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Extremals in the problem of minimum time obstacle avoidance for a 2D double integrator system*

by

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Abstract: This paper provides an analysis of time optimal control problem of motion of a material point in the plane outside the given circle, without friction. The point is controlled by a force whose absolute value is limited by one. The closure of exterior of the circle plays the role of the state constraint. The analysis of the problem is based on the minimum principle.

Keywords: Pontryagin's principle, necessary optimality condition, state constraint, optimal control, bang-bang control

1. Introduction

In Osmolovskii, Figura and Kośka (2013), we have analyzed the following optimization problem of mechanics: a material point (a spacecraft) of the mass equal to one moves in the plane without friction. The point is controlled by a force, whose absolute value is limited by one. The initial position of the point and the initial vector of its velocity are fixed. The final position and the final vector of velocity are fixed, too. The problem is to minimize the time of the motion. In Osmolovskii, Figura and Kośka (2013), this problem was completely solved. Namely, we have described all extremals of the problem.

In the present paper we study the problem posed by A. Milyutin in the early 1970s: to minimize the time of the motion of a material point in the plane, under a bounded force and in the presence of a forbidden area that has the form of a circle $|x| \leq r$ ($r > 0$). Again, the initial position and velocity and the final position and velocity are fixed. So, we consider the problem with state

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constraint: $|x| \geq r$. We see that the state constraint is nonlinear and the set of admissible positions is non-convex. Moreover, we will see that the order of the state constraint is equal to two. All this leads to certain difficulties in the problem. We offer here its partial solution.

The maximum principle for problems with state constraints was obtained by A.Ya. Dubovitskii and A.A. Milyutin in the mid 1960s (see, e.g., Dubovitskii and Milyutin, 1965). Its specificity is that the costate variable is a function of bounded variation and can have jumps at the state boundary. The jumps occur starting with the order of two, and certainly the discontinuity of the costate variable essentially complicates the solution of the problem. Only a few problems with state constraints have been analytically solved up to now.

We analyze the conditions of the maximum principle (we write it in the form of minimum principle) and obtain analytical formulas for the interior and boundary subarcs separately, and formulas for the first six derivatives of the function $\rho(t) = \langle x(t), x(t)\rangle$ at the junction point of these subarcs. For the interior case we present a complete description of all three possible types of extremals through elementary functions (this description was given earlier in Osmolovskii, Figura and Kośka (2013)). For the boundary case, only the rotation with a constant angular speed is analyzed. It is shown that the first five derivatives of $\rho(t)$ at the junction point vanish, while the 6th right derivative is positive, and the trajectory leaves the state boundary. Hence, when leaving the constraint, an extremal is tangent up to order 5 (with variation of the expected sign at order 6). We do not consider the question of "landing" of the extremal on the boundary of the set of admissible positions, since it is similar to the question of leaving the boundary.

2. Statement of the problem

Now let us give a formal description of the problem. Let the position of the point at time t be $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$ and its velocity be $y(t) = (y_1(t), y_2(t)) \in \mathbb{R}^2$. Denote by $u(t) = (u_1(t), u_2(t)) \in \mathbb{R}^2$ the vector of force at time t. We call $u(t)$ the control. There is a control constraint $|u(t)| \leq 1$, where $|u| = \sqrt{\langle u, u\rangle}$. As usual, by $\langle x, x'\rangle$ we denote the scalar product $x_1x_1' + x_2x_2'$ of vectors $x = (x_1, x_2)$ and $x' = (x_1', x_2')$ in $\mathbb{R}^2$.

