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REMARKS ON DAMPED SCHRÖDINGER EQUATION OF CHOQUARD TYPE

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Abstract. This paper is devoted to the Schrödinger–Choquard equation with linear damping. Global existence and scattering are proved depending on the size of the damping coefficient.

Keywords: damped Choquard equation, global existence, scattering, invariant sets.

Mathematics Subject Classification: 35Q55.

1. INTRODUCTION

Consider the Cauchy problem for the Schrödinger equation of Choquard type:

$$\begin{cases} i\dot{u} + \triangle u + i\gamma u = -(I_\alpha * |u|^p)|u|^{p-2}u, \\ u(0,\cdot) = u_0, \end{cases} \tag{1.1}$$

where $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ for some $N \geq 3$, $\gamma \geq 0$, $p > 1$ and $0 < \alpha < N$. The Riesz-potential is defined on $\mathbb{R}^N$ by

$$I_\alpha := \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}}2^\alpha|\cdot|^{N-\alpha}} := \frac{\mathcal{K}}{|\cdot|^{N-\alpha}}.$$

The free operator associated to the damped Schrödinger equation [21] stands for

$$U_\gamma(t)\psi := e^{-\gamma t}\mathcal{F}^{-1}(e^{-it|\cdot|^2})\psi, \quad \psi \in H^1(\mathbb{R}^N).$$

Here and hereafter, we denote

$$p_* := 1 + \frac{2+\alpha}{N} \quad \text{and} \quad p^* := 1 + \frac{2+\alpha}{N-2}.$$

We make the following standard assumption

$$1+\frac{\alpha}{N}<p<p^*.$$

For any solutions to (1.1), let us define the following quantities called mass and energy:

$$M(u(t)) := \int_{\mathbb{R}^N} |u(t)|^2 dx,$$

$$E(u(t)) := \int_{\mathbb{R}^N} \Big\{ |\nabla u(t)|^2 - \frac{1}{p}(I_\alpha * |u(t)|^p)|u(t)|^p \Big\} dx.$$

The classical damped Schrödinger equation

$$i\dot{u} + \triangle u + i\gamma u = -|u|^{2(p-1)}u, \tag{1.2}$$

arises in various areas of nonlinear optics, plasma physics and fluid mechanics, see [1, 2, 11, 13, 29, 30]. Recently, in [21] M. Ohta and G. Todorova established that the Cauchy problem associated to (1.2) is well posed in the energy space and the solution is global for large damping. For other modifications of the classical equation (1.2), see also [8, 9, 24, 25]. The Schrödinger equation with nonlinear damping of power type has been studied in [5], and in a recent work [3] on bounded domains.

For $\gamma = 0$, in the physical case $N = 3, p = 2, \alpha = 2$, equation (1.1) has several origin such as quantum mechanics [15], Hartree-Fock theory to describe an electron trapped on its own hole [17] and non-relativistic quantum theory [14]. In [23], equation (1.1) is used to describe self-gravitating matter in a programme in which quantum state reduction is understood as a gravitational phenomenon. Recently, there are many works interested in the general form of (1.1) ($\gamma = 0$ and $2 \leq p < \frac{N+\alpha}{N-2}$ if $N \geq 3$). Indeed, in [7, 10, 12, 26, 27] authors discussed local and global well-posedness, existence of blow-up solutions, scattering and strong instability of standing waves for the non-damped Schrödinger–Choquard equation. It is thus quite natural to complete the nonlinear Choquard equation by a linear dissipative term to take into account some dissipation phenomena. This paper seems to be the first treating well-posedness issues for the damped Schrödinger–Choquard problem (1.1).

The purpose of this manuscript is two-fold. Firstly, we show that the Cauchy problem (1.1) is well-posed in the energy space. Secondly, we prove that a large damping prevents finite-time blow-up of solutions. Finally, global existence with arbitrary damping is obtained for the above problem with data in some invariant set.

This paper is organized as follows: Section 2 summarizes the main results and some technical tools needed in the sequel. In Section 3, we prove that (1.1) is locally well-posed. Section 4 is devoted to show global existence for large damping. Scattering of such global solutions is obtained in Section 5. In Section 6 we obtain global existence, via potential well method, without any assumption on the size of the damping. The Appendix is reserved to prove some useful modified Strichartz estimate.

We close this section with some notations and definitions. We consider the Lebesgue spaces $L^r := L^r(\mathbb{R}^N)$ equipped with norms

$$\|f\|_r := \|f\|_{L^r} = \left(\int_{\mathbb{R}^N} |f(x)|^r dx\right)^{\frac{1}{r}} \quad \text{if } r < \infty,$$

else

$$\|f\|_\infty := \|f\|_{L^\infty} = \operatorname{esssup}_{x\in\mathbb{R}^N} |f(x)|.$$

For vector valued functions

$$\|(f_j)\|_r := \sup_j \|f_j\|_r.$$

When $r = 2$, let $\|f\| := \|f\|_2$. The usual inhomogeneous Sobolev space is denoted by $W^{1,r} := W^{1,r}(\mathbb{R}^N)$ and endowed with the complete norm

$$\|f\|_{W^{1,r}} := (\|f\|^r + \|\nabla f\|^r)^{\frac{1}{r}},$$

in the case $r = 2$ we denote $H^1 := W^{1,2}$ which is equipped with

$$\|f\|_{H^1} := (\|f\|^2 + \|\nabla f\|^2)^{\frac{1}{2}}.$$

If X is an abstract space, the set of continuous functions defined on $[0, T[$ and valued in X is denoted by $C_T(X) := C([0, T), X)$, if necessary the interval of time may be closed. Also, we denote $L^q_I(X) := L^q(I, X)$ where I is an interval of $\mathbb{R}$. The set X_{rd} stands for the set of radial elements in X. Any constant will be denoted by C which may vary from line to line. For simplicity, let

$$\int f(x)dx := \int_{\mathbb{R}^N} f(x)dx \quad \text{and} \quad \int f(x,y)dxdy := \int\int f(x,y)dxdy.$$

Finally, if A and B are positive quantities, we write $A \lesssim B$ to denote $A \leq CB$.

2. MAIN RESULTS AND BACKGROUND

At first, let us introduce the following quantities

$$B := Np - N - \alpha, \quad A := 2p - B.$$

For $a, b \in \mathbb{R}$, let

$$\underline{\mu} = \min(2a + (N-2)b, 2a + Nb), \quad \overline{\mu} = \max(2a + (N-2)b, 2a + Nb)$$

and

$$\mathcal{A} = \{(a,b) \in \mathbb{R}^*_+ \times \mathbb{R} : \overline{\mu} > 0, \underline{\mu} \geq 0 \text{ and } 2a(p-1) + b\alpha > 0\}.$$

We denote also

$$v_{a,b}^{\lambda} := \lambda^a v(\lambda^{-b}\cdot), \quad \mathcal{L}_{a,b}(v) := (\partial_\lambda v_{a,b}^{\lambda})|_{\lambda=1}.$$

Let $S := M + E$ and define the so called constraint

$$\begin{aligned} K_{a,b}(v) :&= \mathcal{L}_{a,b}S(v) \\ &= (2a + (N-2)b)\|\nabla v\|^2 + (2a + Nb)\|v\|^2 \\ &\quad - \frac{1}{p}(2ap + b(N+\alpha)) \int (I_\alpha * |v|^p)|v|^p dx. \end{aligned}$$

Next, we define the so called energy subcritical ground state solution of the problem (1.1).

Definition 2.1. Any solution $\phi \in H^1 - \{0\}$ of

$$\triangle\phi - \phi + (I_\alpha * |\phi|^p)|\phi|^{p-2}\phi = 0, \tag{2.1}$$

which minimizes the problem

$$m_{a,b} := \inf_{v \in H^1 - \{0\}} \{S(v) : K_{a,b}(v) = 0\} \tag{2.2}$$

is called ground state of the problem (1.1).

Remark 2.2. Existence of ground states in the energy subcritical case is proved in [4] and [19]. In relation with the above notations, we refers to [27] adapted for the homogeneous case. In particular, it is claimed that $m := m_{a,b}$ is independent of any $(a,b) \in \mathcal{A}$.

