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THE PUTNAM-FUGLEDE PROPERTY FOR PARANORMAL AND $*$-PARANORMAL OPERATORS

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Abstract. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to have the Putnam-Fuglede commutativity property (PF property for short) if $T^*X = XJ$ for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and any isometry $J \in \mathcal{B}(\mathcal{K})$ such that $TX = XJ^*$. The main purpose of this paper is to examine if paranormal operators have the PF property. We prove that $k*$-paranormal operators have the PF property. Furthermore, we give an example of a paranormal without the PF property.

Keywords: power-bounded operators, paranormal operators, $*$-paranormal operators, k-paranormal operators, $k*$-paranormal operators, the Putnam-Fuglede theorem.

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1. TERMINOLOGY

Throughout what follows, $\mathbb{Z}$ stands for the set of all integers, $\mathbb{Z}_-$ for the set of all negative integers, $\mathbb{N}$ for the set of all non-negative integers and $\mathbb{N}_+$ for the set of all positive integers. Complex Hilbert spaces are denoted by $\mathcal{H}$ and $\mathcal{K}$ and the inner product is denoted by $\langle \cdot, - \rangle$. Moreover, we denote by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ the set of all bounded operators from $\mathcal{H}$ into $\mathcal{K}$. To simplify, we put $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$. The identity operator on $\mathcal{H}$ is denoted by $Id_{\mathcal{H}}$. If X is a subset of $\mathcal{H}$, then $spanX$ stands for the linear span of X and $\overline{X}$ stands for the closure of X. We say that $T \in \mathcal{B}(\mathcal{H})$ is a *contraction* if $\|Tx\| \leq \|x\|$ for each $x \in \mathcal{H}$. By a *power-bounded* operator we mean $T \in \mathcal{B}(\mathcal{H})$ such that the sequence $\{\|T^n\|\}_{n \in \mathbb{N}}$ is bounded. An operator T is said to be *completely nonunitary* if T restricted to every reducing subspace of $\mathcal{H}$ is nonunitary. As usual, T^* stands for the adjoint of T. We now recall some known classes of operators defined on a Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *hyponormal* if $T^*T \geq TT^*$, or equivalently, $\|T^*x\| \leq \|Tx\|$ for each $x \in \mathcal{H}$. An operator T is (p,k)*-quasihyponormal* if $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$. Usually $(p,0)$-quasihyponormal operators are known as *p-hyponormal* operators. Another generalization of hyponormal operators are

$k*$-paranormal operators. An operator $T \in \mathcal{B}(\mathcal{H})$ is *$k*$-paranormal* if $\|T^*x\|^k \leq \|T^k x\|$ for each $x \in \mathcal{H}$ such that $\|x\| = 1$. Moreover, by a *k-paranormal* operator we mean an operator $T \in \mathcal{B}(\mathcal{H})$ which satisfies $\|Tx\|^k \leq \|T^k x\|$ for each $x \in \mathcal{H}$ such that $\|x\| = 1$. For $k = 2$, k-paranormal and $k*$-paranormal operators are called simply *paranormal* and *$*$-paranormal* operators, respectively.

The inclusion relations between the above-mentioned classes of operators are shown in Figure 1 (cf. [7, 11]).

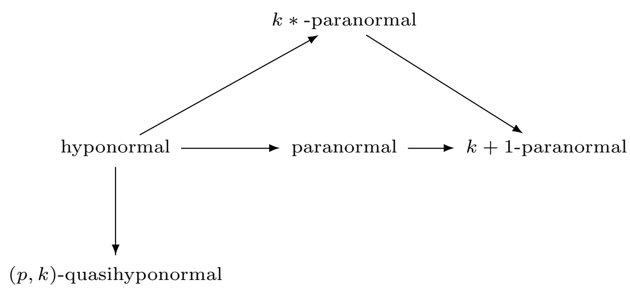

Fig. 1. Inclusions between classes of operators

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *of class $C_{0\cdot}$* if $\liminf_{n\to\infty} \|T^n x\| = 0$ for each $x \in \mathcal{H}$. Note that a power-bounded operator T is of class $C_{0\cdot}$ if and only if it is strongly stable, i.e. $T^n \to 0$ in the strong operator topology (see [12]). Furthermore, we say that T is *of class $C_{\cdot 0}$* if its adjoint is of class $C_{0\cdot}$.

