THEORETICAL RESULTS AND NUMERICAL STUDY ON THE NONLINEAR REFLECTION AND TRANSMISSION OF PLANE SOUND WAVES

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The comparison between theoretical and numerical solutions of the reflection/transmission problem for the acoustic plane wave incident normally on the discontinuity surface between two nonlinear lossy media is presented. Numerical calculations made under the assumption that the two media have the same impedance, allow for singling out the effect of nonlinearities in the description of the reflection and transmission phenomena, so that they agree with theoretical predictions. It is shown that theoretically obtained and numerically calculated results mutually confirm themselves.

INTRODUCTION

The reflection and transmission of acoustic disturbances is a fundamental problem of the propagation theory in inhomogeneous media. Even in the linear propagation case beyond the known classical solutions, the problems which we meet here are difficult and the corresponding literature is wide [1]. On the contrary, in the case of nonlinear description of this problem, the acoustic literature is very scant. Few theoretical and experimental papers presented in Refs. [2-4] are worth mentioning. Though the titles of these papers suggest undertake of this problem but next their essence was eliminated by wrong suppositions or as a small effect. They stimulated the author to perform this study. The author has not found in the literature the solution of the fundamental nonlinear problem: how to determine the reflected and transmitted disturbances as a function of the incident disturbance in the shape of a plane wave incident normally on the plane boundary between two media. That is, to determine nonlinear quantities corresponding to reflection and transmission coefficients known from the linear propagation theory. Their shape results from the propagation equation and from the general continuity conditions of the fields on the boundary. The proper number of equations and unknown variables are incompatible with a definition as in [2] (producing wrong results) or can not be constructed by way of intuitive generalization of ideas (e.g., impedances) known from linear theory. The phenomenon, which will be considered in this paper is a part–often fundamental–of many technologies. For example, in medical ultrasonic diagnostic methods, important information is obtained due to the detection of reflections from boundaries between different tissues. Often equilibrium parameters (impedances) of tissues differ from each other by a very small amount. A question arises– which may be important not only for medical diagnostics–as to how the nonlinear effects impact on the general picture of reflection and transmission phenomenon. The nonlinearity parameter B/A for different soft tissues can vary from 5.8 (cardiac muscle) to 11 (fatty tissue)[6]. Can the media which differ by this parameter only (or generally by third order material parameters) be differentiated from each other due to different reflection? What are the qualitative and quantitative effects of this phenomenon? The aim of this paper is to find a response to the aforementioned question in its simplest one-dimensional geometrical configuration without taking into consideration any transverse disturbances, which may occur in relation to the beam axis.

Derivations of the reflection-transmission (R-T) operators for the plane wave incident normally on the boundary between nonlinear media was presented in detail in the papers [6],[7]. They were shortly recapitulated in chapters I and II. In chapter IV we present numerical solutions describing passage of the pulse plane wave through the boundary between different nonlinear (or linear-nonlinear) media. The main objective of this paper is the comparison of the theoretical predictions with the numerical calculations results.
The selections and applications in the numerical calculations of the characteristics relations (resulting from theory) between material parameters permits on reciprocal verification theory and numerical algorithm (positive for both if results are conformable).

1. ASSUMPTIONS AND PROBLEM FORMULATION

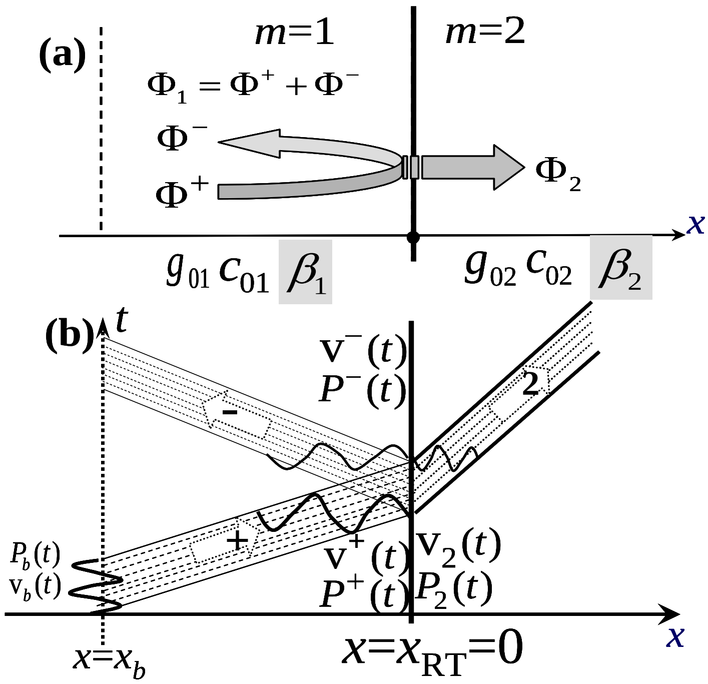

Fig.1. (a) Diagram of the fields decomposition and material parameters distribution in the first and the second media. (b) Time-space diagram of the incident reflected and transmitted pulses.

Two adjacent media with a plane interface between them are considered. Parameters and material function characteristic for the first (left) medium and the second medium will be denoted by $m=1,2$ respectively (Fig.1). The dynamics of the system is one-dimensional. In analysis, a normalized system of variables and parameters will be used. The normalized Cartesian coordinates in space *x* and time *t* and the normalized equilibrium density and sound speed are given respectively by the relations

$$x \equiv x' ko,\ t \equiv t'\Omega o,\ g_{0m} \equiv \rho_{0m}/\rho_0,\ c_{0m} \equiv c_{m0}/c_0,\ \Omega o = koc_0, \qquad (1)$$

where, ρ_{0m}, c_{m0} are the equilibrium density and speed of sound of the *m*-th medium; $\rho_0,\ c_0$ are the density and speed of sound of the reference medium (of course it can happen that $\rho_0=\rho_{01},\ c_0=c_{10}$).

In the case of the isotropic solid medium, third order parameters β_m (normalized by $\rho_0 c_0^2$) describes material and geometrical nonlinearities of the medium. For fluids $\beta_m \equiv (\gamma_m+1)/2$, where γ is the exponent of the adiabate. In the empirical cases $\beta_m \equiv (1+(B/2A)_m)$, $(B/A)_m$ is the nonlinearity parameter. After normalization the formulas analyzed in this paper contain Mach number $q \equiv P_b/\rho_0 c_0^2$ $(=\mathrm{v}_b/c_0)$ where P_b, v_b characteristics pressure and velocity ($P_b \equiv \max_t |P_b(t)|$ see Fig.1). x_{RT} $(=0)$ denotes the coordinate of the boundary plane (Fig.1).