Finally, let us define

$$Q(X) := 2NX^2 - (3N + 2\alpha + 2)X + N + \alpha.$$

Obviously

$$p_{\alpha,N}^{+} := \frac{(3N + 2\alpha + 2) + \sqrt{(N + 2\alpha)^2 + 12N + 8\alpha + 4}}{4N}$$

and

$$p_{\alpha,N}^{-} := \frac{(3N + 2\alpha + 2) - \sqrt{(N + 2\alpha)^2 + 12N + 8\alpha + 4}}{4N}$$

are the roots of Q. Since $Q(p^*) > 0$ and $Q(1 + \frac{\alpha}{N}) < 0$, then

$$p_{\alpha,N}^{-} < 1 + \frac{\alpha}{N} < p_{\alpha,N}^{+} < p^*.$$

For $N = 3$ and $\alpha = 2$, we have $p_{2,3}^{+} \simeq 2.103\,912\,56$.

2.1. MAIN RESULTS

Let us state our first result, the Cauchy problem (1.1) is locally well-posed in the energy space.

Theorem 2.3. *Let $N \geq 3, \max(0, N-4) < \alpha < N, 2 \leq p < p^*$ and $u_0 \in H^1$. Then, there exists $T^*_\gamma := T_{\gamma,u_0} > 0$ and a unique maximal solution $u \in C_{T^*_\gamma}(H^1)$ to the problem* (1.1). *In addition, we have:*

(1) $u \in L^q_{loc}([0, T^*_\gamma), W^{1,r})$, *where* $(q, r) = (\frac{4p}{Np-N-\alpha}, \frac{2Np}{N+\alpha})$,
(2) $M(u(t)) = e^{-\gamma t} M(u_0)$ *and* $\frac{d}{dt} S(u(t)) = -\gamma K_{1,0}(u(t))$ *on* $[0, T^*_\gamma)$.

Remark 2.4. We observe that:

(1) the restriction $p \geq 2$ is due to some contraction arguments used in the proof of Theorem 2.3,
(2) non-existence of standing waves is a direct consequence of the mass decay.

Secondly, we show that global well-posedness of (1.1) holds for large damping.

Theorem 2.5. *Let $N \geq 3, \max(0, N-4) < \alpha < N, p \geq 2, p^+_{\alpha,N} < p < p^*$ and $u \in C_{T^*_\gamma}(H^1)$ be the maximal solution to* (1.1) *with data $u_0 \in H^1$. Then, there exists a positive real number $\gamma^* := \gamma^*(\|u_0\|_{H^1})$ such that $T^*_\gamma = \infty$ for all $\gamma > \gamma^*$.*

Next, we establish a scattering result about the global solution given by Theorem 2.5.

Theorem 2.6. *Let $N \geq 3, \max(0, N-4) < \alpha < N, \gamma > \gamma^*, p \geq 2, p^+_{\alpha,N} < p < p^*$ and $u \in C(\mathbb{R}_+, H^1)$ be the global solution to* (1.1) *with data $u_0 \in H^1$. Then, there exists $u_+ \in H^1$ such that*

$$\lim_{t \to +\infty} \|u(t) - U_\gamma(t) u_+\|_{H^1} = 0. \tag{2.3}$$

Furthermore, the mapping $\tilde{S} : H^1 \to H^1, u_0 \to u_+$ is continuous and one to one.

The last outcome is about existence of global solution to (1.1) with arbitrary damping. Indeed, whatever the size of the damping, we prove the existence of global solution to (1.1) via some stable sets [22].

Theorem 2.7. *Assume $N \geq 3, \max(0, N-4) < \alpha < N$, $1 + \frac{\alpha}{N} < p < p^*$ and $u \in C_{T^*_\gamma}(H^1)$ be the maximal solution to* (1.1) *with data $u_0 \in H^1$. If there exists $(a, b) \in \mathcal{A}$ and $t_0 \in [0, T^*_\gamma)$ such that*

$$u(t_0) \in \mathcal{B}^+_{a,b} := \{v \in H^1 : S(v) < m \text{ and } K_{a,b}(v) \geq 0\},$$

then u is global.

Remark 2.8. In the sequel, we will prove the following interpolation inequality:

$$\int (I_\alpha * |u|^p)|u|^p dx \lesssim \|\nabla u\|^{Np-N-\alpha} \|u\|^{N-Np+\alpha+2p} \quad \text{for every } u \in H^1.$$

Hence, the energy of any local solution u of (1.1) is well defined. So, the result of Theorem 2.7 holds also in the case $1 + \frac{\alpha}{N} < p < 2$, if one succeeds to prove the local well-posedness of (1.1) in such case.

2.2. BACKGROUND AND TOOLS

We start first by some classical Sobolev injections [18] which give a meaning to the energy and different computations done in this note.

Lemma 2.9. *Let $N \geq 3$. Then:*

(1) $H^1 \hookrightarrow L^q$, *for every* $q \in [2, \frac{2N}{N-2}]$,
(2) *the injection* $H^1_{rd} \hookrightarrow L^q$ *is compact for every* $q \in (2, \frac{2N}{N-2})$.

Recall the Hardy–Littlewood–Sobolev inequality [16].

Lemma 2.10. *Let* $N \geq 3, 0 < \lambda < N$, $1 < r, s < \infty$ *and* $f \in L^r, g \in L^s$. *If* $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$, *then there exists* $C_{N,s,\lambda} > 0$ *such that*

$$\int \frac{f(x)g(y)}{|x-y|^\lambda} dxdy \leq C_{N,s,\lambda} \|f\|_r \|g\|_s.$$

One deduces the following.

Corollary 2.11. *Let* $N \geq 3, 0 < \alpha < N$, $1 < q, r, s < \infty$ *and* $f \in L^r, g \in L^s$. *Then, there exists* $C_{N,s,\alpha} > 0$ *such that:*

(1) *if* $\frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{N}$, *then*

$$\int (I_\alpha * f)(x) g(y) dxdy \leq C_{N,s,\alpha} \|f\|_r \|g\|_s,$$

(2) *if* $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{N}$, *then*

$$\|(I_\alpha * f)g\|_{q'} \leq C_{N,s,\alpha} \|f\|_r \|g\|_s.$$

The second previous result is known as the Hardy–Littlwood–Paley inequality. Existence of ground states and the best constant of a Gagliardo–Nirenerg inequality related to the problem (1.1) are investigated in [27], if one omits the inhomogeneous term.

Proposition 2.12. *Let* $N \geq 3, p_* < p < p^*$ *and* $(a,b) \in \mathcal{A}$. *Then:*

(1) $m := m_{a,b}$ *is nonzero and independent of* (a,b),
(2) *there is a ground state solution to* (2.1) *and* (2.2).

Remark 2.13. If the set $\mathcal{A}$ is relaxed to

$$\mathcal{A}' := \{(a,b) \in \mathbb{R}^*_+ \times \mathbb{R} : 2ap + (N+\alpha)b > \overline{\mu} > 0 \text{ and } \underline{\mu} \geq 0\}.$$

Then, the previous theorem holds for any exponent $1 + \frac{\alpha}{N} < p < p^*$.

Proposition 2.14. *Let $N \geq 3$ and $1+\frac{\alpha}{N} < p < p^*$. Then the following assertions hold.*

(1) *There exists $C_{N,p,\alpha} > 0$ such that*

$$\text{for every } u \in H^1, \int (I_\alpha * |u|^p)|u|^p dx \leq C_{N,p,\alpha}\|u\|^A \|\nabla u\|^B.$$

(2) *The minimization problem*

$$\frac{1}{C_{N,p,\alpha}} = \inf_{v \in H^1 - \{0\}} \frac{\|u\|^A \|\nabla u\|^B}{\int (I_\alpha * |u|^p)|u|^p dx}$$

is attained in some $Q \in H^1$ satisfying

$$C_{N,p,\alpha} = \int (I_\alpha * |Q|^p)|Q|^p dx$$

and

$$-B\triangle Q + AQ - \frac{2p}{C_{N,p,\alpha}}(I_\alpha * |Q|^p)|Q|^{p-2}Q = 0.$$

(3) *Moreover, there is ψ a ground state solution to* (2.1) *such that*

$$C_{N,p,\alpha} = \frac{2p}{A}\Big(\frac{A}{B}\Big)^{\frac{B}{2}} \|\psi\|^{-2(p-1)}.$$

Remark 2.15. In [19], the authors established that equation (2.1) admits a radially symmetric ground state solution in H^1. In addition, any ground state Q has fixed sign and satisfies the following Pohozaev identity:

$$(N-2)\|\nabla Q\|^2 + N\|Q\|^2 = \frac{N+\alpha}{p}\int (I_\alpha * |Q|^p)|Q|^p dx. \tag{2.4}$$

The following result summarise some classical properties of $U_\gamma(t)$, the free damped Schrödinger kernel [21].