Definition 1.1 ([4])**.** An operator $T \in \mathcal{B}(\mathcal{H})$ is said to have the *Putnam-Fuglede commutativity property* (*PF property* for short) if $T^*X = XJ$ for any $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and any isometry $J \in \mathcal{B}(\mathcal{K})$ such that $TX = XJ^*$.

2. INTRODUCTION

In [5] Duggal and Kubrusly have shown that a contraction T has the PF property if and only if T is the orthogonal sum $T = U \oplus C$ of a unitary operator U and an operator C which is a $C_{\cdot 0}$ contraction. In the subsequent section, using isometric asymptotes (see [8]), we show that the above statement is also true for power-bounded operators. Moreover, we give the relation between operators with the PF property and A-isometries.

In [11] we have shown that k-paranormal, $k*$-paranormal, (p, k)-quasihyponormal contractions and contractions of class Q have the PF property (see also [3, 4, 6]). Our main purpose is to answer the question: *Which of the above mentioned operators* (*not necessarily contractions*) *have the PF property?*

In [9] Kim has shown that p-hyponormal operators satisfy the general Putnam-Fuglede property. So in particular they have the PF property. Until now, to the best of our knowledge, nothing was known about the PF property for noncontractive $k*$-paranormal and k-paranormal operators.

In Section 3, we show that $k*$-paranormal operators have the PF property. Finally, in Section 4 we present an example of a paranormal (k-paranormal) operator without the PF property.

3. GENERAL REMARKS ON THE PF PROPERTY

In [5] Duggal and Kubrusly have shown the following result.

Proposition 3.1. *If a nonunitary coisometry is a direct summand of a contraction T, then T does not have the PF property. In particular, if a coisometry has the PF property, then it is a unitary operator.*

Appealing to the above result we now show the following theorem.

Theorem 3.2. *A power-bounded operator $T \in \mathcal{B}(\mathcal{H})$ has the PF property if and only if it is a direct sum of a unitary operator and an operator of class $C_{\cdot 0}$.*

Proof. For $x, y \in \mathcal{H}$, we set $[x,y] := \operatorname{glim}\{\langle T^{*n}x, T^{*n}y\rangle\}_{n\in\mathbb{N}}$, where glim denotes the Banach limit. In this way we obtain a new semi-inner product on $\mathcal{H}$. Thus the factor space $\mathcal{H}/\mathcal{H}_0$, where $\mathcal{H}_0$ stands for the linear manifold $\mathcal{H}_0 := \{x \in \mathcal{H} | [x,x] = 0\}$, is an inner product space endowed with the inner product given by $[x+\mathcal{H}_0, y+\mathcal{H}_0] = [x,y]$ for $x, y \in \mathcal{H}$. Let $\mathcal{K}$ denote the resulting Hilbert space obtained by a completion of $\mathcal{H}/\mathcal{H}_0$. Denote by Q the canonical embedding $Q : \mathcal{H} \ni x \mapsto x + \mathcal{H}_0 \in \mathcal{K}$. Note that

$$\|QT^*x\| = \|Qx\|, \quad x \in \mathcal{H}.$$

Hence there is an isometry $V : \mathcal{K} \to \mathcal{K}$ such that $QT^* = VQ$, so from the PF property we deduce that

$$QT^* = VQ \Longleftrightarrow TQ^* = Q^*V^* \Longrightarrow T^*Q^* = Q^*V \Longleftrightarrow QT = V^*Q.$$

Therefore, $\mathcal{R}(Q^*)$ is an invariant subspace for T and T^*. This means that $\overline{\mathcal{R}(Q^*)}$ is reducing for T.

