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HUBTIC NUMBER IN GRAPHS

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Abstract. The maximum order of partition of the vertex set $V(G)$ into hub sets is called hubtic number of G and denoted by $\xi(G)$. In this paper we determine the hubtic number of some standard graphs. Also we obtain bounds for $\xi(G)$. And we characterize the class of all (p,q) graphs for which $\xi(G)=p$.

Keywords: hubtic number, hub number, partition.

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1. INTRODUCTION

By a graph $G=(V,E)$ we mean a finite and undirected graph without loops and multiple edges. A graph G with p vertices and q edges is called a (p,q) graph, the number p is referred to as the order of a graph G and q is referred to as the size of a graph G. In general, the degree of a vertex v in a graph G denoted by $deg(v)$ is the number of edges of G incident with v. Also $\delta(G)$ denotes the minimum degree among the vertices of G [3]. $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Given any vertex $v \in V(G)$, the graph obtained from G by removing the vertex v and all of its incident edges is denoted by $G-v$. In a tree, a leaf is a vertex of degree one. See [3] for terminology and notations not defined here.

Introduced by Walsh [7], a hub set in a graph G is a set H of vertices in G such that any two vertices outside H are connected by a path whose all internal vertices lie in H. The hub number of G, denoted by $h(G)$, is the minimum size of a hub set in G. A connected hub set in G is a vertex set F such that F is hub set and the subgraph of G induced by F (denoted $G[F]$) is connected. The connected hub number of G, denoted $h_c(G)$, is the minimum size of a connected hub set in G [5]. For more details on the hub studies we refer to [5–7]. Graphs G_1 and G_2 have disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively, their union, $G(V,E)=G_1 \cup G_2$ has, as expected, $V=V_1 \cup V_2$ and $E=E_1 \cup E_2$. Their join is denoted by G_1+G_2 and consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2 [3]. The contraction of a vertex v in G

(denoted by G/v) as being the graph obtained by deleting v and putting a clique on the (open) neighbourhood of v, (note that this operation does not create multiple edges, if two neighbours of v are already adjacent, then they remain simply adjacent) [7].

A set D of vertices in a graph G is called dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D, the minimum cardinality of a dominating set in G is called the domination number $\gamma(G)$ of a graph G [4]. In 1977, E.J. Cockayne and S.T. Hedetniemi introduced the concept of domatic number of graph G and defined by, a D-partition of G is a partition of $V(G)$ into dominating sets, the domatic number of G denoted by $d(G)$ is the maximum order of a D-partition of G [1]. Using the concept of hub set of a graph G and the definition of the domatic number of a graph G, motivated by this, we introduce the concept of hubtic number of a graph G as a new parameter of a graph.

A double star $S_{n,m}$ is the tree obtained from two disjoint stars $K_{1,n-1}$ and $K_{1,m-1}$ by connecting their centers [2]. The chromatic number $\chi(G)$ of a graph G is the minimum number of colors required to assign to the vertices of G in such a way that no two adjacent vertices of G receive the same color [3]. A clique of a graph is a maximal complete subgraph, and $c(G)$ (resp. $C(G)$) will denote the smallest (resp. largest) order of a clique of G [3].

The following results will be useful in the proof of our results.

Theorem 1.1 ([7]). *Let $\Delta(G)$ denote the maximum degree of G. Then if G is a connected graph $h(G) \leqslant |V(G)| - \Delta(G)$, and the inequality is sharp.*

Proposition 1.2 ([1]). *For any graph G,*

$$d(G) \leqslant \delta(G) + 1.$$

Theorem 1.3 ([7]). *Let T be a tree with p vertices and l leaves. Then*

$$h(T) = h_c(T) = p - l.$$

Theorem 1.4 ([5]). *If G is connected, then*

$$h_c(G) \leqslant h(G) + 1.$$

Theorem 1.5 ([7]). *Let S be a subset of $V(G)$. Then G/S is complete if and only if S is a hub set of G.*

2. HUBTIC NUMBER

Definition 2.1. The maximum order of partition of the vertex set $V(G)$ into hub sets is called hubtic number of G, and denoted by $\xi(G)$. A H-partition of a graph G is a partition of $V(G)$ into hub sets.

The ihubtic number $i\xi(G)$ of a graph G is the maximum order of a partition of $V(G)$ into hub sets which are independent. In the following observation we determine the hubtic number of some standard graphs. The proofs of these results are simple and are omitted.