Proposition 2.16. *We have:*

(1) $U_\gamma(t)u_0$ *is the solution to the linear problem associated to* (1.1),
(2) $U_\gamma(t)u_0 + i\int_0^t U_\gamma(t-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds$ *is the solution to* (1.1),
(3) $U_0(t)$ *is an isometry of* L^2,
(4) $\|U_0(t)f\|_r \lesssim t^{-N(\frac{1}{2}-\frac{1}{r})}\|f\|_{r'}, 2 \leq r < \infty$,
(5) $U_\gamma(t) = e^{-\gamma t}U_0(t)$,
(6) $U_\gamma(t+s) = U_\gamma(t)U_\gamma(s)$,
(7) $U_{\gamma_1}(t)U_{\gamma_2}(t) = U_{\gamma_1+\gamma_2}(t)$,
(8) $U_\gamma(t)^* = U_{-\gamma}(-t)$.

Strichartz estimate [6] is a standard tools to control the solutions of a Schrödinger equation in Lebesgue spaces.

Definition 2.17. A couple of real numbers (q,r) is said to be admissible if

$$2 \leq q,r \leq \infty, \quad (q,r) \neq (2,\infty) \quad \text{and} \quad N\Big(\frac{1}{2}-\frac{1}{r}\Big)=\frac{2}{q}.$$

Proposition 2.18. *Let $N \geq 3, T>0$ and $u_0 \in L^2$. Then for any admissible pairs (q,r) and $(\tilde{q},\tilde{r})$, there exists $C_{q,\tilde{q},N}>0$ such that*

$$\|u\|_{L^q_T(L^r)} \leq C_{q,\tilde{q},N}\big(\|u_0\| + \|i\dot{u}+\triangle u\|_{L^{\tilde{q}'}_T(L^{\tilde{r}'})}\big). \tag{2.5}$$

Using Proposition 2.16 and the one dimensional Riesz potential inequality, we give some modified Strichartz estimates which will be proved in the Appendix.

Proposition 2.19. *Let $T>0, N\geq 3$ and $2<r<\frac{2N}{N-2}$. Taking $\theta,\mu \in (1,+\infty)$ such that*

$$N\Big(\frac{1}{2}-\frac{1}{r}\Big)=\frac{1}{\theta}+\frac{1}{\mu}.$$

Then, there exists $C_{N,r,\theta}>0$ such that

$$\Big\|\int_0^t U_\gamma(t-s)f(s)ds\Big\|_{L^\theta_T(L^r)} \leq C_{N,r,\theta}\|f\|_{L^{\mu'}_T(L^{r'})}. \tag{2.6}$$

Corollary 2.20. *Let $T>0, N\geq 3$, $2<r<\frac{2N}{N-2}$ and (q,r) be an admissible pair. Then, there exists $C_{N,q}>0$ such that*

$$\|u\|_{L^q_T(L^r)} \leq C_{N,q}\big(\|u_0\| + \|i\dot{u}+\triangle u + i\gamma u\|_{L^{q'}_T(L^{r'})}\big). \tag{2.7}$$

We end this section by showing the following absorption lemma [28].

Lemma 2.21. *Let $T>0$ and $X \in C_T(\mathbb{R}_+)$ such that $X(0)=0$ and*

$$X(t) \leq a + bX(t)^\theta \ \text{ on } [0,T],$$

where $1<\theta, 0<b$ and $0<a<(1-\frac{1}{\theta})(b\theta)^{-\frac{1}{\theta}}$. Then

$$X(t) \leq \frac{\theta}{\theta-1}a \ \text{ on } [0,T].$$

3. LOCAL WELL-POSEDNESS

In this section we have to establish that the Cauchy problem (1.1) is locally well-posed in H^1. Precisely, we are going to prove Theorem 2.3. For any $T>0$ which will be fixed later, let

$$R := 2\|U_0(\cdot)u_0\|_{L^\infty_T(H^1)\cap L^q_T(W^{1,r})},$$

where

$$(q,r) := \Big(\frac{4p}{Np-N-\alpha}, \frac{2Np}{\alpha+N}\Big).$$

Next, let us define

$$B_T(R) := \{u \in C_T(H^1) \cap L^q_T(W^{1,r}) : \|u\|_{L^\infty_T(H^1)\cap L^q_T(W^{1,r})} \leq R\}.$$

The closed ball is equipped with the complete distance

$$d(u,v) := \|u\|_{L^\infty_T(L^2)\cap L^q_T(L^r)}.$$

Taking $u_0 \in H^1$ and define the function

$$\phi(u)(t) := U_0(t)u_0 + i\int_0^t U_0(t-s)\{i\gamma u + (I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)\}ds.$$

Thanks to Strichartz estimate (2.5), for all $u, v \in B_T(R)$, we have

$$\begin{aligned} d(\phi(u),\phi(v)) = \Bigg\| \int_0^t & U_0(t-s)\{i\gamma(u-v) \\ & + ((I_\alpha * |u|^p)|u|^{p-2}u - (I_\alpha * |v|^p)|v|^{p-2}v)\}ds\Bigg\|_{L^\infty_T(L^2)\cap L^q_T(L^r)} \\ & \lesssim \|i\gamma(u-v) + ((I_\alpha * |u|^p)|u|^{p-2}u - (I_\alpha * |v|^p)|v|^{p-2}v)\|_{L^{q'}_T(L^{r'})} \\ & \lesssim \gamma\|u-v\|_{L^{q'}_T(L^{r'})} + \|(I_\alpha * |u|^p)|u|^{p-2}u - (I_\alpha * |v|^p)|v|^{p-2}v\|_{L^{q'}_T(L^{r'})} \\ & \lesssim I + II. \end{aligned}$$

Since $p > 1 + \frac{\alpha}{N}$, then $\frac{2}{r'} > 1$. By a convexity argument, one have

$$\|u-v\|^{q'}_{r'} \leq \|u-v\|^{q'}.$$

So

$$I \leq \gamma\Big(\int_0^T \|u-v\|^{q'}dt\Big)^{\frac{1}{q'}} \leq \gamma T^{\frac{1}{q'}}\|u-v\|_{L^\infty_T(L^2)} \leq \gamma T^{\frac{1}{q'}} d(u,v).$$

Knowing that $2 \leq p < p^*$, then by Sobolev injections we have $H^1 \hookrightarrow L^r$. Using the Mean Value Theorem and the Hardy–Littlwood–Paley inequality, one gets

$$\begin{aligned}
II &:= \|(I_\alpha * |u|^p)|u|^{p-2}u - (I_\alpha * |v|^p)|v|^{p-2}v\|_{L_T^{q'}(L^{r'})} \\
&\leq \|(I_\alpha * [|u|^p - |v|^p])|u|^{p-2}u\|_{L_T^{q'}(L^{r'})} \\
&\quad + \|(I_\alpha * |v|^p)[|u|^{p-2}u - |v|^{p-2}v]\|_{L_T^{q'}(L^{r'})} \\
&\lesssim \|(I_\alpha * [|u|^{p-1} + |v|^{p-1}]|u-v|)|u|^{p-2}u\|_{L_T^{q'}(L^{r'})} \\
&\quad + \|(I_\alpha * |v|^p)(|u|^{p-2} + |v|^{p-2})|u-v|\|_{L_T^{q'}(L^{r'})} \\
&\lesssim \|(\|u\|_r^{2(p-1)} + \|v\|_r^{2(p-1)})\|u-v\|_r\|_{L_T^{q'}} \\
&\lesssim \|(\|u\|_{H^1}^{2(p-1)} + \|v\|_{H^1}^{2(p-1)})\|u-v\|_r\|_{L_T^{q'}} \\
&\lesssim (\|u\|_{L_T^\infty(H^1)}^{2(p-1)} + \|v\|_{L_T^\infty(H^1)}^{2(p-1)})\|u-v\|_{L_T^{q'}(L^r)} \\
&\lesssim T^{1-\frac{2}{q}} R^{2(p-1)} d(u,v).
\end{aligned}$$

In summary, we obtain

$$d(\Phi(u), \Phi(v)) \lesssim (\gamma T^{\frac{1}{q'}} + T^{1-\frac{2}{q}} R^{2(p-1)}) d(u,v).$$