Observe that, for all $x, y \in \mathcal{H}$, we have

$$[Qx, Qy] = [V^nQx, QT^{*n}y] = \langle Q^*V^nQx, T^{*n}y\rangle = \langle T^{*n}Q^*Qx, T^{*n}y\rangle \tag{3.1}$$

and

$$\operatorname{glim}\{\langle T^{*n}Q^*Qx, T^{*n}y\rangle\}_{n\in\mathbb{N}} = [QQ^*Qx, Qy]. \tag{3.2}$$

A combination of (3.1) and (3.2) yields $[Qx, Qy] = [Q(Q^*Qx), Qy]$. Since $\mathcal{R}(Q)$ is dense in $\mathcal{K}$, it follows that $Q = QQ^*Q$, so $(Q^*Q)^2 = Q^*Q$. This means that $P := Q^*Q$ is a projection. Additionally, by (3.1), we get $QQ^* = Id_{\mathcal{K}}$. Therefore, $\mathcal{R}(Q^*) = \mathcal{R}(Q^*QQ^*) \subset \mathcal{R}(Q^*Q) \subset \mathcal{R}(Q^*)$. As a result, $\mathcal{R}(P) = \mathcal{R}(Q^*)$. On the other hand, for $x \in \mathcal{H}$, we get

$$\|Qx\|^2 = [Qx, Qx] = \langle Q^*Qx, x\rangle = \langle Px, x\rangle = \|Px\|^2,$$

and so $\mathcal{N}(P) = \mathcal{H}_0$. Hence $\mathcal{H} = \mathcal{N}(P) \oplus \mathcal{R}(P) = \mathcal{H}_0 \oplus \mathcal{R}(Q^*)$. Take $x \in \mathcal{R}(Q^*)$. Thus $T^*x \in \mathcal{R}(Q^*)$, because $\mathcal{R}(Q^*)$ is reducing for T. Consequently,

$$\|T^*x\| = \|PT^*x\| = \|QT^*x\| = \|Qx\| = \|Px\| = \|x\|.$$

This finally implies that T^* is an isometry on $\mathcal{R}(Q^*)$, so by Proposition 3.1 it is a unitary operator on $\mathcal{R}(Q^*)$.

It only remains to verify that T^* is strongly stable on $\mathcal{H}_0$. To see this, fix $x \in \mathcal{H}_0$. Then $\liminf_{n\to\infty} \|T^{*n}x\|^2 \le \operatorname{glim}\{\|T^{*n}x\|^2\}_{n\in\mathbb{N}} = 0$. Hence for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $\|T^k x\| < \varepsilon$, so for all $m > k$ we have

$$\|T^m x\| = \|T^{m-k}T^k x\| \le \|T^{m-k}\|\|T^k x\| \le \varepsilon \sup_{n\in\mathbb{N}} \|T^n\|.$$

As a consequence, $\lim_{n\to\infty} \|T^{*n}x\| = 0$.

The converse implication is true for all bounded operators (see the proof of [5, Theorem 1]). □

It is plain that Theorem 3.2 does not hold for all bounded operators. To see this, it suffices to consider the operator $2Id_{\mathcal{H}}$ having the PF property.

Proposition 3.3. *If an operator $T \in \mathcal{B}(\mathcal{H})$ has the PF property, then T is a direct sum of a unitary operator and an operator $G \in \mathcal{B}(\mathcal{H}_0)$, which does not satisfy the equation $GX = XJ^*$ for any nonzero $X \in \mathcal{B}(\mathcal{K}, \mathcal{H}_0)$ and any isometry $J \in \mathcal{B}(\mathcal{K})$.*

