NONSTATIONARY ANALYSIS OF A QUEUEING NETWORK WITH POSITIVE AND NEGATIVE MESSAGES

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Abstract. This paper contains an investigation of an open queueing network with positive and negative messages that can be used to model the behavior of viruses in information and telecommunication systems and networks. The purpose of research is investigation of such a network at the transient behavior. We consider the case when the intensity of the incoming flow of positive and negative messages and service intensity of messages do not depend on time. It is assumed that all queueing systems of network are one-line. We obtained a system difference-differential equations for the state probabilities of the network. To find the state probabilities of the network in the transitional behavior applied a methodology based on the use of the apparatus of multidimensional generating functions. We obtained an expression for generating function. An example is calculated.

Keywords: *information and telecommunications systems and networks, virus, open G-network with negative messages, the transient behavior, the system of difference-differential equations, probability states, generating function*

1. General information

Information and telecommunications systems and networks are becoming more complex due to the need to improve the reliability of transmission and processing of information. Construction and study of the mathematical models to assess the quality of their functioning is an important task. Employment of the classical models of the queueing theory does not always provide adequate results, since it is necessary to take into account the model of the characteristic features of systems and the possible influence of various destabilizing factors, such as a sudden reaction, penetration of viruses or the loss of transmitted or processed data.

It is a network, which in addition to the ordinary flow (positive) messages is considered as additional Poisson flow of negative messages. On admission to the network system the negative message destroys one positive message if any are available in the system, thus reducing the number of positive messages in the system by one. Then the negative message disappears from the network without receiving any service for itself. For example, in computer networks a "positive" message is a task (program), and "negative" applications are computer viruses. This reflects the fact that on admission to a computer network, the virus destroys or causes damage, infects one of executable programs, reducing the number of existing programs or queries in the system by one. Then the virus disappears from the network, without receiving any service for itself. It should be noted that the study of such networks at the stationary behavior was conducted in [1, 2].

2. Formulation of the problem

Consider an open queueing G-network with n single-line queueing systems (QS). In QS S_i from the outside (from the system S_0) an incoming flow of positive (normal) of messages intensity of λ_{0i}^{+} and Poisson flow of negative messages intensity of λ_{0i}^{-}, $i=\overline{1,n}$. All flows of messages entering the network are independent. The service time of the positive messages in the QS S_i exponentially distributed with mean μ_i, $i=\overline{1,n}$. Negative messages coming to some system of the network in which there is at least one positive message instantly destroy (destroy, remove from the network) one of them. On the assumption of an exponential distribution of service time of positive messages may not care about what kind of message is destroyed. After this, it immediately leaves the network itself without getting any maintenance in the QS. Thus, each QS of the network can be served by only positive messages, so in the future, when speaking about the positive messages service, usually for the sake of brevity they are called simply messages [3].

Each positive message is sent to the QS of the S_i with probability p_{0i}^{+}, and the negative - with probability p_{0i}^{-}, $\sum_{i=1}^{n} p_{0i}^{+} = \sum_{i=1}^{n} p_{0i}^{-} = 1$, $i=\overline{1,n}$. A positive message serviced in the QS S_i, with probability p_{ij}^{+} sent to the QS S_j as a positive message, with a probability p_{ij}^{-} - as a negative message, and with probability $p_{i0} = 1 - \sum_{j=1}^{n} \left(p_{ij}^{+} + p_{ij}^{-}\right)$ leaving from the network to the external environment (QS S_0), $i, j = \overline{1,n}$.

The state of the network meaning the vector $k(t) = (k,t) = (k_1, k_2, \ldots, k_n, t)$, where k_i - the number of messages at the moment of time t at the system S_i, $i=\overline{1,n}$.

Lemma. *Probable states of considered network satisfy the system of difference--differential equations (DDE):*

$$\frac{dP(k,t)}{dt}=-\sum_{i=1}^{n}\left[\lambda_{0i}^{+}p_{0i}^{+}+\lambda_{0i}^{-}p_{0i}^{-}+\mu_i\right]u(k_i)P(k,t)+$$
$$+\sum_{i=1}^{n}\lambda_{0i}^{+}p_{0i}^{+}u(k_i)P(k-I_i,t)+\sum_{i=1}^{n}\left[\lambda_{0i}^{-}p_{0i}^{-}+\mu_i\left(p_{i0}^{+}+\sum_{j=1}^{n}p_{ij}^{-}\left(1-u\left(k_j\right)\right)\right)\right]P\left(k+I_i,t\right)+$$
$$+\sum_{i,j=1}^{n}\mu_i\left[p_{ij}^{+}u\left(k_j\right)P\left(k+I_i-I_j,t\right)+p_{ij}^{-}P\left(k+I_i+I_j,t\right)\right], \tag{1}$$

where I_i - a vector of dimension n, consisting of zeros, except for the component with number of i, which is equal to 1, $i=\overline{1,n}$; $u(x)=\begin{cases}1, x>0\\0, x\le 0\end{cases}$ - Heaviside function.