Observation 2.2. *1. For any complete graph K_p,*

$$\xi(K_p) = p.$$

2. For any cycle C_p,

$$\xi(C_p) = \begin{cases} 3, & \textit{if } p = 3, \\ 4, & \textit{if } p = 4, \\ 2, & \textit{if } p = 5, 6, \\ 1, & \textit{if } p \geqslant 7. \end{cases}$$

3. For any path P_p,

$$\xi(P_p) = \begin{cases} 2, & \textit{if } p = 2, \\ 3, & \textit{if } p = 3, \\ 2, & \textit{if } p = 4, \\ 1, & \textit{if } p \geqslant 5. \end{cases}$$

4. For the wheel graph $W_{1,p-1}$, $p \geqslant 4$,

$$\xi(W_{1,p-1}) = \begin{cases} 4, & \textit{if } p = 4, \\ 5, & \textit{if } p = 5, \\ 3, & \textit{if } p = 6, 7, \\ 2, & \textit{if } p \geqslant 8. \end{cases}$$

5. For the star $K_{1,p-1}$, $p \geqslant 4$,

$$\xi(K_{1,p-1}) = 2.$$

6. For the double star $S_{n,m}$,

$$\xi(S_{n,m}) = 2.$$

7. For the complete bipartite graph $K_{n,m}$, $m, n \geqslant 3$,

$$\xi(K_{n,m}) = min\{n, m\}.$$

Observation 2.3. *For any graph G, $1 \leqslant \xi(G) \leqslant p$. If $h(G) > 0$, then*

$$\xi(G) \leqslant \left\lfloor \frac{p}{h(G)} \right\rfloor.$$

Proof. Let $H = \{H_1, H_2, H_3, \ldots, H_{\xi(G)}\}$ be the hubtic partition of graph G. Clearly, $|H_i| \geqslant h(G)$ for all $i = 1, 2, \ldots, \xi(G)$, so

$$p = \sum_{i=1}^{\xi(G)} |H_i| \geqslant \xi(G)h(G).$$

Hence the assertion follows. □

Theorem 2.4. *Let G be a tree with at least 3 non-leaf vertices. Then $\xi(G) = 1$.*

Proof. Suppose that G is a tree with at least 3 non-leaf vertices. We discuss the following cases:

Case 1. Let H be a set of all non-leaf vertices. Clearly, any path between two leaves does not pass through another leaf. So, H is a hub set of G, and by Theorem 1.3, it is a minimum hub set. Now, suppose that $B \subseteq V(G) - H$ is a hub set of G. Since G is a tree with at least 3 non-leaf vertices, take any two non-adjacent vertices $u, v \in H$. Since all vertices in B are leaves, then there is no path between u and v with all internal vertices in B. This is a contradiction. Hence $\xi(G) = 1$.

Case 2. Suppose that H is a hub set of G but not containing all non-leaf vertices. Since G has at least three non-leaf vertices, let $\{v_1, v_2, v_3\}$ be non-leaf vertices and $v_1v_3 \notin E(G)$, let l_1, l_3 be leaves adjacent to v_1 and v_3, respectively. Clearly, $G[\{l_1, v_1, v_2, v_3, l_3\}]$ is a path P_5. Since $h(P_5) = 3$, then H contains at least three vertices from P_5. Then any other hub set of G must intersects H since $|P_5| = 5$, therefore $\xi(G) = 1$. □

Proposition 2.5. *For any graph G,*

$$\xi(G) \leqslant \delta(G) + 2.$$

Proof. Let $H = \{H_1, H_2, H_3, \ldots, H_{\xi(G)}\}$ be the hubtic partition of graph G. Let $v \in H_j$ for some $1 \leqslant j \leqslant \xi(G)$ such that v is not adjacent to any vertex in two hub sets H_n, H_m and $n \neq m \neq j$. Then G/H_n is not complete, since v is not adjacent to any vertex in H_n and in H_m. So by Theorem 1.5, H_n is not a hub set of G, which is a contradiction, therefore v is adjacent to at least $\xi(G) - 2$ vertices. Then $\xi(G) \leqslant \delta(G) + 2$. □

By Propositions 1.2 and 2.5, we get the next result.

Remark 2.6. For any graph G,

$$\xi(G) + d(G) \leqslant 2\delta(G) + 3.$$

Theorem 2.7. *For any non-regular graph G of order p,*

$$\xi(G) + \xi(\overline{G}) \leqslant p + 2.$$

Proof. By Proposition 2.5, $\xi(G) \leqslant \delta(G) + 2$, and $\xi(\overline{G}) \leqslant \delta(\overline{G}) + 2 < \Delta(\overline{G}) + 2$. Then

$$\xi(G) + \xi(\overline{G}) < \delta(G) + \Delta(\overline{G}) + 4 = (p - 1) + 4 = p + 3.$$

This yields the desired conclusion. □

Theorem 2.8. *Let G be a non-regular graph of order p. Then*

$$\xi(G) + h(G) \leqslant p + 1.$$

Proof. By Theorem 1.1, $h(G) \leqslant p-\Delta(G)$. Since G is non-regular, then $h(G) < p-\delta(G)$. Also from Proposition 2.5 we deduce that $\xi(G) \leqslant \delta(G)+2$. Now we have $\xi(G) \leqslant \delta(G)+2$ and $h(G) < p - \delta(G)$. It follows that $\xi(G) + h(G) < p + 2$. The proof is finished. □

By the previous theorem and Theorem 1.4, the next corollary is obvious.