The following representations of the fields in media (nonlinear and lossy) will be used.

$$\Phi_1(x,t)=\Phi^+(t-\tfrac{x}{c_{01}},x)+\Phi^-(t+\tfrac{x}{c_{01}},-x), \qquad x \le x_{RT}, \qquad m=1 \qquad (2)$$

$$\Phi_2(x,t)=\Phi_2(t-\tfrac{x}{c_{02}},x), \qquad x \ge x_{RT}, \qquad m=2, \qquad (3)$$

where $\Phi_1=\Phi^+ + \Phi^-$ denotes the composition of the incident Φ^+ and reflected Φ^- waves.

From the definition $\mathbf{v} \equiv \nabla\Phi$, it means that the velocity field is linear in respect to the potential. Therefore, the following decompositions of the velocity, corresponding to (2), (3), are valid

$$\mathrm{v}_1(x,t)=\mathrm{v}^+(t-\tfrac{x}{c_{01}},x)+\mathrm{v}^-(t+\tfrac{x}{c_{01}},-x), \qquad (4)$$

$$\mathrm{v}_2(x,t)=\mathrm{v}_2(t-\tfrac{x}{c_{02}},x). \qquad (5)$$

The relation between acoustical pressure and potential follows from the definition $\mathbf{v} \equiv \nabla\Phi$ and from the general equations of motion of the continuous medium, and is not linear, $P_1=\mathsf{P}_1[\Phi_1]=\mathsf{P}_1[\Phi^+ + \Phi^-] \ne \mathsf{P}_1[\Phi^+]+\mathsf{P}_1[\Phi^-]=P^+ + P^-$; nevertheless $\mathsf{P}_m \cong -g_{0m}\partial_t$ is a very good approximation. We would like to stress that the allowance of the neglecting terms in the P_m operator have no affect on the obtained results [6,7]. Similarly, the following decompositions of the acoustical pressure are obtained:

$$P_1(x,t)=P^+(t-\tfrac{x}{c_{01}},x)+P^-(t+\tfrac{x}{c_{01}},-x), \qquad (6)$$

$$P_2(x,t)=P_2(t-\tfrac{x}{c_{02}},x). \qquad (7)$$

Firstly, we would like to determine the relations between $\mathrm{v}^+,\mathrm{v}^-,\mathrm{v}_2$ and P^+,P^-,P_2 on the interface at $x=x_{RT}=0$. Then we will research the boundary conditions for $\mathrm{v}^-,\mathrm{v}_2$ and P^-,P_2 on the plane between two different nonlinear and lossy media. This problem will be solved if we find the functions R,T (operators) such that,

$$\mathrm{v}^- = \mathsf{R}'_{\mathrm{v}}\left[\{m\};\mathrm{v}^+\right]=\mathsf{R}_{\mathrm{v}}\left[\{m\};\mathrm{v}^+\right]\circ \mathrm{v}^+ \qquad (8)$$

$$\mathrm{v}_2 = \mathsf{T}'_{\mathrm{v}}\left[\{m\};\mathrm{v}^+\right]=\mathsf{T}_{\mathrm{v}}\left[\{m\};\mathrm{v}^+\right]\circ \mathrm{v}^+ \qquad (9)$$

where $\mathsf{R}_{\mathrm{v}}[\cdot;\cdot]$, $\mathsf{T}_{\mathrm{v}}[\cdot;\cdot]$ are the reflection and transmission operators; in [the] general case, $\circ$ - denotes operation characteristics for the operator. Generally, in almost all considered cases, $\circ \equiv \otimes$ is convolution (in time or Fourier frequency domain), but sometimes in special cases $\circ \equiv \cdot$ is the ordinary multiplication; $\{m\}$ denotes a set of the material parameters which characterizes the media. In linear case we have a "classical" problem of reflection and transmission of the plane wave. For no absorbing media $\mathrm{v}^- = \hat{\mathsf{R}}_{\mathrm{v}}\left[\{m\}\right] \circ \mathrm{v}^+ = R_{\mathrm{v}}\left[\{m\}\right] \cdot \mathrm{v}^+$,

$\hat{\mathrm{v}}_2 = \hat{\mathsf{T}}_{\mathrm{v}}\left[\{m\}\right] \circ \mathrm{v}^+ = \hat{T}_{\mathrm{v}}\left[\{m\}\right] \cdot \hat{\mathrm{v}}^+$. R,T are reflection and transmission coefficients.

As a base of our theoretical description, we assume equation (24) referred in [7]. It describes finite amplitude potential disturbances in lossy media. For the m-th homogenous phase of the medium, it takes the form

$$\partial_{tt}\Phi_m - c_m^2 \partial_{xx}\Phi_m + 2\mathcal{A}_m \partial_t \Phi_m + q_m \partial_t \left(\partial_t \Phi_m\right)^2 = 0 \ + o^2, \quad q_m \equiv q\,\beta_m / c_{0m}^2 \quad m=1,2 \,. \qquad (10)$$

Generally $\mathcal{A}_m$ is the convolution type operator of absorption $\mathcal{A}_m \Phi \equiv A_m(t) \otimes \Phi(x,t)$. For classical absorption $\mathcal{A}_m = -\alpha_2^m \partial_{xx} \simeq -\alpha_2^m \partial_{tt} + o^2$, α_2^m is the dimensionless hybrid viscosity. o^l is a small quantity of the order l with respect to q or $\alpha = \|\mathcal{A}\|$, $o^l = o(q+\alpha)^l$.

Secondly, we would like compare theoretical with numerical calculations results. In numerical calculations we will use unitary description of the nonlinear disturbances in the heterogeneous stationary media. It heave the form

$$\partial_{tt}P - c^2 g \partial_x \tfrac{1}{g} \partial_x P + 2\mathcal{A}\partial_t P - \bar{q}\partial_{tt}\left(P\right)^2 = 0 \ + o^2, \quad \bar{q} \equiv q\,\beta / gc^2 \ . \qquad (11)$$

In our case $g(x) \equiv g_{0m}$; $c(x) \equiv c_{0m}$; $\bar{q}(x) \equiv q\beta_m / g_{0m}c_{0m}^2$; $P(x,t) \equiv P_m(x,t) = -g_{0m}\partial_t \Phi_m$ for $x \in$ m-th phase of the medium. Then Eqs.(11) reduces to Eqs.(10) for $x \in$ (m-th phase of the medium).