Proof. Suppose that $TX = XV^*$ for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and some isometry $V \in \mathcal{B}(\mathcal{K})$. Owing to the PF property we have $T^*X = XV$. By this, for all $x, y \in \mathcal{H}$, we get

$$\begin{aligned}\langle XX^*x, y\rangle &= \langle X^*x, X^*y\rangle = \langle VX^*x, VX^*y\rangle = \langle VX^*x, X^*T^*y\rangle = \\ &= \langle XVX^*x, T^*y\rangle = \langle T^*XX^*x, T^*y\rangle = \langle XX^*x, TT^*y\rangle.\end{aligned}$$

Hence $TT^* = Id$ on $\mathcal{R}(XX^*)$, but $\overline{\mathcal{R}(XX^*)} = \overline{\mathcal{R}(X)}$ is reducing for T, so $T|_{\overline{\mathcal{R}(X)}}$ is a coisometry.

Let $T = T' \oplus C$ with respect to $\mathcal{H} = \overline{\mathcal{R}(X)} \oplus \mathcal{N}(X^*)$. It turns out that T' has the PF property. Indeed, if $YT'^* = JY$ for some $Y \in \mathcal{B}(\mathcal{H}, \mathcal{K}')$ and some isometry $J \in \mathcal{B}(\mathcal{K}')$, then

$$(Y \oplus 0)T^* = (Y \oplus 0)(T'^* \oplus C^*) = (J \oplus Id)(Y \oplus 0).$$

Hence, by the PF property of T, we get

$$(Y \oplus 0)(T' \oplus C) = (J^* \oplus Id)(Y \oplus 0),$$

and so $YT' = J^*Y$. This means that T' also has the PF property. As a consequence, by Proposition 3.1, $T' = T|_{\mathcal{R}(X^*)}$ is a unitary operator.

Next, let $G \in \mathcal{B}(\mathcal{H}_0)$ be a completely nonunitary part of T. By the above argument, G has the PF property. Hence if G satisfies $GX = XJ^*$ for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H}_0)$ and some isometry $J \in \mathcal{B}(\mathcal{K})$, then $X = 0$, which finishes the proof. □

Remark 3.4. In view of Proposition 3.3 a completely nonunitary operator $T \in \mathcal{B}(\mathcal{H})$ has the PF property if and only if there does not exist $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and an isometry $J \in \mathcal{B}(\mathcal{K})$ such that $TX = XJ^*$.

If T satisfies the PF property in a nontrivial way (that is, with a nonzero X), then we have

$$\|(XX^*)^{\frac{1}{2}}x\| = \|X^*x\| = \|X^*T^*x\| = \|(XX^*)^{\frac{1}{2}}T^*x\|, \quad x \in \mathcal{H}.$$

Thus there exists an isometry V such that $V(XX^*)^{\frac{1}{2}} = (XX^*)^{\frac{1}{2}}T^*$. This means that T^* is an A-isometry for $A = XX^*$ (for more information about A-isometries see [2, 13]). Hence the relation between the PF property and A-isometries can be formulated as follows.

Proposition 3.5. *A completely nonunitary operator T has the PF property if and only if its adjoint is not an A-isometry for any positive operator A.*

4. THE PF PROPERTY FOR $k*$-PARANORMAL OPERATORS

In this section we show that a $k*$-paranormal operator has PF property even if it is not a contraction.

Theorem 4.1. *Each $k*$-paranormal operator has the PF property.*

Proof. Take $k \geq 2$. Let $T \in \mathcal{B}(\mathcal{H})$ be a $(k-1)*$-paranormal operator. Suppose that there exist an operator $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and an isometry $V \in \mathcal{B}(\mathcal{K})$ such that

$$TX = XV^*. \tag{4.1}$$

Take $x_0 \in \mathcal{R}(X)$. There exists $x \in \mathcal{K}$ such that $x_0 = Xx$. Let us define the sequence $\{x_n\}_{n\in\mathbb{Z}}$ as follows:

$$x_n := \begin{cases} XV^n x, & n > 0, \\ XV^{*-n}x, & n < 0. \end{cases}$$

By (4.1), it is immediate that $Tx_{n+1} = x_n$. Additionally, we have

$$\|x_n\| \leq \max\{\|XV^{|n|}x\|, \|XV^{*|n|}x\|\} \leq \|X\|\|x\| =: M, \quad n \in \mathbb{Z}.$$