It is interesting to remark that the condition $p < p^*$ gives $1 - \frac{2}{q} > 0$. Taking $v = 0$ and T small enough, we get

$$\begin{aligned}
\|\Phi(u)\|_{L_T^\infty(L^2)\cap L_T^q(L^r)} &\leq \|U_0(t)u_0\|_{L_T^\infty(L^2)\cap L_T^q(L^r)} + C(\gamma T^{\frac{1}{q'}} + T^{1-\frac{2}{q}} R^{2(p-1)})R \\
&\leq \frac{R}{2} + C(\gamma T^{\frac{1}{q'}} + T^{1-\frac{2}{q}} R^{2(p-1)})R \\
&\leq \frac{R}{2} + \frac{R}{2} = R.
\end{aligned}$$

Now, it remains to estimate $\|\Phi(u)\|_{L_T^\infty(\dot{H}^1)\cap L_T^q(\dot{W}^{1,r})}$. Once again, using Strichartz estimate (2.5), it follows that

$$\begin{aligned}
\|\Phi(u)\|_{L_T^\infty(\dot{H}^1)\cap L_T^q(\dot{W}^{1,r})} &\lesssim \frac{R}{2} + \gamma T^{\frac{1}{q'}} \|\nabla u\|_{L_T^\infty(L^2)} + \|\nabla((I_\alpha * |u|^p)|u|^{p-2}u)\|_{L_T^{q'}(L^{r'})} \\
&\lesssim \frac{R}{2} + \gamma T^{\frac{1}{q'}} R + \|(I_\alpha * |u|^{p-1}|\nabla u|)|u|^{p-1}\|_{L_T^{q'}(L^{r'})} \\
&\quad + \|(I_\alpha * |u|^p)|u|^{p-2}|\nabla u|\|_{L_T^{q'}(L^{r'})}.
\end{aligned}$$

Let

$$III := \|\Phi(u)\|_{L_T^\infty(\dot{H}^1)\cap L_T^q(\dot{W}^{1,r})} - \Big(\frac{R}{2} + \gamma T^{\frac{1}{q'}} R\Big).$$

By using the Hardy–Littlwood–Paley inequality, Sobolev injection and Hölder estimate, we obtain

$$\begin{aligned}
III &\lesssim \big\|\|u\|_r^{2(p-1)}\|\nabla u\|_r\big\|_{L_T^{q'}}\\
&\lesssim \big\|\|u\|_{H^1}^{2(p-1)}\|u\|_{W^{1,r}}\big\|_{L_T^{q'}}\\
&\lesssim \|u\|_{L^\infty(H^1)}^{2(p-1)}\|u\|_{L_T^{q'}(W^{1,r})}\\
&\lesssim T^{1-\frac{2}{q}}\|u\|_{L^\infty(H^1)}^{2(p-1)}\|u\|_{L_T^{q}(W^{1,r})}\\
&\lesssim T^{1-\frac{2}{q}}R^{2p-1}.
\end{aligned}$$

Consequently,

$$\|\Phi(u)\|_{L_T^\infty(\dot H^1)\cap L_T^q(\dot W^{1,r})} \lesssim \frac{R}{2} + (\gamma T^{\frac{1}{q'}} + T^{1-\frac{2}{q}}R^{2(p-1)})R.$$

Therefore, the functional Φ is a contraction of $B_T(R)$ for some $T>0$ small enough. So, fixed point arguments prove the existence of a unique local solution in $B_T(R)$ to the main problem (1.1). The rest of this section is devoted to establish uniqueness of solutions to (1.1) in H^1. Let $T>0$ and $u,v\in C_T(H^1)$ be two solutions of (1.1). So, $w:=u-v$ is solution to the following Cauchy problem

$$\begin{cases} i\dot w + \triangle w + i\gamma w = (I_\alpha * |v|^p)|v|^{p-2}v - (I_\alpha * |u|^p)|u|^{p-2}u,\\ w(0,\cdot)=0.\end{cases}$$

Taking $\tau\in(0,T)$, with a continuity argument we can suppose τ small enough such that

$$\max(\|u\|_{L_\tau^\infty(H^1)}, \|v\|_{L_\tau^\infty(H^1)}) \le 1.$$

Using Strichartz estimate (2.5) and arguing as previously, one gets

$$\begin{aligned}
\|w\|_{L_\tau^\infty(L^2)\cap L_\tau^q(L^r)} &\lesssim \gamma\|w\|_{L_\tau^{q'}(L^{r'})} + \|(I_\alpha * |v|^p)|v|^{p-2}v - (I_\alpha * |u|^p)|u|^{p-2}u\|_{L_\tau^{q'}(L^{r'})}\\
&\lesssim \gamma\tau^{\frac{1}{q'}}\|w\|_{L_\tau^\infty(L^2)} + \tau^{1-\frac{2}{q}}(\|u\|_{L_\tau^\infty(H^1)}^{2(p-1)} + \|v\|_{L_\tau^\infty(H^1)}^{2(p-1)})\|w\|_{L_\tau^q(L^r)}\\
&\lesssim (\gamma\tau^{\frac{1}{q'}} + \tau^{1-\frac{2}{q}})\|w\|_{L_\tau^\infty(L^2)\cap L_\tau^q(L^r)}.
\end{aligned}$$

Thus, uniqueness follows for small time τ and then on $[0,T)$ with a standard translation argument.

4. GLOBAL WELL-POSEDNESS

This section is devoted to prove Theorem 2.5 about existence of a global solution to (1.1) in the energy space, for large damping coefficient. Let $\gamma>0$ and $u\in C_{T_\gamma^*}(H^1)$ be the maximal solution to (1.1). Fix

$$q:=\frac{4p}{Np-N-\alpha},\quad r:=\frac{2Np}{N+\alpha}\quad\text{and}\quad \theta:=\frac{2q(p-1)}{q-2}.$$

Since $2 \leq p < p^*$, then $q > 2$ which implies $\theta > 0$ and $r \in]0, \frac{2N}{N-2}[$, so $H^1 \hookrightarrow L^r$. Thanks to Proposition 2.16, we have

$$\begin{aligned}
\|U_\gamma(\cdot)u_0\|_{L^\theta_{T^*_\gamma}(L^r)} &\lesssim \Big(\int_0^{T^*_\gamma} e^{-\gamma\theta s} \|U_0(s)u_0\|_r^\theta ds \Big)^{\frac{1}{\theta}} \\
&\lesssim \Big(\int_0^{T^*_\gamma} e^{-\gamma\theta s} \|u_0\|_{H^1}^\theta ds \Big)^{\frac{1}{\theta}} \\
&\lesssim \|u_0\|_{H^1} \Big(\int_0^{+\infty} e^{-\gamma\theta s} ds \Big)^{\frac{1}{\theta}} \\
&\lesssim \frac{\|u_0\|_{H^1}}{\gamma\theta}.
\end{aligned}$$

We deduce that

$$\lim_{\gamma\to+\infty} \|U_\gamma(\cdot)u_o\|_{L^\theta_{T^*_\gamma}(L^r)} = 0. \tag{4.1}$$

Let $T \in [0, T^*_\gamma)$, by using Strichartz estimate (2.7) and the Hardy–Littlwood–Paley inequality, we obtain

$$\begin{aligned}
\|u\|_{L^q_T(L^r)} &\lesssim \|u_0\| + \|(I_\alpha * |u|^p)|u|^{p-2}u\|_{L^{q'}_T(L^{r'})} \\
&\lesssim \|u_0\| + \|\|u\|_r^{2(p-1)}\|u\|_r\|_{L^{q'}_T}
\end{aligned}$$

Since $\frac{1}{q'} = \frac{1}{q} + \frac{2p-2}{\theta}$, then the Hölder inequality gives

$$\|u\|_{L^q_T(L^r)} \lesssim \|u_0\| + \|u\|^{2(p-1)}_{L^\theta_T(L^r)} \|u\|_{L^q_T(L^r)}.$$