Functions

$$\tau^+(t,x) = t - \frac{x}{c_{01}}, \ \tau^-(t,x) = t + \frac{x}{c_{01}} \ , \qquad x \le 0 \qquad (12)$$

$$\tau_2(t,x) = t - \frac{x}{c_{02}}, \qquad x \ge 0 \qquad (13)$$

are a set of characteristics of the d'Alambertian operator $\Box_m \equiv c_{0m}^2 \partial_{xx} - \partial_{tt}$, then for linear ($q \equiv 0$) and lossless media ($\mathcal{A}_m \equiv 0$), Φ_1, Φ_2 depends only on the characteristics and $\Box_2 \Phi_2(\tau_2) \equiv 0$ $\Box_1 \Phi_1 \equiv 0 \equiv \Box_1(\Phi^+(\tau^+) + \Phi^-(\tau^-))$. In the characteristics coordinates ($\tau_{,2}^{+,-}; x$) $\partial_x \Phi^{+,-} = 0 = \partial_x \Phi_2$.

In the solutions of the equations with absorbing and nonlinear terms $\mathcal{A}_m \neq 0$, $q \neq 0$ (of the range o^1) dependence on the additional coordinate (in our case x) should occurs in the space scale $\lambda_{\alpha,q} \equiv \min(1/\alpha, 1/q)$ (1/q-normalized shock distance; $\alpha = \|\mathcal{A}\|$). In the coordinates containing characteristics (Eq.(12),(13)) $\partial_x(\Phi_1(\bullet,x),\Phi_2(\bullet,x)) = o^1$ however, $\partial_{xx}(\Phi_1;\Phi_2) = o^2$ (see.[6,7]). Then, after substitution Eqs.(2,3) into Eqs.(10), and using coordinates with characteristic (retarded time) we obtain

$$\partial_x\left(\Phi^+ - \Phi^-\right) = -\frac{1}{c_{01}}\bar{\mathcal{A}}_1\left(\partial_t\Phi^+ + \partial_t\Phi^-\right) - \frac{q_1}{2c_{01}}\left(\partial_t\Phi^+ + \partial_t\Phi^-\right)^2 + o^2 \ ; \ \Phi^{+,-} = \Phi^{+,-}(\tau^{+,-}, x), \qquad (14)$$

$$\partial_x\Phi_2 = -\frac{1}{c_{02}}\bar{\mathcal{A}}_2\partial_t\Phi_2 - \frac{q_2}{2c_{02}}\left(\partial_t\Phi_2\right)^2 + o^2; \qquad \Phi_2 = \Phi_2(\tau_2, x) \qquad (15)$$

$$\partial_{\tau^+}\Phi^{+,-} = \partial_{\tau^-}\Phi^{+,-} = \partial_t\Phi^{+,-}, \quad \partial_{\tau_2}\Phi_2 = \partial_t\Phi_2$$

where $\bar{\mathcal{A}}_m \equiv \int \mathcal{A}_m \ \bar{\mathcal{A}}_m \partial_t \Phi = \bar{A}_m(t) \otimes (\partial_t \Phi)$, for a classically absorbing medium $\bar{\mathcal{A}}_m = -\alpha_2^m \partial_t$.

We see that equation (15) is the Burger's equation for $P_2 = -g_{02}\partial_t \Phi_2$. Equation (28) may be interpreted as the nonlinear coupled by means of the term $(q_1/z_{01})\partial_t\left(P^+P^-\right)$ of two Burger's equations for P^+ and P^-.

For assumed shapes Φ^+, Φ^-, Φ_2, acoustical pressures and velocities take the form,

$$P^+ = -g_{01}\partial_t \Phi^+ = -g_{01}\partial_{\tau^+}\Phi^+ \tag{16}$$

$$P^- = -g_{01}\partial_t \Phi^- = -g_{01}\partial_{\tau^-}\Phi^- \tag{17}$$

$$P_2 = -g_{02}\partial_t \Phi_2 = -g_{02}\partial_{\tau_2}\Phi_2 \tag{18}$$

$$\mathrm{v}^+ = \partial_x \Phi^+ = -\frac{1}{c_{01}}\partial_t \Phi^+ + \partial_x \Phi^+ = -\frac{1}{c_{01}}\partial_{\tau^+}\Phi^+ + \partial_x \Phi^+ = \frac{P^+}{z_{01}} + \partial_x \Phi^+, \tag{19}$$

$$\mathrm{v}^- = \partial_x \Phi^- = \frac{1}{c_{01}}\partial_t \Phi^- + \partial_x \Phi^- = \frac{1}{c_{01}}\partial_{\tau^-}\Phi^- + \partial_x \Phi^- = -\frac{P^-}{z_{01}} + \partial_x \Phi^- \tag{20}$$

$$\mathrm{v}_2 = \partial_x \Phi_2 = -\frac{1}{c_{02}}\partial_t \Phi_2 + \partial_x \Phi_2 = -\frac{1}{c_{02}}\partial_{\tau_2}\Phi_2 + \partial_x \Phi_2 = \frac{P_2}{z_{02}} + \partial_x \Phi_2 \tag{21}$$

where: $z_{0m} \equiv g_{0m}c_{m0}$, are equilibrium impedances. From (19), (20), (21) we obtain

$$P^+ = z_{01}(\mathrm{v}^+ - \partial_x \Phi^+) \tag{22}$$

$$P^- = -z_{01}(\mathrm{v}^- - \partial_x \Phi^-) \tag{23}$$

$$P_2 = z_{02}(\mathrm{v}_2 - \partial_x \Phi_2) \tag{24}$$

The functions $\partial_x\Phi^+, \partial_x\Phi^-, \partial_x\Phi_2$ show substantial differences between impedance relations for an ideal linear medium, and for a lossy or nonlinear one.

2.CONTINUITY CONDITIONS. REFLECTION-TRANSMISSION OPERATORS

In this section all functions and relations on the interface are considered and analyzed. On this surface at $x = x_{RT} = 0$ all functions and relations depend only on time t.