Proof. In view of the exponential service times of messages, a random process $k(t)=(k,t)$ is a Markov chain with a countable number of states. The possible transitions in the state $(k,t+\Delta t)$ for the time Δt:

1) from the state $(k-I_i,t)$ with the probability $\lambda_{0i}^{+}p_{0i}^{+}u(k_i)\Delta t+o(\Delta t)$, $i=\overline{1,n}$;
2) from the state $(k+I_i,t)$ with the probability

$$\left(\mu_i p_{i0}^{+}+\lambda_{0i}^{-}p_{0i}^{-}+\mu_i p_{ij}^{-}\left(1-u\left(k_j\right)\right)\right)\Delta t+o(\Delta t),\ i=\overline{1,n};$$

3) from the state $(k+I_i-I_j,t)$ with the probability

$$\mu_i p_{ij}^{+}u\left(k_j\right)\Delta t+o(\Delta t),\ i=\overline{1,n};$$

4) from the state $\left(k+I_i+I_j,t\right)$ with the probability $\mu_i p_{ij}^{-}\Delta t+o(\Delta t)$, $i=\overline{1,n}$;
5) from the state (k,t) with the probability

$$\left(1-\sum_{i=1}^{n}\left[\lambda_{0i}^{+}p_{0i}^{+}+\lambda_{0i}^{-}p_{0i}^{-}+\mu_i\right]u(k_i)\right)\Delta t+o(\Delta t),\ i=\overline{1,n};$$

6) of the remaining states with a probability $o(\Delta t)$ $o(\Delta t)$.

Then, using the formula of total probability, we can obtain

$$P(k,t+\Delta t)=\sum_{i=1}^{n}\lambda_{0i}^{+}p_{0i}^{+}u(k_i)P(k-I_i,t)\Delta t+$$
$$+\sum_{i,j=1}^{n}\left[\mu_i p_{i0}^{+}+\lambda_{0i}^{-}p_{0i}^{-}+\mu_i p_{ij}^{-}\left(1-u\left(k_j\right)\right)\right]P\left(k+I_i,t\right)\Delta t+$$
$$+\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}u\left(k_j\right)P\left(k+I_i-I_j,t\right)\Delta t+\sum_{i,j=1}^{n}p_{ij}^{-}P\left(k+I_i+I_j,t\right)\Delta t+$$
$$+\left(1-\sum_{i=1}^{n}\left[\lambda_{0i}^{+}p_{0i}^{+}+\lambda_{0i}^{-}p_{0i}^{-}+\mu_i\right]u(k_i)\right)\Delta t+o(\Delta t).$$

Dividing both sides of this relationship by Δt and taking the limit $\Delta t \to 0$, we obtain a system of equations for the state probabilities of the network (6). The lemma is proved.

3. Finding network state probabilities

Suppose that all systems of the network operating in high load, i.e. $k_i(t) > 0$, $\forall t > 0$, $i = \overline{1,n}$, then the system of DDE (1) takes the form

$$\frac{dP(k,t)}{dt} = -\sum_{i=1}^{n}\left(\lambda_{0i}^{+}p_{0i}^{+} + \lambda_{0i}^{-}p_{0i}^{-} + \mu_i\right)P(k,t) + \sum_{i=1}^{n}\lambda_{0i}^{+}p_{0i}^{+}P(k - I_i, t) +$$
$$+\sum_{i=1}^{n}\left(\mu_i p_{i0}^{+} + \lambda_{0i}^{-}p_{0i}^{-}\right)P\left(k + I_i, t\right) + \sum_{i,j=1}^{n}\mu_i\left[p_{ij}^{+}P\left(k + I_i - I_j, t\right) + p_{ij}^{-}P\left(k + I_i + I_j, t\right)\right]. \quad (2)$$