Hence the sequence $\{\|x_n\|\}_{n\in\mathbb{Z}}$ is bounded by M. Each $k*$-paranormal operator is $(k+1)$-paranormal (cf. [11, Proposition 4.8]), so T is a k-paranormal operator. Thus

$$\|x_n\|^k = \|Tx_{n+1}\|^k \leq \|T^k x_{n+1}\|\|x_{n+1}\|^{k-1} = \|x_{n-k+1}\|\|x_{n+1}\|^{k-1}, \quad n \in \mathbb{Z}.$$

Using the mean inequality and putting $n + k - 1$ instead of n we get

$$\|x_{n+k-1}\| \leq \frac{\|x_n\| + (k-1)\|x_{n+k}\|}{k}, \quad n \in \mathbb{Z}.$$

Next, multiplying both sides of this inequality by k and subtracting $\|x_n\| + \|x_{n+1}\| + \|x_{n+2}\| + \ldots + \|x_{n+k-1}\|$ we obtain

$$\begin{aligned}-\|x_n\| - \|x_{n+1}\| - \|x_{n+2}\| + \ldots - \|x_{n+k-2}\| + (k-1)\|x_{n+k-1}\| \leq \\ \leq -\|x_{n+1}\| - \|x_{n+2}\| - \|x_{n+3}\| + \ldots - \|x_{n+k-1}\| + (k-1)\|x_{n+k}\|.\end{aligned}$$

Thus the sequence $\{A_n\}_{n\in\mathbb{Z}}$, where

$$A_n := -\|x_n\| - \|x_{n+1}\| - \|x_{n+2}\| + \ldots - \|x_{n+k-2}\| + (k-1)\|x_{n+k-1}\|, \quad n \in \mathbb{Z},$$

is increasing. We now show that the sequence $\{A_n\}_{n\in\mathbb{Z}}$ is constant equal to 0. Let us observe that for small enough l and big enough m we get

$$\begin{aligned}\Big|\sum_{n=l}^{m} A_n\Big| &= \Big|\sum_{n=l}^{m}(-\|x_n\| - \|x_{n+1}\| + \ldots - \|x_{n+k-2}\| + (k-1)\|x_{n+k-1}\|)\Big| = \\ &= |(k-1)\|x_{m+k-1}\| + (k-2)\|x_{m+k-2}\| + \ldots + \|x_{m+1}\| - \\ &\quad - (\|x_l\| + 2\|x_{l+1}\| + 3\|x_{l+2}\| + \ldots + (k-1)\|x_{l+k-2}\|)| \leq Mk(k-1).\end{aligned}$$

Thus

$$(m-l+1)A_l = \sum_{n=l}^{m} A_l \leq \sum_{n=l}^{m} A_n \leq Mk(k-1).$$

Hence

$$A_l = \lim_{m\to\infty} \frac{m-l+1}{m} A_l \leq \lim_{m\to\infty} \frac{Mk(k-1)}{m} = 0.$$

Similarly, we deduce that

$$(m-l+1)A_m = \sum_{n=l}^{m} A_m \geq \sum_{n=l}^{m} A_n \geq -Mk(k-1).$$

Therefore,

$$A_m = \lim_{l\to-\infty} \frac{m-l+1}{-l} A_m \geq \lim_{l\to-\infty} \frac{-Mk(k-1)}{-l} = 0.$$