Taking $\mu \in \mathbb{R}$ such that $\frac{1}{\mu} = \frac{2}{q} - \frac{1}{\theta}$, one can see that $\mu > 1$. Indeed, the condition $\frac{1}{\mu} > 0$ is equivalent to $\frac{2}{q} > \frac{1}{2p-1}$ which is satisfied because $p^+_{\alpha,N} < p < p^*$. In addition, the condition $\frac{1}{\mu} < 1$ is equivalent to $qp > 2p-1$ which is satisfied because $q > 2$. Now, applying Strichartz estimate (2.6) and then using the Hardy–Littlwood–Paley inequality, we get

$$\begin{aligned}
\|u\|_{L^\theta_T(L^r)} &\lesssim \|U_\gamma(\cdot)u_0\|_{L^\theta_T(L^r)} + \|(I_\alpha * |u|^p)|u|^{p-2}u\|_{L^{\mu'}_T(L^{r'})} \\
&\lesssim \|U_\gamma(\cdot)u_0\|_{L^\theta_{T^*_\gamma}(L^r)} + \|\|(I_\alpha * |u|^p)|u|^{p-2}u\|_{r'}\|_{L^{\mu'}_T} \\
&\lesssim \|U_\gamma(\cdot)u_0\|_{L^\theta_{T^*_\gamma}(L^r)} + \|\|u\|_r^{2p-1}\|_{L^{\mu'}_T}.
\end{aligned}$$

We compute

$$\begin{aligned}
\mu'(2p-1) &= \frac{\mu}{\mu-1}(2p-1) = \frac{2p-1}{1-\frac{1}{\mu}} = \frac{2p-1}{1-(\frac{2}{q}-\frac{1}{\theta})} \\
&= \frac{2p-1}{(1-\frac{2}{q})+\frac{1}{\theta}} = \frac{2p-1}{\frac{2p-2}{\theta}+\frac{1}{\theta}} = \theta.
\end{aligned}$$

Then

$$\|u\|_{L^\theta_T(L^r)} \lesssim \|U_\gamma(\cdot)u_0\|_{L^\theta_{T^*_\gamma}(L^r)} + \|u\|^{2p-1}_{L^\theta_T(L^r)}.$$

Taking account of (4.1) and applying Lemma 2.21 with the previous estimate for γ large enough, we get

$$\|u\|_{L^\theta_T(L^r)} \lesssim \|U_\gamma(\cdot)u_0\|_{L^\theta_{T^*_\gamma}(L^r)} \quad \text{on } [0, T^*_\gamma). \tag{4.2}$$

Thus, by using Hardy–Littlwood–Paley and Hölder inequalities, it follows that

$$\begin{aligned}
\|(I_\alpha * |u|^p)|u|^{p-2}u\|_{L^{q'}_T(\dot W^{1,r'})} &= \|\nabla((I_\alpha * |u|^p)|u|^{p-2}u)\|_{L^{q'}_T(L^{r'})} \\
&\lesssim \|(I_\alpha * |u|^{p-2}\Re(\overline{u}\nabla u))|u|^{p-2}u\|_{L^{q'}_T(L^{r'})} \\
&\quad + \|(I_\alpha * |u|^p)|u|^{p-2}\nabla u\|_{L^{q'}_T(L^{r'})} \\
&\lesssim \|\|u\|_r^{2(p-1)}\|\nabla u\|_r\|_{L^{q'}_T} \\
&\lesssim \|u\|^{2(p-1)}_{L^\theta_T(L^r)}\|\nabla u\|_{L^q_T(L^r)} \\
&\lesssim \|U_\gamma(\cdot)u_0\|^{2(p-1)}_{L^\theta_{T^*_\gamma}(L^r)}\|u\|_{L^q_T(\dot W^{1,r})}.
\end{aligned}$$

So, by Strichartz estimate (2.7), one obtains

$$\begin{aligned}
\|u\|_{L^\infty_T(H^1)\cap L^q_T(W^{1,r})} &\lesssim \|u_0\|_{H^1} + \|U_\gamma(\cdot)u_0\|^{2(p-1)}_{L^\theta_{T^*_\gamma}(L^r)}\|u\|_{L^q_T(W^{1,r})} \\
&\le C\big(\|u_0\|_{H^1} + \|U_\gamma(\cdot)u_0\|^{2(p-1)}_{L^\theta_{T^*_\gamma}(L^r)}\|u\|_{L^\infty_T(H^1)\cap L^q_T(W^{1,r})}\big).
\end{aligned}$$

For γ large enough, one gets

$$\|u\|_{L^\infty_T(H^1)\cap L^q_T(W^{1,r})} \le \frac{C}{1 - C\|U_\gamma(\cdot)u_0\|^{2(p-1)}_{L^\theta_{T^*_\gamma}(L^r)}}\|u_0\|_{H^1} \quad \text{on } [0, T^*_\gamma). \tag{4.3}$$

Therefore $\|u\|_{L^\infty_{T^*_\gamma}(H^1)} < \infty$ and then $T^*_\gamma = \infty$. This finishes the proof.

5. SCATTERING

Our aim in this section is to prove Theorem 2.6. Mainly, we prove scattering for global solutions to (1.1) given by Theorem 2.5. Thanks to Proposition 2.16, it is sufficient to show that

$$\lim_{t\to+\infty} \|U_{-\gamma}(-t)u(t) - u_+\|_{H^1} = 0.$$

For this purpose, we are going to prove that $v(t) := U_{-\gamma}(-t)u(t)$ satisfies the Cauchy criteria in H^1. We have

$$v(t) = u_0 + i\int_0^t U_\gamma(-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds.$$

Taking $\tilde{t} > t$, using the Strichartz estimate, Hardy–Littlwood–Paley and Hölder inequalities, we obtain

$$\begin{aligned}
\|v(t) - v(\tilde{t})\|_{H^1} &\lesssim \| \int_{\tilde{t}}^{t} U_\gamma(-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds\|_{H^1} \\
&\lesssim \|(I_\alpha * |u|^p)|u|^{p-2}u\|_{L^{q'}_{[t,\tilde{t}]}(W^{1,r'})} \\
&\lesssim \|u\|^{2(p-1)}_{L^\theta_{[t,\tilde{t}]}(L^r)} \|u\|_{L^q_{[t,\tilde{t}]}(W^{1,r})}.
\end{aligned}$$

Thanks to (4.2) and (4.3), we get

$$u \in L^\theta_{(0,+\infty)}(L^r) \cap L^q_{(0,+\infty)}(W^{1,r}).$$

Therefore, the function v satisfies the Cauchy criteria in H^1 and then it is sufficient to take

$$u_+ := \lim_{t\to+\infty} v(t) = u_0 + i \int_0^{+\infty} U_\gamma(-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds.$$

Now, our goal is to establish that the scattering mapping $\tilde{S}$ is one to one. If $u_+ \in H^1$, we have to prove that there exists $u \in C(\mathbb{R}_+, H^1)$ solution to problem (1.1) with data $u(0)$ such that (2.3) is verified. For this purpose, we use fixed point argument at infinity. Let us define

$$\phi(u)(t) := U_\gamma(t)u_+ + i \int_t^{+\infty} U_\gamma(t-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds.$$

Let $T > 0$ and $C_{q,N}$ be the constant produced in Strichartz estimate (2.7), we define

$$\begin{aligned}
B_T := \Big\{ & u \in C([T, +\infty), H^1) : \\
& \max(\|u\|_{L^\infty_{(T,+\infty)}(H^1)}, \|u\|_{L^q_{(T,+\infty)}(W^{1,r})}) \leq 2C_{q,N}\|u_+\|_{H^1}, \\
& \|u\|_{L^\theta_{(T,+\infty)}(L^r) \cap L^q_{(T,+\infty)}(L^r)} \leq 2\|U_\gamma(\cdot)u_+\|_{L^q_{(T,+\infty)}(L^r)} \Big\}.
\end{aligned}$$

The set B_T is equipped with the complete distance ([6])

$$d(u, v) := \|u - v\|_{L^q_{(T,+\infty)}(L^r)}.$$

Taking account of the fact that

$$\lim_{T\to+\infty} \|U_\gamma(\cdot)u_+\|_{L^\theta_{(T,+\infty)}(L^r)} = 0,$$

using Strichartz estimate (2.6), the Hardy–Littlwood–Paley and Hölder inequalities, for T large enough, we get

$$\begin{aligned}\|\phi(u)\|_{L^\theta_{(T,+\infty)}(L^r)} &\le \|U_\gamma(\cdot)u_+\|_{L^\theta_{(T,+\infty)}(L^r)} + C\|(I_\alpha * |u|^p)|u|^{p-2}u\|_{L^{\mu'}_{(T,+\infty)}(L^{r'})}\\ &\le \|U_\gamma(\cdot)u_+\|_{L^\theta_{(T,+\infty)}(L^r)} + C\|u\|^{2p-1}_{L^\theta_{(T,+\infty)}(L^r)}\\ &\le \|U_\gamma(\cdot)u_+\|_{L^\theta_{(T,+\infty)}(L^r)} + C2^{2p-1}\|U_\gamma(\cdot)u_+\|^{2p-1}_{L^\theta_{(T,+\infty)}(L^r)}\\ &\le 2\|U_\gamma(\cdot)u_+\|_{L^\theta_{(T,+\infty)}(L^r)}\\ &\le 2\|u_+\|_{H^1}.\end{aligned}$$