We assume continuity conditions in the conventional form

$$P_1 \equiv P^+ + P^- = P_2, \tag{25}$$

$$\mathrm{v}_1 \equiv \mathrm{v}^+ + \mathrm{v}^- = \mathrm{v}_2 \quad . \tag{26}$$

Applying (22), (23), (24) to reduce P^+, P^-, P_2 from (25), we obtain

$$z_{01}\left((\mathrm{v}^+ - \mathrm{v}^-) - \partial_x(\Phi^+ - \Phi^-)\right) = z_{02}\left(\mathrm{v}_2 - \partial_x \Phi_2\right) \tag{27}$$

$$\mathrm{v}^+ + \mathrm{v}^- = \mathrm{v}_2 \quad . \tag{28}$$

The needed terms $\partial_x(\Phi^+ - \Phi^-)$, $\partial_x\Phi_2$ were obtained from (14), (15) for $x \to x_{RT}, x = x_{RT}$. Using (16),(17),(18) and (22), (23), (24) , we obtain

$$\left(\partial_x\Phi^+ - \partial_x\Phi^-\right) = \bar{\mathcal{A}}_1\left(\mathrm{v}^+ - \mathrm{v}^-\right) - \frac{q_1 c_{01}}{2}\left(\mathrm{v}^+ - \mathrm{v}^-\right)^2 + o^2 \tag{29}$$

$$\partial_x\Phi_2 = \bar{\mathcal{A}}_2\,\mathrm{v}_2 - \frac{q_2 c_{02}}{2}\mathrm{v}_2^{\,2} + o^2. \tag{30}$$

Substituting (29), (30) into (27) and using (28), we obtain

$$\left[z_1 + z_2 + w\cdot v^+\right]v^- = \left[z_1 - z_2 + \tfrac{1}{2}u\cdot v^+\right]v^+ + \tfrac{1}{2}u\cdot(v^-)^2 \ , \tag{31}$$

where: z_1, z_2 are the operators of the linear impedance $z_m \equiv z_{0m}(1-\bar{\mathcal{A}}_m)$, $m=1,2$. $w \equiv z_{01}c_{01}q_1 + z_{02}c_{02}q_2$, $u \equiv z_{01}c_{01}q_1 - z_{02}c_{02}q_2$. Due to absorption, Eq.(31) may be an integral-differential nonlinear equation in time domain, or a nonlinear convolution equation in the Fourier frequency domain. We obtain the simplest case with regard to the absorption when assuming classical absorption $\bar{\mathcal{A}}_m = -\alpha_2^m \partial_t$ for both media (Riccati equation).

The abstract (operational) solution of equation (31) can be presented as

$$v^- = R'_v\left[\{m\};v^+\right] \equiv \frac{1}{2w}\left[1-\sqrt{1-4wR_0v^+}\right] = R_v\left[\{m\};v^+\right]\circ v^+ \tag{32}$$

$$R_v\left[\{m\};v^+\right] \equiv \frac{1}{2w\circ R_0\,v^+}\left[1-\sqrt{1-4w\circ R_0\,v^+}\right]R_0 \, , \tag{33}$$

$$T_v\left[\{m\};v^+\right] = 1 + R_v\left[\{m\};v^+\right] \quad , \tag{34}$$

$$R_0 \equiv \frac{z_1 - z_2 + \frac{1}{2}u\,v^+}{z_1 + z_2 + wv^+}\, , \ w \equiv \frac{\frac{1}{2}u\cdot}{z_1 + z_2 + w\cdot v^+} \ . \tag{35}$$

The factorizations of w and R_0 operators was present in [6,7]. If absorption is absent, $z_m = z_{0m}$ (or negligibility). $w[\cdot]$ and $R_0[\cdot]$ factorize themselves in ordinary functions, and the operational solutions (32),(33),(34),(35) give the factorized (algebraic) solution, $\circ = \cdot$. Expanding the above formulas in respect to q and keeping the terms of and o^1 only, we obtain

$$v^-(t) = \left(\frac{z_{01}-z_{02}}{z_{01}+z_{02}} + \frac{2z_{01}^2 z_{02}^2 q}{\left(z_{01}+z_{02}\right)^3}\left(\frac{\beta_1}{z_{01}c_{01}} - \frac{\beta_2}{z_{02}c_{02}}\right)v^+(t)\right)v^+(t) \ , \tag{36}$$

$$v_2(t) = \left(\frac{2z_{01}}{z_{01}+z_{02}} + \frac{2z_{01}^2 z_{02}^2 q}{\left(z_{01}+z_{02}\right)^3}\left(\frac{\beta_1}{z_{01}c_{01}} - \frac{\beta_2}{z_{02}c_{02}}\right)v^+(t)\right)v^+(t) \ . \tag{37}$$

We rewrite the above formulas in the form

$$v^- = R_v\, v^+ = \left(R_v + r_v\, v^+\right)v^+ \tag{38}$$

$$R_v \equiv \frac{z_{01} - z_{02}}{z_{01} + z_{02}}\, , \qquad r_v \equiv \frac{q 2 z_{01}^2 z_{02}^2}{\left(z_{01}+z_{02}\right)^3}\left(\frac{\beta_1}{z_{01}c_{01}} - \frac{\beta_2}{z_{02}c_{02}}\right), \tag{39}$$

$$v_2 = T_v\, v^+ = \left(T_v + r_v\, v^+\right)v^+ \ , \ T_v = 1 + R_v \ , \ T_v = 1 + R_v = \frac{2z_{01}}{z_{01}+z_{02}} \ . \tag{40}$$

In similar way, apply Eqs.(22), (23), (24) to reduce v^+, v^-, v_2 from (26); we obtain reflection R_P and transmission $T_P = 1 + R_P$ operators for treasure P^+. However the procedure is more complicated see [6,7].