Denote by $\Psi_n(z,t)$, where $z = (z_1, z_2, ..., z_n)$, n-n-dimensional generating function:

$$\Psi_n(z,t) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k_1,k_2,..,k_n,t)z_1^{k_1}z_2^{k_2}\cdot ...\cdot z_n^{k_n} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k,t)\prod_{i=1}^{n}z_i^{k_i}.$$

Multiplied (2) on $\prod_{l=1}^{n}z_l^{k_l}$ and adding together all possible values k_l from 0 to $+\infty$, $l = \overline{1,n}$, obtain:

$$\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}\frac{dP(k,t)}{dt}\prod_{l=1}^{n}z_l^{k_l} = -\sum_{i=1}^{n}\left(\lambda_{0i}^{+}p_{0i}^{+} + \lambda_{0i}^{-}p_{0i}^{-} + \mu_i\right)\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k,t)\prod_{l=1}^{n}z_l^{k_l} +$$
$$+\sum_{i=1}^{n}\lambda_{0i}^{+}p_{0i}^{+}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k - I_i, t)\prod_{l=1}^{n}z_l^{k_l} +$$
$$+\sum_{i=1}^{n}\left(\mu_i p_{i0}^{+} + \lambda_{0i}^{-}p_{0i}^{-}\right)\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k + I_i, t)\prod_{l=1}^{n}z_l^{k_l} +$$
$$+\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k + I_i - I_j, t)\prod_{l=1}^{n}z_l^{k_l} +$$
$$+\sum_{i,j=1}^{n}\mu_i p_{ij}^{-}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k + I_i + I_j, t)\prod_{l=1}^{n}z_l^{k_l}. \quad (3)$$

Consider some sums, contained on the right side of (3). Let

$$\sum\nolimits_1(z,t) = \sum_{i=1}^{n}\lambda_{0i}^{+}p_{0i}^{+}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k - I_i, t)\prod_{l=1}^{n}z_l^{k_l}.$$

Then

$$\sum\nolimits_1(z,t)=\sum_{i=1}^{n}\lambda_{0i}^{+}p_{0i}^{+}z_i\sum_{\substack{k_j=0\\ j=\overline{1,n},\,j\neq i}}^{\infty}\sum_{k_i=1}^{\infty}P(k-I_i,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}z_i^{k_i-1}=$$

$$=\sum_{i=1}^{n}\lambda_{0i}^{+}p_{0i}^{+}z_i\sum_{\substack{k_j=0\\ j=\overline{1,n}}}^{\infty}P(k,t)\prod_{l=1}^{n}z_l^{k_l}=\sum_{i=1}^{n}\lambda_{0i}^{+}p_{0i}^{+}z_i\Psi_n(z,t).$$

For sum of $\sum_2(z,t)=\sum_{i=1}^{n}\left(\mu_i p_{i0}^{+}+\lambda_{0i}^{-}p_{0i}^{-}\right)\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k+I_i,t)\prod_{l=1}^{n}z_l^{k_l}$ have:

$$\sum\nolimits_2(z,t)=\sum_{i=1}^{n}\frac{\mu_i p_{i0}^{+}+\lambda_{0i}^{-}p_{0i}^{-}}{z_i}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k+I_i,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}z_i^{k_i+1}=$$

$$=\sum_{i=1}^{n}\frac{\mu_i p_{i0}^{+}+\lambda_{0i}^{-}p_{0i}^{-}}{z_i}\sum_{\substack{k_j=0\\ j=\overline{1,n},\,j\neq i}}^{\infty}\sum_{k_i=1}^{\infty}P(k,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}z_i^{k_i}=$$

$$=\sum_{i=1}^{n}\frac{\mu_i p_{i0}^{+}+\lambda_{0i}^{-}p_{0i}^{-}}{z_i}\Psi_n(z,t)-$$

$$-\sum_{i=1}^{n}\frac{\mu_i p_{i0}^{+}+\lambda_{0i}^{-}p_{0i}^{-}}{z_i}\sum_{\substack{k_j=0\\ j=\overline{1,n},\,j\neq i}}^{\infty}P(k_1,...,k_{i-1},0,k_{i+1}..,k_n,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}.$$