Hence $A_n = 0$, thus $\|x_n\| + \|x_{n+1}\| + \ldots + \|x_{n+k-2}\| = (k-1)\|x_{n+k-1}\|$. As before, we get

$$\|x_{n+k-1}\| = \frac{\|x_n\| + (k-1)\|x_{n+k}\|}{k}. \tag{4.2}$$

On the other hand,

$$\frac{\|x_n\| + (k-1)\|x_{n+k}\|}{k} \geq \sqrt[k]{\|x_n\|\|x_{n+k}\|^{k-1}} \geq \|x_{n+k-1}\|,$$

so we have equality in the mean inequality. It follows that $\|x_n\| = \|x_{n+k}\|$, so by (4.2) we obtain $\|x_n\| = \|x_{n+1}\|$ for each $n \in \mathbb{N}$. In particular, $\|Tx_0\| = \|x_{-1}\| = \|x_0\|$. Thus $\|Tx_0\| = \|x_0\|$ for each $x_0 \in \overline{\mathcal{R}(X)}$. This means that T is an isometry on the invariant subspace $\overline{\mathcal{R}(X)}$.

Since $TX = XV^*$, it follows that $T\mid_{\overline{\mathcal{R}(X)}}$ is a surjection. Thus it is a unitary operator. We can express T with respect to $\mathcal{H} = \overline{\mathcal{R}(X)} \oplus \mathcal{N}(X^*)$ as

$$T = \begin{bmatrix} T_{11} & T_{21} \\ 0 & T_{22} \end{bmatrix},$$

where $T_{11} = T\mid_{\overline{\mathcal{R}(X)}}$, $T_{21} \in \mathcal{B}(\mathcal{N}(X^*), \overline{\mathcal{R}(X)})$ and $T_{21} \in \mathcal{B}(\mathcal{N}(X^*))$. Hence

$$T^* = \begin{bmatrix} T_{11}^* & 0 \\ T_{21}^* & T_{22}^* \end{bmatrix}.$$

Taking into account that T is $(k-1)$∗-paranormal, we have

$$\begin{aligned}(1 + \|T_{21}^* x\|^2)^{k-1} = (\|x\|^2 + \|T_{21}^* x\|^2)^{k-1} = \|T^*(x,0)\|^{2(k-1)} \le \|T^{k-1}(x,0)\|^2 = \\ = \|T_{11}^{k-1} x\|^2 = \|x\|^2 = 1\end{aligned}$$

for each $x \in \overline{\mathcal{R}(X)}$ such that $\|x\| = 1$. As a result, $T_{21} = 0$ and $T = T_{11} \oplus T_{22}$. Since $TX = XV^*$, it follows that

$$XV = Id\mid_{\overline{\mathcal{R}(X)}} XV = T_{11}^* T_{11} XV = T^*TXV = T^*XV^*V = T^*X.$$

This completes the proof. □

As an immediate consequence of Theorem 4.1 we obtain the following corollary.

Corollary 4.2. *Each ∗-paranormal operator has the PF property.*

5. THE PF PROPERTY FOR PARANORMAL OPERATORS

An operator T is log-hyponormal if $\log(T^*T) \ge \log(TT^*)$. Mecheri have shown that log-hyponormal operators satisfy the Putnam-Fuglede theorem (see [10]). In particular, this fact implies that all log-hyponormal operators have the PF property.

In [1] Andô has shown (see Theorem 2 therein) that each log-hyponormal operator which satisfies $\mathcal{N}(T) = \mathcal{N}(T^*)$ is paranormal. Thus the log-hyponormal operators are not far from being paranormal. Hence it can be surprising that there exists a paranormal operator without the PF property. In this section we give a suitable example of such an operator. To do this first we prove the following lemma.