$$P^-(t) = \left(\frac{z_{02}-z_{01}}{z_{01}+z_{02}} + \frac{2z_{01} z_{02}^2 q}{\left(z_{01}+z_{02}\right)^3}\left(\frac{\beta_2}{z_{02}c_{02}} - \frac{\beta_1}{z_{01}c_{01}}\right)P^+(t)\right)P^+(t)\, , \tag{41}$$

$$P_2(t) = \left(\frac{2z_{02}}{z_{01}+z_{02}} + \frac{2z_{01} z_{02}^2 q}{\left(z_{01}+z_{02}\right)^3}\left(\frac{\beta_2}{z_{02}c_{02}} - \frac{\beta_1}{z_{01}c_{01}}\right)P^+(t)\right)P^+(t)\, , \tag{42}$$

$$P^{-}=\mathsf{R}_{p}P^{+}=\left(R_{P}+r_{P}P^{+}\right)P^{+},\ P_{2}=\mathsf{T}_{p}P^{+}=\left(T_{P}+r_{P}P^{+}\right)P^{+}\ ,\mathsf{T}_{P}=1+\mathsf{R}_{P} \tag{43}$$

$$R_{P}=\frac{z_{02}-z_{01}}{z_{01}+z_{02}}=-R_{\mathrm{v}},\qquad r_{P}\equiv\frac{2z_{01}z_{02}^{2}q}{\left(z_{01}+z_{02}\right)^{3}}\left(\frac{\beta_{2}}{z_{02}c_{02}}-\frac{\beta_{1}}{z_{01}c_{01}}\right)=-r_{\mathrm{v}}\frac{1}{z_{01}}+o^{2}\ . \tag{44}$$

3. DISCUSSION

The properties of nonlinear part of the boundary conditions for reflected-transmitted waves are described by r_P, the nonlinear reflection-transmission coefficient. Let us notice that the lack of differences of nonlinear parameters in both media ($\beta_1=\beta_2$) is not a sufficient condition for the vanishing of the nonlinear component of the reflection R_p and transmission T_P operators. This condition has the form

$$r_P=0\quad\rightarrow\quad\frac{\beta_2}{z_{02}c_{02}}=\frac{\beta_1}{z_{01}c_{01}}\quad . \tag{45}$$

However, in this case, $\mathsf{R}_P=R_P$, where

$$R_P=\frac{z_{02}-z_{01}}{z_{02}+z_{01}}=\frac{\beta_2c_{01}-\beta_1c_{02}}{\beta_1c_{02}+\beta_2c_{01}}\ . \tag{46}$$

This implies in the case $\beta_1\neq\beta_2$ that, in spite of a linear dependence between the reflected and incident disturbances on the interface $x=x_{RT}$, the phenomenon of the interaction with the interface preserves its nonlinear character, as before.

If one component of interconnected media is linear (either $\beta_1=0$ or $\beta_2=0$), nonlinear reflection ($r_P\neq 0$) also occurs. Let medium m=1 be linear and not dissipative. Pulses propagating in this medium preserve their shapes given by boundary conditions (boundary pulse time shape). For incident waves this is $P_b(t)$ (see Fig.1.b). The solution of Eq.(10) with m=1 for incident waves have the form

$$P^{+}(t,x)=P_b\left(t-\tfrac{x-x_b}{c_{01}}\right)\qquad x_b\le x\le x_{RT}=0\,. \tag{47}$$

The moving left (reflected) solution of Eq.(10) has the form $P^{-}(t,x)=P^{-}(t+\tfrac{x}{c_{01}})$, where $P^{-}(t)\equiv P^{-}(t,x=0)$ is the boundary condition. Accordingly, in Eqs.(41),(43), the boundary conditions for reflected in the $x=x_{RT}=0$ wave have the form

$$P^{-}(t)=\mathsf{R}_{p}P^{+}(t)=\left(R_P+r_PP^{+}(t)\right)P^{+}(t)=\left(R_P+r_PP_b(t-\tfrac{-x_b}{c_{01}})\right)P_b(t-\tfrac{-x_b}{c_{01}})\,. \tag{48}$$

The solution for reflected wave takes the shape,

$$P^{-}(t,x)=P^{-}(t+\tfrac{x}{c_{01}})=R_PP_b(t+\tfrac{x+x_b}{c_{01}})+r_PP_b(t+\tfrac{x+x_b}{c_{01}})^2\qquad x<0\quad . \tag{49}$$

In the case of nonlinear or absorbing media, the shapes of reflecting (transmitting) and propagating pulses still change. Nevertheless, these changes are small on distances $|\delta x|<<\lambda_{\alpha,q}$. Then, propagating near the surface ($x=x_{RT}=0$), the reflecting-transmitting short pulses almost preserve the shape given by boundary conditions (Reflection-transmission operators)

$$P^{-}(t,x)=P^{-}(t+\tfrac{x}{c_{01}},-x)\cong R_PP^{+}(t+\tfrac{x}{c_{01}})+r_PP^{+}(t+\tfrac{x}{c_{01}})^2\ ,\ \ |x-x_{RT}|\sim(\text{pulse length})<<\lambda_{\alpha,q}\,. \tag{50}$$

Because the Mach number q is a small quantity even for relatively strong acoustical disturbances (q=0.001 for water and $P_b=2.25\mathrm{MPa}$), the effects of the nonlinear reflection may by observed separately if $z_{02}=z_{01}, R_P=0$ ($z_{02}\cong z_{01}, |R_P|<|r_P|$). In this case

$$P^-(t,x)=P^-(t+\frac{x}{c_{01}})=r_P\cdot P^+(t+\frac{x}{c_{01}})^2 \quad , \; r_P=\frac{q}{4}\left(\frac{\beta_2}{c_{02}}-\frac{\beta_1}{c_{01}}\right), because \; z_{01}=1 \; . \qquad (51)$$

If $\beta_1 \neq 0$, " = " should be replaced by $\cong$ in Eq.(51). In next section we compare results of the numerical solution of Eq.(11) with theoretical one, especially as expressed by the formula(51).

4. NUMERICAL CALCULATIONS

The numerical solutions of Eq.(11) are the response of the solving kernel for material parameters and on changing these parameters. The solving kernel is a general numerical image of the nonlinear and inhomogeneous (respect coefficients) operator (11), in particular, not containing any theoretically received information, on the basis of Eq.(10), about continuity conditions and the shapes of the (R-T) operators.

Incident pulses $P^+(t,x)$ were excited by boundary pulse $P^+(t,x=x_b)\equiv P_b(t)$ (see Fig.1), where $P_b(t)=P_{0b}\cdot Env(t-t_{en})\cdot\sin(2\pi\nu_c\cdot(t-t_c))$, t_c=0, $t_{en}=2/\nu_c$,is a four cycles length sine wave with triangle envelope *Env*(), carrier frequency ν_c=5 MHz (λ_c=0.3 mm in water), and maximal amplitude P_{0b}=1.125 MPa.