For sum of $\sum_3(z,t)=\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}P(k+I_i-I_j,t)\prod_{l=1}^{n}z_l^{k_l}$ obtain:

$$\sum\nolimits_3(z,t)=$$

$$=\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}\sum_{\substack{k_m=0\\ m=\overline{1,n},m\neq j}}^{\infty}\sum_{k_j=1}^{\infty}P(k_1,...,k_{i-1},k_i+1,k_{i+1},...,k_{j-1},k_j-1,k_{j+1},...,k_n,t)\times$$

$$\times\prod_{\substack{l=1\\ l\neq i,j}}^{n}z_l^{k_l}z_i^{k_i+1}z_j^{k_j-1}=\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}\sum_{\substack{k_m=0\\ m=\overline{1,n},m\neq i}}^{\infty}\sum_{k_i=1}^{\infty}P(k,t)\prod_{l=1}^{n}z_l^{k_l}=$$

$$=\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}\Psi_n(z,t)-$$

$$-\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}\sum_{\substack{k_m=0\\ m=\overline{1,n},m\neq j}}^{\infty}P(k_1,...,k_{i-1},0,k_{i+1},...,k_n,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}.$$

And finally, for the last sum

$$\sum\nolimits_4(z,t)=\sum_{i,j=1}^{n}\mu_i p_{ij}^{-}\sum_{k_1=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}P(k+I_i+I_j,t)\prod_{l=1}^{n}z_l^{k_l}$$

we will have:

$$\sum\nolimits_4(z,t)=$$

$$=\sum_{i,j=1}^{n}\mu_i p_{ij}^{-}\frac{1}{z_i z_j}\sum_{\substack{k_m=0\\ m=\overline{1,n},m\neq j}}^{\infty}\sum_{k_j=1}^{\infty}P(k_1,\ldots,k_{i-1},k_i-1,k_{i+1},\ldots,k_{j-1},k_j-1,k_{j+1},\ldots,k_n,t)\times$$

$$\times\prod_{\substack{l=1\\ l\neq i,j}}^{n}z_l^{k_l}z_i^{k_i-1}z_j^{k_j-1}=\sum_{i,j=1}^{n}\mu_i p_{ij}^{-}\frac{1}{z_i z_j}\sum_{\substack{k_m=0\\ m=\overline{1,n},m\neq i}}^{\infty}\sum_{k_i=1}^{\infty}P(k,t)\prod_{l=1}^{n}z_l^{k_l}=$$

$$=\sum_{i,j=1}^{n}\mu_i p_{ij}^{-}\frac{1}{z_i z_j}\Psi_n(z,t)-$$

$$-\sum_{i,j=1}^{n}\mu_i p_{ij}^{-}\frac{1}{z_i z_j}\sum_{\substack{k_m=0\\ m=\overline{1,n},m\neq j}}^{\infty}P(k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}.$$

Thus for the generating function the inhomogeneous linear differential equations (DE)

$$\frac{d\Psi_n(z,t)}{dt}=-\left[\sum_{i=1}^{n}\left(\lambda_{0i}^{+}p_{0i}^{+}+\lambda_{0i}^{-}p_{0i}^{-}+\mu_i\right)-\sum_{i=1}^{n}\lambda_{0i}^{+}p_{0i}^{+}z_i-\sum_{i=1}^{n}\frac{\mu_i p_{i0}^{+}+\lambda_{0i}^{-}p_{0i}^{-}}{z_i}-\right.$$

$$\left.-\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}-\sum_{i,j=1}^{n}\mu_i p_{ij}^{-}\frac{1}{z_i z_j}\right]\Psi_n(z,t)-$$

$$-\sum_{i=1}^{n}\frac{\mu_i p_{i0}^{+}+\lambda_{0i}^{-}p_{0i}^{-}}{z_i}\sum_{\substack{k_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}P(k_1,\ldots,k_{i-1},0,k_{i+1}..,k_n,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}-$$

$$-\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}\sum_{\substack{k_m=0\\ m=\overline{1,n},m\neq j}}^{\infty}P(k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}-$$

$$-\sum_{i,j=1}^{n}\mu_i p_{ij}^{-}\frac{1}{z_i z_j}\sum_{\substack{k_m=0\\ m=\overline{1,n},m\neq j}}^{\infty}P(k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n,t)\prod_{\substack{l=1\\ l\neq i}}^{n}z_l^{k_l}. \quad (4)$$