Once again, arguing as previously, for T large enough one gets

$$\begin{aligned}&\|\phi(u)\|_{L^q_{(T,+\infty)}(W^{1,r})\cap L^\infty_{(T,+\infty)}(H^1)}\\ &\le C_{q,N}\big(\|u_+\|_{H^1} + \|(I_\alpha * |u|^p)|u|^{p-2}u\|_{L^{q'}_{(T,+\infty)}(W^{1,r'})}\big)\\ &\le C_{q,N}\big(\|u_+\|_{H^1} + C\|u\|^{2(p-1)}_{L^\theta_{(T,+\infty)}(L^r)}\|u\|_{L^q_{(T,+\infty)}(W^{1,r})}\big)\\ &\le C_{q,N}\|u_+\|_{H^1}\big(1 + C\|U_\gamma(\cdot)u_+\|^{2(p-1)}_{L^\theta_{(T,+\infty)}(L^r)}\big)\\ &\le 2C_{q,N}\|u_+\|_{H^1}.\end{aligned}$$

According to Strichartz estimate (2.7), one has $U_\gamma(\cdot)u_+ \in L^q_{(0,+\infty)}(L^r)$, then

$$\lim_{T\to\infty}\|U_\gamma(\cdot)u_+\|_{L^q_{(T,+\infty)}(L^r)} = 0. \tag{5.1}$$

So, in a similar way as previous, for T large enough we get

$$\begin{aligned}\|\phi(u)\|_{L^q_{(T,+\infty)}(L^r)} &\le \|U_\gamma(\cdot)u_+\|_{L^q_{(T,+\infty)}(L^r)} + C\|U_\gamma(\cdot)u_+\|^{2p-1}_{L^q_{(T,+\infty)}(L^r)}\\ &\le 2\|U_\gamma(\cdot)u_+\|_{L^q_{(T,+\infty)}(L^r)}.\end{aligned}$$

In conclusion, B_T is conserved by ϕ for T sufficiently large. It remains then to prove that ϕ is a contraction on B_T. Taking $u, v \in B_T$, using the Mean Value Theorem and compute as previously, it follows that

$$\begin{aligned}\|\phi(u)-\phi(v)\|_{L^q_{(T,+\infty)}(L^r)} &\lesssim \|(I_\alpha * |u|^p)|u|^{p-2}u - (I_\alpha * |v|^p)|v|^{p-2}v\|_{L^{q'}_{(T,+\infty)}(L^{r'})}\\ &\lesssim \|(I_\alpha * [|u|^p - |v|^p])|u|^{p-2}u\|_{L^{q'}_{(T,+\infty)}(L^{r'})}\\ &\quad + \|(I_\alpha * |v|^p)[|u|^{p-2}u - |v|^{p-2}v]\|_{L^{q'}_{(T,+\infty)}(L^{r'})}\\ &\lesssim \|(I_\alpha * [|u|^{p-1} + |v|^{p-1}]|u-v|)|u|^{p-2}u\|_{L^{q'}_{(T,+\infty)}(L^{r'})}\\ &\quad + \|(I_\alpha * |v|^p)[|u|^{p-2} + |v|^{p-2}]|u-v|\|_{L^{q'}_{(T,+\infty)}(L^{r'})}\\ &\lesssim \big(\|u\|^{2(p-1)}_{L^\theta_{(T,+\infty)}(L^r)} + \|v\|^{2(p-1)}_{L^\theta_{(T,+\infty)}(L^r)}\big)\|u-v\|_{L^q_{(T,+\infty)}(L^r)}\\ &\lesssim \|U_\gamma(\cdot)u_+\|^{2(p-1)}_{L^q_{(T,+\infty)}(L^r)}\|u-v\|_{L^q_{(T,+\infty)}(L^r)}.\end{aligned}$$

Thanks to (5.1), for T large enough, the functional ϕ defines a contraction on B_T. Therefore, for some $T_+ > 0$, ϕ admits a unique fixed point in B_{T_+} which satisfies

$$u(t) = U_\gamma(t)u_+ + i\int_t^{+\infty} U_\gamma(t-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds, \quad t \geq T_+. \tag{5.2}$$

Now, let us define $\psi := U_\gamma(-T_+)u(T_+)$. Since

$$\begin{aligned} U_\gamma(t)\psi &= U_\gamma(t)(u_+ + i\int_{T_+}^{+\infty} U_\gamma(-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds) \\ &= U_\gamma(t)u_+ + i\int_{T_+}^{+\infty} U_\gamma(t-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds, \end{aligned}$$

then

$$\begin{aligned} u(t) &= U_\gamma(t)\psi - i\int_{T_+}^{+\infty} U_\gamma(t-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds \\ &\quad + i\int_t^{+\infty} U_\gamma(t-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds \\ &= U_\gamma(t)\psi - i\int_{T_+}^{t} U_\gamma(t-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds. \end{aligned}$$

Hence, u resolves the problem (1.1) on $(T_+, +\infty)$ and by Theorem 2.5 u is the global solution to (1.1) with initial data ψ. Furthermore, using (5.2), one has for t large enough

$$\begin{aligned} \|u(t) - U_\gamma(t)u_+\|_{H^1} &= \left\| i\int_t^{+\infty} U_\gamma(t-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds \right\|_{H^1} \\ &\lesssim \left\| \int_\tau^{+\infty} U_\gamma(\tau-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds \right\|_{L^\infty_{(t,+\infty)}(H^1)} \\ &\lesssim \|(I_\alpha * |u|^p)|u|^{p-2}u\|_{L^{q'}_{(t,+\infty)}(W^{1,r'})} \\ &\lesssim \|u\|^{2(p-1)}_{L^\theta_{(t,+\infty)}(L^r)} \|u\|_{L^q_{(t,+\infty)}(W^{1,r})}. \end{aligned}$$

Thanks to (4.2) and (4.3), we have $u \in L^\theta_{(0,+\infty)}(L^r) \cap L^q_{(0,+\infty)}(W^{1,r})$, then by the previous estimate one obtains

$$\lim_{t \to +\infty} \|u(t) - U_\gamma(t)u_+\|_{H^1} = 0.$$

Finally, it remains to prove uniqueness of such solution u. Let $v \in C(\mathbb{R}_+, H^1)$ be another solution to the first equation in (1.1) such that

$$\lim_{t\to+\infty} \|v(t) - U_\gamma(t)u_+\|_{H^1} = 0.$$

With the integral formula of Duhamel, v satisfies

$$v(t) = U_\gamma(t)u_+ + i \int_t^{+\infty} U_\gamma(t-s)(I_\alpha * |u(s)|^p)|u(s)|^{p-2}u(s)ds.$$

Arguing as previously, we get

$$\|u-v\|_{L^q_{(t,+\infty)}(L^r)} \lesssim \|U_\gamma(\cdot)u_+\|^{2(p-1)}_{L^q_{(t,+\infty)}(L^r)} \|u-v\|_{L^q_{(t,+\infty)}(L^r)}.$$

Using (5.1), yields for t large enough

$$\|u-v\|_{L^q_{(t,+\infty)}(L^r)} \leq \frac{1}{2}\|u-v\|_{L^q_{(t,+\infty)}(L^r)}.$$

This close the question of uniqueness. The continuity of the scattering mapping $\tilde{S}$ is a consequence of previous computations.

6. GLOBAL SOLUTIONS VIA INVARIANT SETS

In this section, we are going to prove Theorem 2.7. Precisely, the solutions of (1.1) are global if the initial data belongs to some invariant sets. For this purpose, we need to prove at first some auxiliary results.

6.1. INTERMEDIATE RESULTS

Let us start by showing the continuity of the energy functional S on H^1.