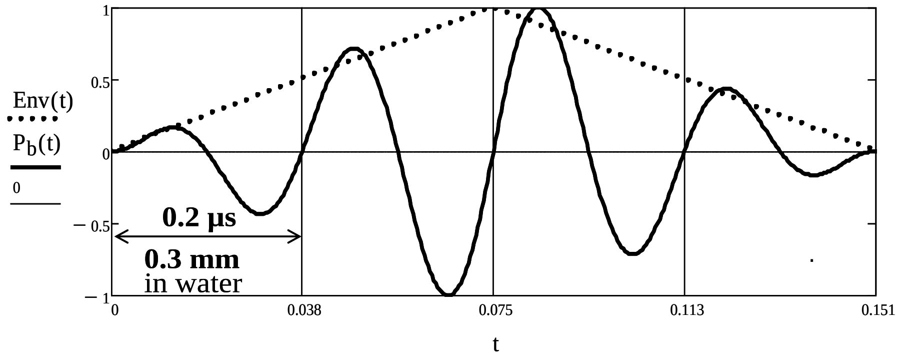

Fig. 2. Boundary pulse time shape for excitation of the incident $P^+(t,x)$ pulses.

The numerical calculations were performed for medium consisting of three layers of 27mm, 3mm and 21mm thickness, respectively (total thickness was 51[mm]=170 λ_c). The equilibrium material parameters of the first layer (as for water) were used for normalization, then $g_{01}=1, c_{01}=1$. We introduced a third layer which copies properties of the first one. This is an additional test for the numerical algorithm and a method for shoving effects of the sign changes of the differences between equivalent materials parameters (z,β,c). In a normalized system, the Mach number $q\equiv P_{0b}/\rho_0 c_0^2=P_{0b}/\rho_{01}c_{01}^2=0.0005$ and $\max_t|P_b(t)|=1$. Normalized carrier frequency and wavelength was equal to ν=170 and $\lambda=2\pi/170$. Because (pulse length)= $4\lambda=8\pi/170<<\lambda_{\alpha,q}=1/q\nu=1/q\cdot170$, then conditions provide for applying of the formulas (50,51), interpretations of the space shapes of numerical solutions as preserving time shapes of the boundary conditions Eqs.(41,42) for reflecting-transmitting pulses are satisfying. Amplitude level of the incident pulses in $x=x_{RT}=0$ were $\max_t|P^+(t)|=1$.

A. Linear propagation $\beta_m=0$.

As we se on Figs.3.,4. Calculated (R-T) pulses heaves amplitudes and phases (depending on signs of differences) in accordance with theory. Additionally, because $c_{02} = 5/3 > c_{01} = c_{03}$, we observe that wave length in m=2 is higher then in m=1,3.

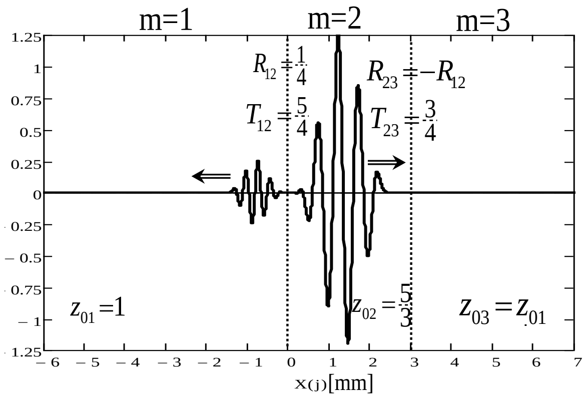

Fig. 3. The reflected and transmitted pulses near first (R-T) plane in x_{RT}=0.

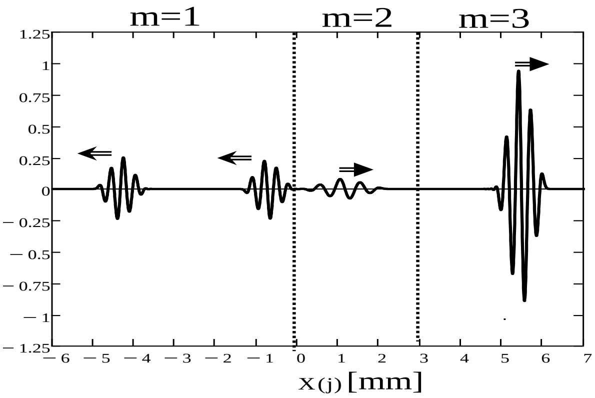

Fig. 4. R-T pulses. Second and next reflections after passing of the second medium by incident pulse.

B. Nonlinear propagation

In this case the (*R-T*) operators are the same like in A. However $\beta_1 = \beta_2 = \beta_3 = 3$ and $r_{p12} \neq 0$ bat $R_{12}, T_{12} \gg r_{12} = q(9/16)$ and effect of nonlinear reflection is masking by linear one. The same effects like in linear propagation case (R-T) are observed. Numerical results are in accordance with theoretically predicted.

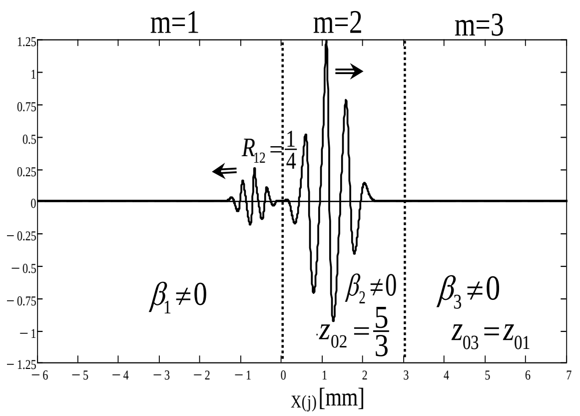

Fig. 5. The reflected and transmitted pulses after near first (R-T) plane in x_{RT}=0.

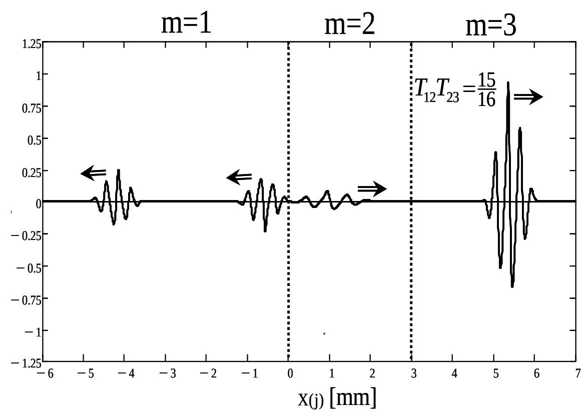

Fig. 6. R-T pulses. Second and next reflections passing of the second medium by incident pulse.