Since all of the QS networks operate under high load conditions, the last two expressions in the form of the sums in equation (4) will be zero, and it becomes homogeneous:

$$\frac{d\Psi_n(z,t)}{dt} = -\left[\sum_{i=1}^{n}\left(\lambda_{0i}^{+}p_{0i}^{+} + \lambda_{0i}^{-}p_{0i}^{-} + \mu_i\right) - \sum_{i=1}^{n}\lambda_{0i}^{+}p_{0i}^{+}z_i - \sum_{i=1}^{n}\frac{\mu_i p_{i0}^{+} + \lambda_{0i}^{-}p_{0i}^{-}}{z_i} - \right.$$
$$\left. - \sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i} - \sum_{i,j=1}^{n}\mu_i p_{ij}^{-}\frac{1}{z_i z_j}\right]\Psi_n(z,t).$$

Its solution has the form

$$\Psi_n(z,t) = C_n \exp\left\{-\left[\sum_{i=1}^{n}\left(\lambda_{0i}^{+}p_{0i}^{+} + \lambda_{0i}^{-}p_{0i}^{-} + \mu_i\right) - \sum_{i=1}^{n}\lambda_{0i}^{+}p_{0i}^{+}z_i - \sum_{i=1}^{n}\frac{1}{z_i}\left(\mu_i p_{i0}^{+} + \lambda_{0i}^{-}p_{0i}^{-} - \right.\right.\right.$$
$$\left.\left.\left. - \mu_i\sum_{j=1}^{n}p_{ij}^{+}z_j - \mu_i\sum_{j=1}^{n}p_{ij}^{-}\frac{1}{z_j}\right)\right]t\right\}. \quad (5)$$

We assume that at the initial moment of time network is in state

$$(\alpha_1,\alpha_2,...,\alpha_n,0),\ \alpha_i > 0\,,\ i = \overline{1,n}\,,$$
$$P(\alpha_1,\alpha_2,...,\alpha_n,0) = 1,\ P(k_1,k_2,...,k_n,0) = 0, \forall\alpha_i \neq k_i\,,\ i = \overline{1,n}\,.$$

Then the initial condition for the last equation (5) will be $\Psi_n(z,0) = P(\alpha_1,\alpha_2,...,\alpha_n,0)\prod_{l=1}^{n} z_l^{\alpha_l} = \prod_{l=1}^{n} z_l^{\alpha_l}$. Using it, we obtain $C_n = 1$.

Thus, the expression for the generating function $\Psi_n(z,t)$ has the form

$$\Psi_n(z,t) = a_0(t)\exp\left\{\sum_{i=1}^{n}\lambda_{0i}^{+}p_{0i}^{+}z_i t\right\}\exp\left\{\sum_{i=1}^{n}\frac{\mu_i p_{i0}^{+} + \lambda_{0i}^{-}p_{0i}^{-}}{z_i}t\right\}\times$$
$$\times\exp\left\{\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}t\right\}\exp\left\{\sum_{i,j=1}^{n}\mu_i p_{ij}^{-}\frac{1}{z_i z_j}t\right\}\prod_{l=1}^{n} z_l^{\alpha_l}, \quad (6)$$

where

$$a_0(t) = \exp\left\{-\sum_{i=1}^{n}\left(\lambda_{0i}^{+}p_{0i}^{+} + \lambda_{0i}^{-}p_{0i}^{-} + \mu_i\right)t\right\}. \quad (7)$$

Transform (6) to a form suitable for finding the state probabilities of the network, expanding its member exhibitors in the Maclaurin series. Then the following statement

Theorem. *The expression for the generating function has the form*

$$\Psi_n(z,t)=a_0(t)\sum_{l_1=0}^{\infty}...\sum_{l_n=0}^{\infty}\sum_{q_1=0}^{\infty}...\sum_{q_n=0}^{\infty}\sum_{r_1=0}^{\infty}...\sum_{r_n=0}^{\infty}\sum_{u_1=0}^{\infty}...\sum_{u_n=0}^{\infty}t^{\sum_{i=1}^{n}(l_i+q_i+r_i+u_i)}\times$$

$$\times\prod_{i=1}^{n}\left[\frac{\lambda_{0i}^{+\,l_n}p_{0i}^{+\,l_1}\left(\mu_i p_{i0}^{+}+\lambda_{0i}^{-}p_{0i}^{-}\right)^{q_i}\mu_i^{r_i+u_i}\left(\prod_{j=1}^{n}p_{ij}^{+}\right)^{r_i}\left(\prod_{j=1}^{n}p_{ij}^{-}\right)^{u_i}}{l_i!q_i!r_i!\mathrm{u}_i!}z_i^{\alpha_i+l_i-q_i-r_i+R-u_i-U}\right],\ (8)$$

where $R=\sum_{i=1}^{n}r_i$, $U=\sum_{i=1}^{n}u_i$.