Lemma 5.1. *There are real bounded sequences $\{x_n\}_{n\in\mathbb{N}_+}$ and $\{y_n\}_{n\in\mathbb{N}_+}$ such that*

$$\begin{cases} x_n = y_n x_{n+1}, & n \in \mathbb{N}_+, \\ y_{n+1} = (x_{n+1}^2 + 1) y_n, & n \in \mathbb{N}_+. \end{cases}$$

Proof. Let us define a sequence $\{y_n\}_{n\in\mathbb{N}_+}$ such that

$$\begin{cases} y_1 = 1, y_2 = 2, \\ y_{n+2} = \frac{y_{n+1} - y_n + y_n y_{n+1}}{y_n}, & n \in \mathbb{N}_+. \end{cases}$$

By the definition of $\{y_n\}_{n\in\mathbb{N}_+}$, we have

$$y_{n+2} - y_{n+1} = \frac{y_{n+1} - y_n}{y_n}, \quad n \in \mathbb{N}_+, \tag{5.1}$$

so by induction we can deduce that the sequence $\{y_n\}_{n\in\mathbb{N}_+}$ is positive and increasing. Hence $y_n \geq 2$ for $n \geq 2$. Moreover, $y_3 - y_2 = 3 - 2 = 1$, so again using (5.1) we can easily show that $y_{n+1} - y_n \leq \frac{1}{2^{n-2}}$ for $n \geq 2$. Thus $y_n = \sum_{i=2}^{n}(y_i - y_{i-1}) + y_1 \leq 4$. It follows that the positive sequence $\{y_n\}_{n\in\mathbb{N}_+}$ is bounded. Now, if we set $x_{n+1} = \sqrt{\frac{y_{n+1}}{y_n} - 1}$ and $x_1 = 1$, it is easy to see that these two sequences satisfy the desired condition. □

Example 5.2. Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{e_n\}_{n\in\mathbb{Z}} \cup \{f_i\}_{i\in\mathbb{N}_+}$. Let us define an operator $S \in \mathcal{B}(\mathcal{H})$ by the formula

$$\begin{cases} S(e_n) = e_{n+1}, & n \in \mathbb{Z}, \\ S(f_k) = x_k e_k + y_k f_{k+1}, & k = 1, 2, \ldots, \end{cases}$$

where $\{x_n\}_{n\in\mathbb{N}_+}$ and $\{y_n\}_{n\in\mathbb{N}_+}$ are as in Lemma 5.1.

Since the sequences $\{x_n\}_{n\in\mathbb{N}_+}$ and $\{y_n\}_{n\in\mathbb{N}_+}$ are bounded, the operator S is bounded. We can express this operator with the help of the graph given in Figure 2.

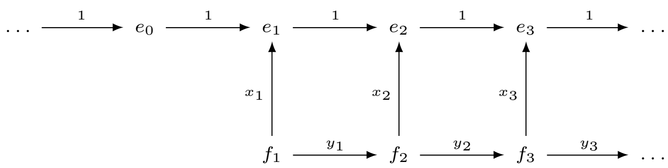

Fig. 2. The graph representation of the operator S

An alternative definition of paranormal operators is as follows: An operator $T \in \mathcal{B}(\mathcal{H})$ is paranormal if and only if $|T^2|^2 - 2\lambda|T|^2 + \lambda^2 Id_{\mathcal{H}}$ is nonnegative for all $\lambda > 0$ (cf. [1]). This means that T is paranormal if and only if $\|T^2h\|^2 - 2\lambda\|Th\|^2 + \lambda^2\|h\|^2 \geq 0$ for all $\lambda > 0$ and $h \in \mathcal{H}$.

Using this definition we now show that S is paranormal. Let us fix an arbitrary $h = \sum_{n\in\mathbb{Z}} \alpha_n e_n + \sum_{k\in\mathbb{N}_+} \beta_k f_k$ and $\lambda > 0$. Hence $h = \sum_{n\in\mathbb{N}} h_n + g$, where $g := \sum_{k\in\mathbb{Z}_-} \alpha_k e_k$ and $h_n := \alpha_n e_n + \beta_{n+1} f_{n+1}$. First, observe that $\|S^2 g\| = \|Sg\| = \|g\|$.