C. Nonlinear reflection. Linear-Nonlinear media. $\beta_1 = 0, R_P = 0; r_{p12} \neq 0$

In this case, $z_{01} = z_{02}, c_{01} = c_{02} = 1$, $\beta_1 = 0, \beta_2 = 4$, $R_P = 0, r_{p12} = (1/4)q\beta_2 = q = 0.0005$. Then - accordingly [?according to]theory- in reflecting waves, only the effects of the nonlinear reflection should occur only. Reflected waves propagate linearly and should preserve the shape predicted by formula (51). This implies should be proportional to "square" of incident wave shape. In the Fig.7 scale, the plots of the incident-transmitted pulses are out of the scale (vertical lines). For better visualization and comparison, the plots $\mathbf{0.0005} \times P_2(t-x)$

($\cong 0.0005 \times P^{+}(t-x)$) are displayed in bold. The numerically calculated shape and amplitude level of the reflecting pulse are in full accordance with theoretical predictions (see. Fig. 7).

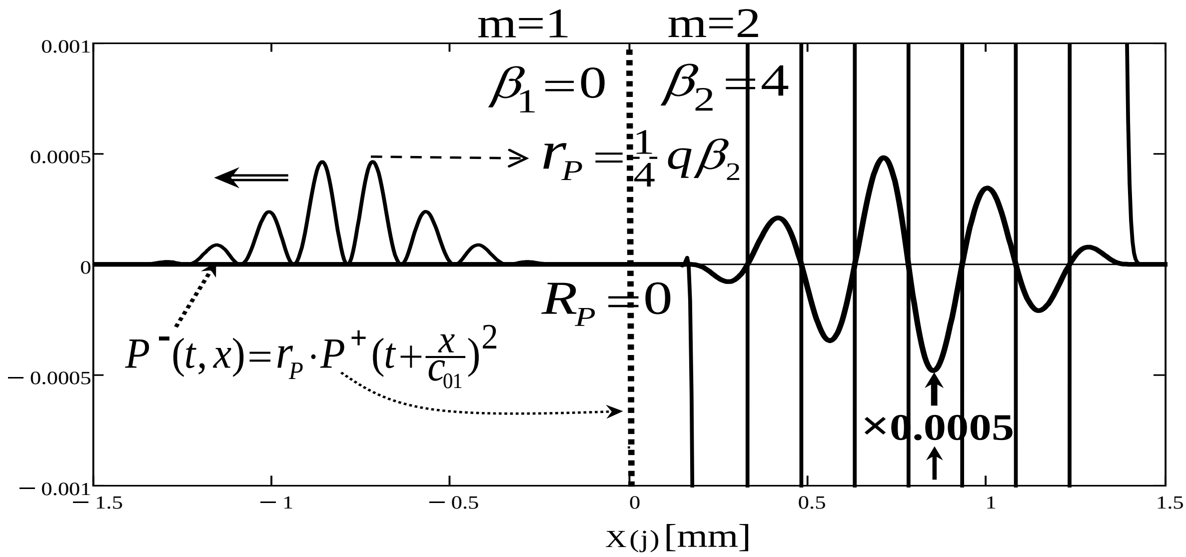

Fig. 7. Nonlinear reflection. Linear-nonlinear media.

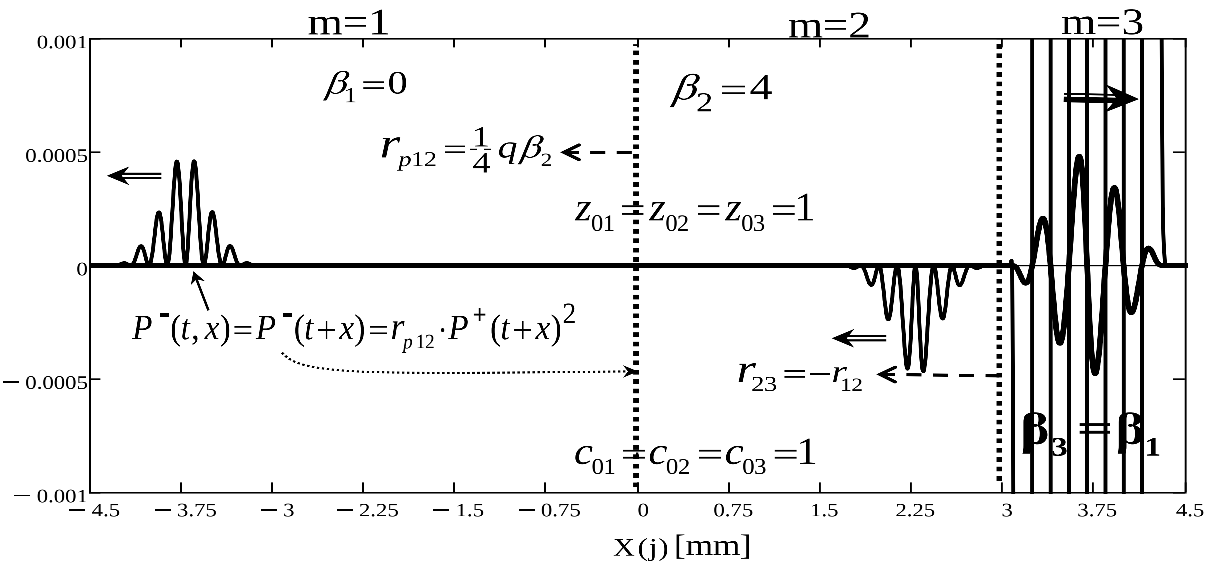

Fig. 8. Nonlinear reflection. Effect of sign changes of r_P. $r_{p23} = 0.25q(\beta_3 - \beta_2) = -0.25q\beta_2 = -r_{p12}$

D. Nonlinear reflection. Nonlinear-Nonlinear media. $\beta_m \neq 0, R_P = 0; r_{p12} \neq 0$

In this case, as above $z_{01} = z_{02}, c_{01} = c_{02} = 1$, but $\beta_1 = 3, \beta_2 = 7, \beta_3 = 3$, $R_P = 0$, $r_{p12} = (1/4)q(\beta_2 - \beta_1) = q = 0.0005$. Then according to theory, in reflecting waves, only the effects of the nonlinear reflection should occur. However, reflected waves propagate nonlinearly but near the (R-T) planes they should nevertheless preserve the shape predicted by formula (51) in very good approximation. This implies (as in C) that they should be proportional to the "square" of the incident wave shape. The numerically calculated shape and amplitude level of the reflecting pulse are in full accordance with theoretical predictions (see Fig.9).