Proof. From relation (6) follows that

$$\Psi_n(z,t)=a_0(t)a_1(z,t)a_2(z,t)a_3(z,t)a_4(z,t)\prod_{l=1}^{n}z_l^{\alpha_l},$$

where

$$a_1(z,t)=\exp\left\{\sum_{i=1}^{n}\lambda_{0i}^{+}p_{0i}^{+}z_i t\right\}=\prod_{i=1}^{n}\exp\left\{\lambda_{0i}^{+}p_{0i}^{+}z_i t\right\}=\prod_{i=1}^{n}\sum_{l_i=0}^{\infty}\frac{\left[\lambda_{0i}^{+}p_{0i}^{+}z_i t\right]^{l_i}}{l_i!}=$$

$$=\sum_{l_1=0}^{\infty}...\sum_{l_n=0}^{\infty}\prod_{i=1}^{n}\frac{\left[\lambda_{0i}^{+}p_{0i}^{+}z_i t\right]^{l_i}}{l_i!}=\sum_{l_1=0}^{\infty}...\sum_{l_n=0}^{\infty}\frac{t^{l_1+l_2+...+l_n}}{l_1!l_2!\cdot...\cdot l_n!}\lambda_{01}^{+\,l_1}\cdot...\cdot\lambda_{0n}^{+\,l_n}p_{01}^{+\,l_1}\cdot...\cdot p_{0n}^{+\,l_n}z_1^{l_1}\cdot...\cdot z_n^{l_n},$$

$$a_2(z,t)=\exp\left\{\sum_{i=1}^{n}\frac{\mu_i p_{i0}^{+}+\lambda_{0i}^{-}p_{0i}^{-}}{z_i}t\right\}=\prod_{i=1}^{n}\exp\left\{\frac{\mu_i p_{i0}^{+}+\lambda_{0i}^{-}p_{0i}^{-}}{z_i}t\right\}=$$

$$=\sum_{q_1=0}^{\infty}...\sum_{q_n=0}^{\infty}t^{q_1+...+q_n}\frac{\left(\mu_1 p_{10}^{+}+\lambda_{01}^{-}p_{01}^{-}\right)^{q_1}\cdot...\cdot\left(\mu_n p_{n0}^{+}+\lambda_{0n}^{-}p_{0n}^{-}\right)^{q_n}}{q_1!\cdot...\cdot q_n!}z_1^{-q_1}\cdot...\cdot z_n^{-q_n},$$

$$a_3(z,t)=\exp\left\{\sum_{i,j=1}^{n}\mu_i p_{ij}^{+}\frac{z_j}{z_i}t\right\}=\prod_{i=1}^{n}\prod_{j=1}^{n}\exp\left\{\mu_i p_{ij}^{+}\frac{z_j}{z_i}t\right\}=$$

$$=\prod_{i=1}^{n}\prod_{j=1}^{n}\sum_{r_i=0}^{\infty}\frac{\left[\mu_i p_{ij}^{+}tz_j z_i^{-1}\right]^{r_i}}{r_i!}=\sum_{r_1=0}^{\infty}..\sum_{r_n=0}^{\infty}\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{\left[\mu_i p_{ij}^{+}tz_j z_i^{-1}\right]^{r_i}}{r_i!}=$$

$$=\sum_{r_1=0}^{\infty}\dots\sum_{r_n=0}^{\infty}t^{r_1+\dots+r_n}\frac{\left(\mu_1\prod_{j=1}^{n}p_{1j}^{+}\right)^{r_1}\cdot\ldots\cdot\left(\mu_n\prod_{j=1}^{n}p_{nj}^{+}\right)^{r_n}}{r_1!\cdot\ldots\cdot r_n!}\times$$

$$\times z_1^{r_1+r_2+\dots+r_n}z_2^{r_1+r_2+\dots+r_n}\cdot\ldots\cdot z_n^{r_1+r_2+\dots+r_n}z_1^{-r_1}z_2^{-r_2}\dots z_n^{-r_n}=$$