Lemma 6.1. *Let $1+\frac{\alpha}{N} < p < p^*$. The functional $u \to S(u)$ satisfies the local Lipschitz condition on H^1. Precisely, for all $u, v \in H^1$ we have*

$$|S(u) - S(v)| \lesssim (\|u\|_{H^1} + \|v\|_{H^1} + \|u\|^{p-1}_{H^1} + \|v\|^{p-1}_{H^1})\|u-v\|_{H^1}.$$

Proof. Let

$$I := \int (I_\alpha * |u|^p)|u|^p dx - \int (I_\alpha * |v|^p)|v|^p dx.$$

We have

$$\begin{aligned} I &= \int (I_\alpha * (|u|^p - |v|^p))|u|^p dx + \int (I_\alpha * |v|^p)(|u|^p - |v|^p) dx \\ &= \int (I_\alpha * (|u|^p - |v|^p))|u|^p dx + \int (I_\alpha * (|u|^p - |v|^p))|v|^p dx. \end{aligned}$$

Thanks to the Hardy–Littlewood–Sobolev inequality, we get

$$I \lesssim \{\||u|^p\|_{\frac{2N}{\alpha+N}} + \||v|^p\|_{\frac{2N}{\alpha+N}}\}\|(|u|^p - |v|^p)\|_{\frac{2N}{\alpha+N}}.$$

Next, by using the Mean Value Theorem, the function $x \to x^{\frac{2N}{\alpha+N}}$ is convex and the Hölder inequality, one gets

$$\begin{aligned}\||u|^p - |v|^p\|_{\frac{2N}{\alpha+N}} &\lesssim \|\{|u|^{p-1} + |v|^{p-1}\}|u-v|\|_{\frac{2N}{\alpha+N}} \\ &\lesssim \left\|\{|u|^{\frac{2N(p-1)}{\alpha+N}} + |v|^{\frac{2N(p-1)}{\alpha+N}}\}|u-v|^{\frac{2N}{\alpha+N}}\right\|_1^{\frac{\alpha+N}{2N}} \\ &\lesssim \left(\|u\|_{\frac{2Np}{\alpha+N}}^{\frac{2N(p-1)}{\alpha+N}} + \|v\|_{\frac{2Np}{\alpha+N}}^{\frac{2N(p-1)}{\alpha+N}}\right)^{\frac{\alpha+N}{2N}} \|u-v\|_{\frac{2Np}{\alpha+N}}.\end{aligned}$$

Applying the inequality $(a+b)^\rho \le 2(a^\rho + b^\rho)$ for a, b nonnegative and $0 \le \rho \le 2$, we deduce

$$\||u|^p - |v|^p\|_{\frac{2N}{\alpha+N}} \lesssim (\|u\|_{\frac{2Np}{\alpha+N}}^{p-1} + \|v\|_{\frac{2Np}{\alpha+N}}^{p-1})\|u-v\|_{\frac{2Np}{\alpha+N}}.$$

Since $1 + \frac{\alpha}{N} < p < p^*$, then $2 < \frac{2Np}{\alpha+N} < \frac{2N}{N-2}$. So, by Sobolev injections one gets

$$\||u|^p - |v|^p\|_{\frac{2N}{\alpha+N}} \lesssim (\|u\|_{H^1}^{p-1} + \|v\|_{H^1}^{p-1})\|u-v\|_{H^1}.$$

Moreover, we have

$$\||u|^p\|_{\frac{2N}{\alpha+N}} = \|u\|^p_{\frac{2N}{\alpha+N}} \lesssim \|u\|^p_{H^1},$$

and

$$|\|u\|^2_{H^1} - \|v\|^2_{H^1}| \le (\|u\|_{H^1} + \|v\|_{H^1})\|u-v\|_{H^1}.$$

This finishes the proof by combining previous inequalities. □

Remark 6.2. In the same way, the functional $K_{a,b}$ is continuous on H^1 for any $(a, b) \in \mathcal{A}$.

Lemma 6.3. *Taking $(a, b) \in \mathcal{A}$, $u \in H^1$ and denote $u^\lambda := e^{a\lambda}u(e^{-b\lambda}\cdot)$. Then, for all $\lambda \le 0$ we have $S(u^\lambda) \le S(u)$.*

Proof. Denote $\mu := 2ap + N(b+\alpha)$ and write

$$\begin{aligned}S(u^\lambda) &= \|u^\lambda\|^2 + \|\nabla u^\lambda\|^2 - \frac{1}{p}\int (I_\alpha * |u^\lambda|^p)|u^\lambda|^p dx \\ &= e^{\lambda(2a+Nb)}\|u\|^2 + e^{\lambda(2a+(N-2)b)}\|\nabla u\|^2 - \frac{1}{p}e^{\lambda\mu}\int (I_\alpha * |u|^p)|u|^p dx \\ &= e^{\lambda\mu}\{e^{(-\lambda)(\mu-(2a+Nb))}\|u\|^2 + e^{(-\lambda)(\mu-(2a+(N-2)b))}\|\nabla u\|^2 \\ &\quad - \frac{1}{p}\int (I_\alpha * |u|^p)|u|^p dx\} \\ &\le e^{\lambda\mu}e^{(-\lambda)(\mu-\underline{\mu})}S(u) \le e^{\lambda\underline{\mu}}S(u) \le S(u).\end{aligned}$$

□

Let us define

$$\mathcal{B}^-_{a,b} := \{v \in H^1 : S(v) < m \text{ and } K_{a,b}(v) < 0\}.$$

Lemma 6.4. *The sets $\mathcal{B}^+_{a,b}$ and $\mathcal{B}^-_{a,b}$ are independent of $(1, \frac{-2}{N}) \neq (a,b) \in \mathcal{A}$.*

Proof. Recall that m is independent of $(a,b) \in \mathcal{A}$ ([27]). Then, the reunion $\mathcal{B}^+_{a,b} \cup \mathcal{B}^-_{a,b}$ is independent of $(a,b) \in \mathcal{A}$. So, it is sufficient to prove that $\mathcal{B}^+_{a,b}$ is independent of (a,b). Obviously $S(u) \leq \|u\|^2_{H^1}$, then $S(u) < m$ for $\|u\|_{H^1}$ small enough. Since $(a,b) \neq (1, \frac{-2}{N})$ then $2a + Nb > 0$, it follows that

$$2a + (N-2)b > 2a - \frac{2a}{N}(N-2) = \frac{4a}{N} > 0.$$

Hence, one has $\overline{\mu} > \underline{\mu} > 0$. Thanks to the Gagliardo–Nirenberg inequality, there exists $C > 0$ such that

$$\int (I_\alpha * |u|^p)|u|^p dx \leq C\|u\|^A \|\nabla u\|^B.$$

Therefore

$$K_{a,b}(u) \geq \underline{\mu}\|u\|^2_{H^1} - C\|u\|^A\|\nabla u\|^B \geq \|u\|^2_{H^1}(\underline{\mu} - C\|u\|^{2(p-1)}_{H^1}).$$

Consequently, $K_{a,b}(u) \geq 0$ for $\|u\|_{H^1}$ small enough. From previous, one deduce that $\mathcal{B}^+_{a,b}$ contains an open small ball $B(0,\delta)$ of H^1 centered on 0 and of radius $\delta > 0$. According to the definition of m, if $S(u) < m$ and $K_{a,b}(u) = 0$ then $u = 0$. Hence

$$\mathcal{B}^+_{a,b} = B(0,\delta) \cup \{u \in H^1 : S(u) < m \text{ and } K_{a,b}(u) > 0\}.$$

Thanks to Lemma 6.1, the set $\mathcal{B}^+_{a,b}$ is open in H^1. We take $u \in \mathcal{B}^+_{a,b}$ such that $K_{a,b}(u) > 0$ and denote $u^\lambda := e^{a\lambda}u(e^{-b\lambda}\cdot)$. By using Lemma 6.3 one gets $S(u^\lambda) \leq S(u) < m$ for all $\lambda \leq 0$. In all cases, the scaling $\{u^\lambda, \lambda \leq 0\}$ defines a continuous path from u to 0 in H^1. Therefore, the set $\mathcal{B}^+_{a,b}$ is contracted to 0 and then it is connected. Now, let $(1, \frac{-2}{N}) \neq (a', b') \in \mathcal{A}$ and write

$$\mathcal{B}^+_{a,b} = \mathcal{B}^+_{a,b} \cap (\mathcal{B}^+_{a',b'} \cup \mathcal{B}^-_{a',b'}) = (\mathcal{B}^+_{a,b} \cap \mathcal{B}^+_{a',b'}) \cup (\mathcal{B}^+_{a,b} \cap \mathcal{B}^-_{a',b'}).$$

Thanks to Lemma 6.1, the set $\mathcal{B}^-_{a',b'}$ is also an open set in H^1. Since $0 \in \mathcal{B}^+_{a,b} \cap \mathcal{B}^+_{a',b'}$, then a connectivity argument gives $\mathcal{B}^+_{a,b} \subset \mathcal{B}^+_{a,b} \cap \mathcal{B}^+_{a',b'} \subset \mathcal{B}^+_{a',b'}$. Conversely, one gets $\mathcal{B}^+_{a',b'} \subset \mathcal{B}^+_{a,b}$. This finishes the proof. □

Lemma 6.5. *For all $(1, \frac{-2}{N}) \neq (a,b) \in \mathcal{A}$, the set $\mathcal{B}^+_{a,b}$ is invariant under the flow of* (1.1).