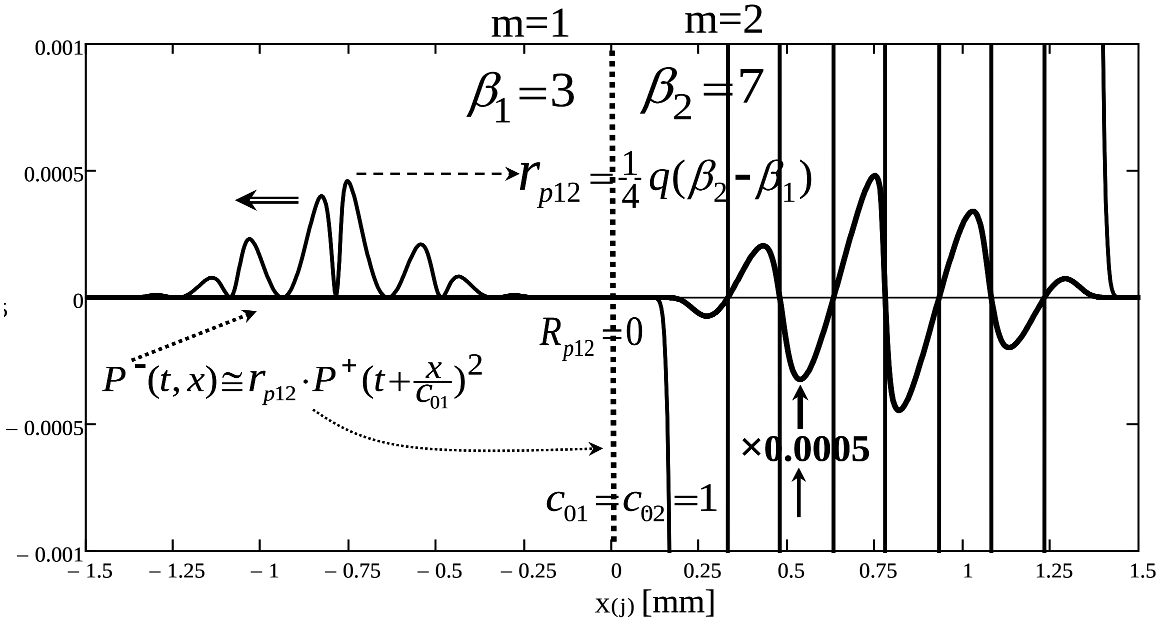

Fig. 9. Nonlinear reflection, $R_P = 0$, $\beta_1 = 3, \beta_2 = 7, \beta_3 = 3$

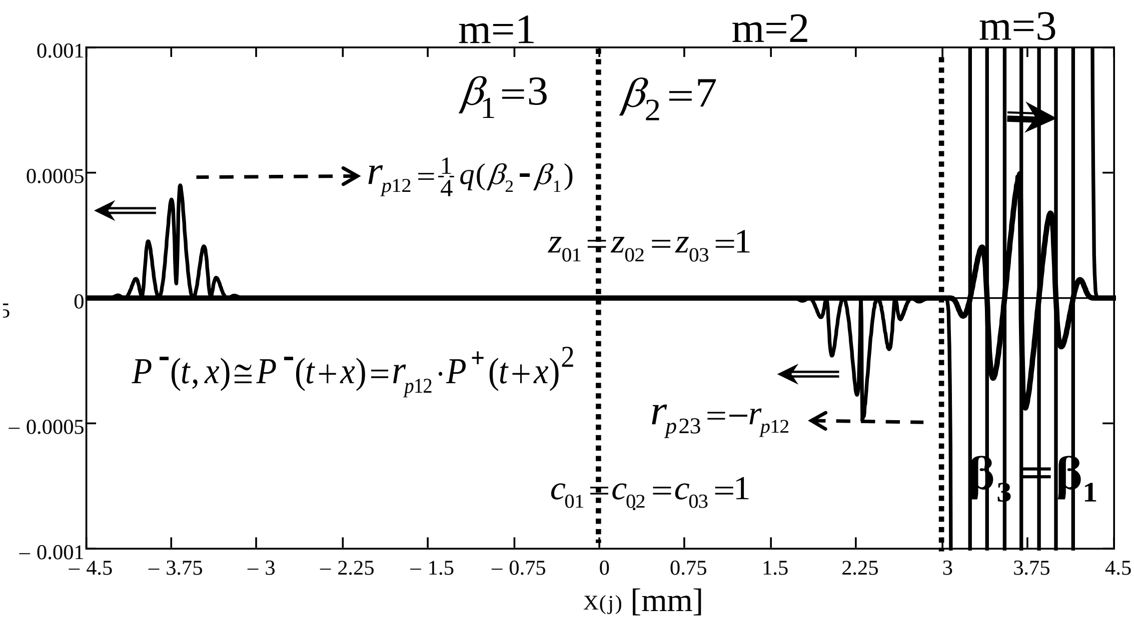

Fig.10.Nonlinear reflection. Effect of sign changes of the r_P . $r_{p23} = 0.25q(\beta_3 - \beta_2) = -0.25q(\beta_2 - \beta_1) = -r_{p12}$

5. CONCLUSIONS

The numerically calculated and theoretically obtained results mutually confirmed themselves.

A much broader discussion on the phenomenon of the nonlinear reflection and transmission was performed in [6,7]. Nevertheless, we would like turn attention to two features of the nonlinear reflection-transmission phenomenon: firstly, nonlinear (R-T) occurs if only one from both media is nonlinear and, secondly, nonlinear (R-T) is nonlocal in the Fourier frequency domain. The reflection and transmission of every Fourier component of the incident wave depends on the all remaining components, as follows from the properties of the auto correlation.

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