$$=\sum_{r_1=0}^{\infty}\dots\sum_{r_n=0}^{\infty}t^{r_1+\dots+r_n}\frac{\left(\mu_1\prod_{j=1}^{n}p_{1j}^{+}\right)^{r_1}\cdot\ldots\cdot\left(\mu_n\prod_{j=1}^{n}p_{nj}^{+}\right)^{r_n}}{r_1!\cdot\ldots\cdot r_n!}z_1^{R-r_1}\cdot\ldots\cdot z_n^{R-r_n},$$

$$a_4(z,t)=\exp\left\{\sum_{i,j=1}^{n}\mu_i p_{ij}^{-}\frac{1}{z_i z_j}t\right\}=\prod_{i=1}^{n}\prod_{j=1}^{n}\exp\left\{\mu_i p_{ij}^{-}\frac{1}{z_i z_j}t\right\}=$$

$$=\prod_{i=1}^{n}\prod_{j=1}^{n}\sum_{u_i=0}^{\infty}\frac{\left[\mu_i p_{ij}^{-}tz_i^{-1}z_j^{-1}\right]^{u_i}}{u_i!}=\sum_{u_1=0}^{\infty}\dots\sum_{u_n=0}^{\infty}\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{\left[\mu_i p_{ij}^{-}tz_i^{-1}z_j^{-1}\right]^{u_i}}{u_i!}=$$

$$=\sum_{u_1=0}^{\infty}\dots\sum_{u_n=0}^{\infty}t^{u_1+\dots+u_n}\frac{\left(\mu_1\prod_{j=1}^{n}p_{1j}^{-}\right)^{u_1}\cdot\ldots\cdot\left(\mu_n\prod_{j=1}^{n}p_{nj}^{-}\right)^{u_n}}{u_1!\cdot\ldots\cdot u_n!}\times$$

$$\times z_1^{-u_1-u_2-\dots-u_n}z_2^{-u_1-u_2-\dots-u_n}\cdot\ldots\cdot z_n^{-u_1-u_2-\dots-u_n}z_1^{-u_1}z_2^{-u_2}\dots z_n^{-u_n}=$$

$$=\sum_{u_1=0}^{\infty}\dots\sum_{u_n=0}^{\infty}t^{u_1+\dots+u_n}\frac{\left(\mu_1\prod_{j=1}^{n}p_{1j}^{-}\right)^{u_1}\cdot\ldots\cdot\left(\mu_n\prod_{j=1}^{n}p_{nj}^{-}\right)^{u_n}}{u_1!\cdot\ldots\cdot u_n!}z_1^{-(U+u_1)}\cdot\ldots\cdot z_n^{-(U+u_n)}.$$

Multiplying $a_0(t)$, $a_1(z,t)$, $a_2(z,t)$, $a_3(z,t)$, $a_4(z,t)$ and $\prod_{l=1}^{n}z_l^{\alpha_l}$ we get the expression (8).

Example

Let the number of QS at network equal $n=10$. The intensity of the input stream of positive and negative messages λ_{0i}^{+} and λ_{0i}^{-} are equal respectively $\lambda_{01}^{+}=1$, $\lambda_{03}^{+}=31$, $\lambda_{06}^{+}=3$, $\lambda_{07}^{+}=2$, $\lambda_{09}^{+}=15$, $\lambda_{010}^{+}=2$, $\lambda_{02}^{+}=\lambda_{04}^{+}=\lambda_{05}^{+}=\lambda_{08}^{+}=0$, $\lambda_{02}^{-}=\lambda_{04}^{-}=\lambda_{05}^{-}=\lambda_{08}^{-}=0$, $\lambda_{01}^{-}=\lambda_{03}^{-}=\lambda_{06}^{-}=\lambda_{07}^{-}=\lambda_{09}^{-}=\lambda_{010}^{-}=2$. The intensities of message service μ_i equal: $\mu_1=\mu_2=\mu_3=12$, $\mu_4=1$, $\mu_5=32$, $\mu_6=4$, $\mu_7=13$, $\mu_8=13$, $\mu_9=7$, $\mu_{10}=8$. Let the probabilities p_{0i}^{+}, with which the positive message is sent to QS S_i equal $p_{01}^{+} = p_{03}^{+} = p_{06}^{+} = p_{07}^{+} = p_{09}^{+} = p_{010}^{+} = 1/6$, $p_{02}^{+} = p_{04}^{+} = p_{05}^{+} = p_{08}^{+} = 0$, and similar probabilities of negative messages equal $p_{01}^{-} = 1/22$, $p_{03}^{-} = 1/11$, $p_{06}^{-} = 3/22$, $p_{07}^{-} = 1/11$, $p_{09}^{-} = 3/11$, $p_{010}^{-} = 4/11$, $p_{02}^{-} = p_{04}^{-} = p_{05}^{-} = p_{06}^{-} = p_{08}^{-} = 0$. Probabilities p_{ij}^{+} equal respectively: $p_{12}^{+} = p_{23}^{+} = p_{36}^{+} = p_{410}^{+} = p_{54}^{+} = p_{710}^{+} = p_{87}^{+} = p_{98}^{+} = p_{101}^{+} = 1$, $p_{65}^{+} = p_{69}^{+} = 1/2$, others are zero. With probability $p_{60} = p_{100} = 1/2$ message outgoing from network to an external environment. Expression (7) in this case takes the form: $a_0(t) = \exp\{-49t\} = e^{-49t}$.

