11), (4.13) and Theorem 5.7.6 of [12, p.449], imply that we can find $\rho \in (0,1)$, small such that

$$\varphi_0(u_0) < \inf\{\varphi_0(u) : \|u-u_0\| = \rho\} = m_0. \tag{4.14}$$

Hypothesis H_2(ii) implies that if $u \in \mathrm{int}\, C_+$, then

$$\varphi_0(tu) \to -\infty \text{ as } t \to +\infty. \tag{4.15}$$

Arguing as in the Claim in the proof of Proposition 4 of [14] and using hypothesis H_2(iii), we show that

$$\varphi_0(\cdot) \text{ satisfies the } C\text{-condition}. \tag{4.16}$$

Then (4.14), (4.15) and (4.16) permit the use of the mountain pass theorem. So, we can find $\hat{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$\begin{aligned} &\hat{u} \in K_{\varphi_0},\ m_0 \leq \varphi_0(\hat{u})\\ &\Rightarrow\ \hat{u} \notin \{0, u_0\} \text{ (see (4.12), (4.14) and is a positive solution of (1.1))} \end{aligned}$$

The proof is now complete. □

Finally, we can state the following multiplicity theorem for problem (1.1).

Theorem 4.3. *If hypothesis H_0, H_1, H_2 hold, then problem* (1.1) *has at least two positive solutions*

$$u_0, \hat{u} \in \mathrm{int}\, C_+,\ u_0 \neq \hat{u},\ u_0(z) < \gamma \textit{ for all } z \in \bar{\Omega}.$$

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