Thus $\|S^2g\|^2-2\lambda\|Sg\|^2+\lambda^2\|g\|^2 = (\lambda-1)^2\|g\|^2$. Using the relation between $\{x_n\}_{n\in\mathbb{N}_+}$ and $\{y_n\}_{n\in\mathbb{N}_+}$ we can conduct the following calculations:

$$\begin{aligned}
&\|S^2h_n\|^2 - 2\lambda\|Sh_n\|^2 + \lambda^2\|h_n\|^2 = \\
&= \|S^2(\alpha_n e_n + \beta_{n+1}f_{n+1})\|^2 - \\
&\quad - 2\lambda\|S(\alpha_n e_n + \beta_{n+1}f_{n+1})\|^2 + \lambda^2\|(\alpha_n e_n + \beta_{n+1}f_{n+1})\|^2 = \\
&= \|(2x_{n+1}\beta_{n+1} + \alpha_n)e_{n+2} + y_{n+1}y_{n+2}\beta_{n+1}f_{n+3}\|^2 - \\
&\quad - 2\lambda\|(x_{n+1}\beta_{n+1} + \alpha_n)e_{n+1} + y_{n+1}\beta_{n+1}f_{n+2}\|^2 + \\
&\quad + \lambda^2\|(\alpha_n e_n + \beta_{n+1}f_{n+1})\|^2 = |2x_{n+1}\beta_{n+1} + \alpha_n|^2 + y_{n+1}^2y_{n+2}^2 - \\
&\quad - 2\lambda(|x_{n+1}\beta_{n+1} + \alpha_n|^2 + |y_{n+1}\beta_{n+1}|^2) + \lambda^2(|\alpha_n|^2 + |\beta_{n+1}|^2) = \\
&= |(1-\lambda)\alpha_n + 2x_{n+2}y_{n+1}\beta_{n+1}|^2 + |\beta_{n+1}|^2(\lambda - (x_{n+2}^2y_{n+1}^2 + y_{n+1}^2))^2 + \\
&\quad + |\beta_{n+1}|^2(y_{n+1}^2y_{n+2}^2 - (x_{n+2}^2y_{n+1}^2 + y_{n+1}^2)^2).
\end{aligned}$$

The last part of this formula is equal to 0. Hence

$$\|S^2h_n\|^2 - 2\lambda\|Sh_n\|^2 + \lambda^2\|h_n\|^2 \geq 0.$$

Finally, let us observe that each of the sets $\{g\}\cup\{h_n\}_{n\in\mathbb{N}}$, $\{Sg\}\cup\{Sh_n\}_{n\in\mathbb{N}}$ and $\{S^2g\}\cup\{S^2h_n\}_{n\in\mathbb{N}}$ is orthogonal. Thus

$$\begin{aligned}
\|S^2h\|^2 - 2\lambda\|Sh\|^2 + \lambda^2\|h\|^2 &= \sum_{n\in\mathbb{N}}(\|S^2h_n\|^2 - 2\lambda\|Sh_n\|^2 + \lambda^2\|h_n\|^2) + \\
&\quad + \|S^2g\|^2 - 2\lambda\|Sg\|^2 + \lambda^2\|g\|^2 \geq 0.
\end{aligned}$$

This means that S is paranormal.

It remains to show that the operator S does not have the PF property. Indeed, S satisfies the equality $PS^* = UP$, where P is orthogonal projection on $E := \overline{span\{e_n | n\in\mathbb{Z}\}}$ and U is a direct sum of a bilateral backward shift on E and the identity operator, but

$$PSf_1 = x_1e_1 \neq 0 = UPf_1.$$

A part of Theorem 7.1.7 from [7] says that a paranormal operator is k-paranormal for each $k\in\mathbb{N}$. Owing to this result we see that the operator S from Example 5.2 is k-paranormal for each $k = 2, 3, \ldots$. On the other hand, due to Theorem 4.1, S is not $k*$-paranormal for any $k\in\mathbb{N}_+$.

In [11] it was proved that each $k*$-paranormal operator is $(k+1)$-paranormal. We conclude the paper with the ensuing observation.

Remark 5.3. The class of $k*$-paranormal operators is not equal to the class of $(k+1)$-paranormal operators.

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