For example, we need to find the probability of the state $P(2,2,...,2,t)$. It is the coefficient of $z_1 z_2 \cdot ... \cdot z_n$ in the expansion of $\Psi_n(z,t)$ in multiple series (8), so the degrees at z_i must satisfy the relation $\alpha_i + l_i - q_i - r_i + R - u_i - U = 2$, $i = \overline{1,n}$, this implies that

$$q_i = \alpha_i + l_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} r_j - \sum_{\substack{j=1 \\ j \neq i}}^{n} u_j - 2, \quad q_i + r_i + u_i = \alpha_i + l_i + \sum_{j=1}^{n} r_j - \sum_{j=1}^{n} u_j - 2,$$

$$l_i + q_i + r_i + u_i = \alpha_i + 2l_i + \sum_{j=1}^{n} r_j - \sum_{j=1}^{n} u_j - 2, \; i = \overline{1,n},$$

$$\sum_{i=1}^{n} (l_i + q_i + r_i + u_i) = \sum_{i=1}^{n} (\alpha_i + 2l_i) + n(R - U - 2).$$

Then from (8) obtain that

$$P(2,2,\ldots,2,t) = e^{-49t} \sum_{l_1=0}^{\infty} \ldots \sum_{l_n=0}^{\infty} \sum_{r_1=0}^{\infty} \ldots \sum_{r_n=0}^{\infty} \sum_{u_1=0}^{\infty} \ldots \sum_{u_n=0}^{\infty} t^{\sum_{i=1}^{n}(\alpha_i + 2l_i) + n(R-U-2)} \times$$

$$\times \prod_{i=1}^{n} \left[\frac{\lambda_{0i}^{+\,l_n} p_{0i}^{+\,l_1} \left(\mu_i p_{i0}^{+} + \lambda_{0i}^{-} p_{0i}^{-} \right)^{\alpha_i + l_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} r_j - \sum_{\substack{j=1 \\ j\neq i}}^{n} u_j - 2} \mu_i^{r_i + u_i} \left(\prod_{j=1}^{n} p_{ij}^{+} \right)^{r_i} \left(\prod_{j=1}^{n} p_{ij}^{-} \right)^{u_i}}{l_i! \left(\alpha_i + l_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} r_j - \sum_{\substack{j=1 \\ j\neq i}}^{n} u_j - 2 \right)! r_i! \mathrm{u}_i!} \right], \; n = 10.$$

Figure 1 shows a graph of the probability for different t.

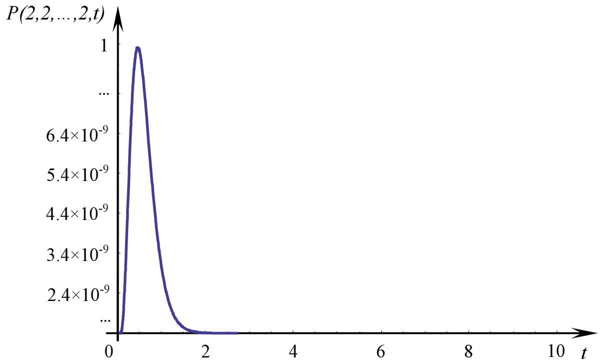

Fig. 1. The chart of the probability of the state $P\ (2, 2, ..., 2, t)$

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