Corollary 2.9. *Let G be a connected graph of order p. Then*

$$\xi(G) + h_c(G) \leqslant p + 2.$$

Lemma 2.10. *Let G be a non-regular graph of order p. Then*

$$h(G) + h(\overline{G}) \leqslant p.$$

Proof. By Theorems 2.7 and 2.8, $\xi(G) + h(G) \leqslant p+1$ and $\xi(\overline{G}) + h(\overline{G}) \leqslant p+1$. Thus we get $\xi(G) + \xi(\overline{G}) + h(G) + h(\overline{G}) \leqslant p + 2 + p$, which implies the assertion. □

Corollary 2.11. *Let G be a connected graph of order p. Then*

$$h_c(G) + h_c(\overline{G}) \leqslant p + 2.$$

Theorem 2.12. *For any graph G of order p, $\xi(G) = p$ if and only if $G \cong K_p$, or G with $\delta \geqslant p - 2$.*

Proof. Suppose that $\xi(G) = p$. Then we have p partition of $V(G)$ into hub sets and every partite set consists of one vertex, we have the following cases:

Case 1. All vertices of G are adjacent, so we can choose every vertex of G as a hub set, hence G should be complete graph K_p. So $\delta = p - 1$.

Case 2. Any vertex of degree $p - 1$ is adjacent to all vertices and hence it constitutes a hub set of G. Since any vertex of degree $p - 2$ is adjacent to all vertices of G except one, so every vertex of them must be a hub set for G, hence $\delta(G) = p - 2$, if we consider any vertex u such that $deg(u) < p-2$. In this case $\{u\}$ is not a hub set for G. So $\xi(G) = p$ only if the graph G satisfies $\delta(G) = p - 2$. The converse is obvious. □

Theorem 2.13. *For any (p, q) graph G, we have*

$$\xi(G) \geqslant \left\lfloor \frac{p}{p - \delta(G)} \right\rfloor.$$

Proof. Let G be a graph. The following cases are considered.
Case 1. G is disconnected graph. Obviously, $\delta \leqslant \frac{p}{2} - 1$. Then $\left\lfloor \frac{p}{p-\delta(G)} \right\rfloor = 1$. Hence $\xi(G) \geqslant \left\lfloor \frac{p}{p-\delta(G)} \right\rfloor$.
Case 2. G is a connected graph. If $G = K_p$, then $\xi(G) = p$. Let $G \neq K_p$, we have two subcases:

Subcase 2.1. $\delta \leqslant \frac{p}{2}$. If inequality is strict, the result is trivial. Let $\delta = \frac{p}{2}$. Then $\lfloor \frac{p}{p-\delta(G)} \rfloor = 2$. In this case, we can get at least two hub sets for G, thus the result follows.

Subcase 2.2. $\delta > \frac{p}{2}$, clearly $\lfloor \frac{p}{p-\delta(G)} \rfloor \leqslant \frac{p}{2}$ and $\xi(G) > \frac{p}{2}$, hence we obtain the assertion. □

Proposition 2.14. *For any two connected graphs G_1 and G_2,*

$$\xi(G_1 \cup G_2) = \begin{cases} 1, & \text{if } G_1 \text{ or } G_2 \text{ is non-complete,} \\ 2, & \text{if } G_1 \text{ and } G_2 \text{ are complete.} \end{cases}$$

Proof. Let G_1, G_2 be both complete graphs. Clearly, $V(G_1)$ is a hub set for $G_1 \cup G_2$ and $V(G_2)$ is a hub set of the same graph, thus $\xi(G_1 \cup G_2) = 2$. If G_1 or G_2 is not complete graph, then any hub set of $G_1 \cup G_2$ must contain all of the vertices of G_1 and any hub set of G_2, therefore $\xi(G_1 \cup G_2) = 1$. □

Corollary 2.15. *If G is disconnected graph with $n \geqslant 3$ components, then $\xi(G) = 1$.*

The inequality $c(G) \leqslant \xi(G)$ is not true for $G \cong C_7$. Therefore we define the following graph.

Definition 2.16. A graph G is said to be inhubable if it has a H-partition in which every partite set is independent.

Proposition 2.17. *Let G be inhubable graph. Then $c(G) \leqslant i\xi(G)$.*

Proof. By definition, chromatic number of G is the minimum order of a partition of $V(G)$ into independent subsets. Clearly that

$$c(G) \leqslant C(G) \leqslant \chi(G) \leqslant i\xi(G).$$ □

3. CONCLUSION

This paper introduces a new variation in the theory of hub number namely hubtic number, and just initiates a study on this notion. We list some interesting problems for further research that we encountered during the course of our investigation.

1. Find a characterization of graphs G for which $d(G) = \xi(G)$.
2. Find a characterization of graphs G for which $\xi(G) = 1$.
3. Obtain more relations between $\xi(G)$ and the other graph parameters, also obtain good bounds for $\xi(G)$.

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