Proof. According to the previous lemma, it is sufficient to prove the desired result for the set $\mathcal{B}^+_{1,0}$. For this purpose, let $u_0 \in \mathcal{B}^+_{1,0}$ and $u \in C_{T^*_\gamma}(H^1)$ be the maximal solution to (1.1). Suppose that there exists $t_0 \in (0, T^*_\gamma)$ such that $u(t_0)$ does not belongs to $\mathcal{B}^+_{1,0}$, means that $K_{1,0}(u(t_0)) < 0$, and $u(t) \in \mathcal{B}^+_{1,0}$ for all $t \in [0, t_0)$. Since that $\frac{d}{dt}S(u(t)) = -\gamma K_{1,0}(u(t))$, then $S(u(t))$ is non-increasing on $[0, t_0)$. Thanks to Lemma 6.1, the function $t \to S(u(t))$ is continuous on $[0, T^*_\gamma)$. Thus

$$\S(u(t)) \leq S(u(0)) < m \quad \text{for all } t \in [0, t_0]. \tag{6.1}$$

Since $K_{1,0}(u(t_0)) < 0$, with a continuity argument, there exists an instant of time $t_1 \in (0, t_0)$ such that $K_{1,0}(u(t_1)) = 0$. With respect to the definition of m, one gets $m \leq S(u(t_1))$. This finishes the proof by contradiction with (6.1). $\square$

6.2. PROOF OF THEOREM 2.7

Without loss of generality, assume $t_0 = 0$ by translation argument. Using the two previous lemmas, one gets $u(t) \in \mathcal{B}^+_{1,1}$ for all $t \in [0, T^*_\gamma)$. Then

$$\begin{aligned} m &> S(u) - \frac{1}{\mu}K_{1,1}(u) \\ &= S(u) - \frac{1}{N+2}K_{1,1}(u) \\ &= \frac{2}{N+2}\|\nabla u\|^2 + \frac{2(p-1)+\alpha}{p(N+2)} \int (I_\alpha * |u|^p)|u|^p dx \\ &\geq \frac{2}{N+2}\|\nabla u\|^2. \end{aligned}$$

Hence

$$\sup_{t\in[0,T^*_\gamma)} \|\nabla u(t)\|^2 < \infty.$$

Since $\|u(t)\|^2 = e^{-\gamma t}\|u_0\|^2 \leq \|u_0\|^2$, then u is bounded in H^1, so $T^*_\gamma = \infty$.

Remark 6.6. Let $p = p_*$ and Q be the positive radially symmetric solution of (2.1), see [19]. If $\|u_0\| < \|Q\|$, then the solution of (1.1) emanating from u_0 is global. Indeed, from the conservation laws and Proposition 2.14, one has

$$\begin{aligned} E(u_0) &= \|\nabla u\|^2 - \frac{1}{p_*} \int (I_\alpha * |u|^p)|u|^p dx \\ &\geq \|\nabla u\|^2 - \frac{C_{N,p_*,\alpha}}{p_*}\|u\|^A\|\nabla u\|^B \\ &\geq \|\nabla u\|^2 - \|Q\|^{2-2p_*}\|u_0\|^{2p_*-2}\|\nabla u\|^2 \\ &\geq (1 - \|Q\|^{2-2p_*}\|u_0\|^{2p_*-2})\|\nabla u\|^2. \end{aligned}$$

Therefore, whenever $\|u_0\| < \|Q\|$, the solution u is bounded and then it is global in H^1. The condition $\|u_0\| < \|Q\|$ is sharp in the following sense: for every real constant $C > 1$, if $u_0(x) := CQ(x)$ then $\|u_0\| > \|Q\|$ and the solution of (1.1) with data u_0 blows up in finite time. Indeed, using the identity (2.4), one gets

$$\frac{1}{p_*} \int (I_\alpha * |Q|^p)|Q|^p dx = \|\nabla Q\|^2.$$

Thus

$$E(u_0) = C^2(1 - C^{2(p_*-1)})\|\nabla Q\|^2 < 0,$$

Let $J(t) := \int |x|^2|u|^2 dx$, according to computations made in [6], the damping term has no effect, so when $p = p_*$ we obtain

$$\begin{aligned} J''(t) &= 8\|\nabla u\|^2 - \frac{4(Np_* - N - \alpha)}{p_*}\int (I_\alpha * |u|^{p_*})|u|^{p_*} dx \\ &= 8E(u_0) - 16J(t) \le 8E(u_0) < 0. \end{aligned}$$

It follows that the solution u with data u_0 blows up in finite time.

7. APPENDIX

Recall the so-called Riesz potential inequality, see [20, Appendix A].

Lemma 7.1. *Let $d \ge 1, q > 1, 0 < \alpha < \frac{d}{q}$ and $\frac{1}{r} = \frac{1}{q} - \frac{\alpha}{d}$. Then, $I_\alpha : L^q \to L^r$ is a bounded operator. Precisely, there exists $C_{d,\alpha,q} > 0$ such that*

$$\|I_\alpha * f\|_r \le C_{d,\alpha,q}\|f\|_q.$$

7.1. PROOF OF PROPOSITION 2.19

Elementary computations give

$$0 < r < \frac{2N}{N-2} \quad \Leftrightarrow \quad 0 < N\Big(\frac{1}{2} - \frac{1}{r}\Big) < 1.$$

Thanks to Proposition 2.16, one have

$$\begin{aligned} \Big\| \int_0^t U_\gamma(t-s)f(s)ds\Big\|_{L^\theta_T(L^r)} &\lesssim \Big\| \int_0^t e^{-\gamma(t-s)}U_0(t-s)f(s)ds\Big\|_{L^\theta_T(L^r)} \\ &\lesssim \Big\| \int_0^T \frac{1}{|t-s|^{\frac{1}{\theta}+\frac{1}{\mu}}}\|f(s)\|_{r'}ds\Big\|_{L^\theta_T} \\ &\lesssim \Big\| \int_0^T I_{\frac{1}{\mu'}-\frac{1}{\theta}} * \|f(s)\|_{r'}ds\Big\|_{L^\theta_T}. \end{aligned}$$

Applying Lemma 7.1 with $d = 1$ and taking account of $\frac{1}{\theta} + \frac{1}{d}(\frac{1}{\mu'} - \frac{1}{\theta}) = \frac{1}{\mu'}$, we obtain

$$\Big\| \int_0^t U_\gamma(t-s)f(s)ds\Big\|_{L^\theta_T(L^r)} \lesssim \|f\|_{L^{\mu'}_T(L^{r'})}.$$

7.2. PROOF OF COROLLARY 2.20

Using Proposition 2.16 and the standard Strichartz estimate, one gets

$$\begin{aligned}\|U_\gamma(t)u_0\|_{L^q_T(L^r)} &= \|e^{-\gamma t}U_0(t)u_0\|_{L^q_T(L^r)}\\ &\lesssim \|U_0(t)u_0\|_{L^q_T(L^r)}\\ &\lesssim \|u_0\|.\end{aligned}$$

On the other hand, applying the previous Proposition with $\theta = \mu = q$, one obtains

$$\Big\|\int_0^t U_\gamma(t-s)f(s)ds\Big\|_{L^q_T(L^r)} \leq \|f\|_{L^{q'}_T(L^{r'})}.$$

In summary, if

$$i\dot{u} + \triangle u + i\gamma u = f$$

with data u_0, then

$$\begin{aligned}\|u\|_{L^q_T(L^r)} &\lesssim \|U_\gamma(t)u_0\|_{L^q_T(L^r)} + \Big\|\int_0^t U_\gamma(t-s)f(s)ds\Big\|_{L^q_T(L^r)}\\ &\lesssim \|u_0\| + \|f\|_{L^{q'}_T(L^{r'})}.\end{aligned}$$

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