The trajectory $(x(t), y(t))$ must satisfy the endpoint constraints: at the initial time $t = 0$, the initial position $x(0)$ should be equal to a given vector $\hat{x}_0 \in \mathbb{R}^2$, and the initial velocity $y(0)$ should be equal to a given vector $\hat{y}_0 \in \mathbb{R}^2$; at the final time $t = T$, the final position $x(T)$ and the final velocity $y(T)$ should be equal to given vectors $\hat{x}_T \in \mathbb{R}^2$ and $\hat{y}_T \in \mathbb{R}^2$, respectively. The motion is forbidden inside the circle $|x| \leq r$ $(r > 0)$. Hence, the inequality $|x(t)| \geq r$ should be fulfilled for all t. The problem is to minimize the time T of the control process $(x(t), y(t), u(t))$, $t \in [0, T]$. Since the mass of the point is equal to one, by Newton's law we have $u(t) = \ddot{x}(t) = \dot{y}(t)$. Thus, the problem has the form:

$$\min\ T, \tag{1}$$

subject to the constraints

$$\begin{aligned} &\dot{x}(t) = y(t), \quad \dot{y}(t) = u(t), \quad |u(t)| \leq 1, \quad |x(t)| \geq r, \quad t \in [0,T], \\ &x(0) = \hat{x}_0, \quad y(0) = \hat{y}_0, \quad x(T) = \hat{x}_T, \qquad y(T) = \hat{y}_T, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^2$, $y \in \mathbb{R}^2$, $u \in \mathbb{R}^2$, $r > 0$. We say that a triple of the functions $(x(t), y(t), u(t))$ considered on the interval of time $[0,T]$ ($T > 0$) is an *admissible process* if the functions $x(t)$ and $y(t)$ are absolutely continuous and the function $u(t)$ is measurable and essentially bounded on $[0,T]$, and all constraints (2) are fulfilled. The minimum is sought among the admissible processes. We assume that $|\hat{x}_0| > r$ and $|\hat{x}_T| > r$.

3. Minimum principle in time optimal control problem with state inequality constraint

3.1. Time optimal control problem

For the convenience of the reader, we formulate here the minimum principle for a class of problems, containing problem (1)–(2). For problems with state constraints, the full set of necessary conditions in the form of the maximum principle was obtained by Dubovitskii and Milyutin in mid 1960s, see, e.g., Dubovitskii and Milyutin (1965). Later on, these conditions were published in other papers and books, e.g. Girsanov (1972), Ioffe and Tikhomirov (1974), Milyutin, Dmitruk, and Osmolovskii (2004). We will formulate them for the following problem:

$$\min\ T \tag{3}$$

subject to

$$\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U, \quad t \in [0,T], \tag{4}$$

$$x(0) = a, \quad x(T) = b, \tag{5}$$

$$g(x(t)) \leq 0, \quad t \in [0,T], \tag{6}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, the functions $f : \mathbb{R}^{n+m} \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \mapsto \mathbb{R}$ are continuously differentiable, $U \subset \mathbb{R}^m$ is an arbitrary set, and $a, b \in \mathbb{R}^n$ are given vectors. Hence, we consider the problem with one *state constraint* (6). We assume that

$$g(a) < 0, \quad g(b) < 0. \tag{7}$$

We say that $(x(t), u(t))$, $t \in [0,T]$ is an *admissible process* if the function $x(t)$ is absolutely continuous on $[0,T]$, the function $u(t)$ is measurable and essentially bounded on $[0,T]$, and all constraints (4)-(6) are satisfied. The differential equation $\dot{x}(t) = f(x(t), u(t))$ and the condition $u(t) \in U$ are assumed to be fulfilled a.e. on $[0,T]$. The minimum of T is sought among all admissible processes. In view of (7), the ends of optimal trajectory do not belong to the boundary of the set of admissible positions.

3.2. Minimum principle

Denote by $\mathbb{R}^{n*}$ the space of n-dimensional row-vectors. To formulate the minimum principle, we introduce the *Pontryagin function* (or the *pre-Hamiltonian*)

$$H(x,u,\lambda) = \lambda f(x,u) \tag{8}$$

and the *augmented Pontryagin function* (or the *augmented pre-Hamiltonian*)

$$H^a(x,u,\lambda,\dot{\mu}) = \lambda f(x,u) + \dot{\mu} g(x), \tag{9}$$

where $\lambda \in \mathbb{R}^{n*}$ and $\dot{\mu} \in \mathbb{R}$. Note that here $\dot{\mu}$ is an arbitrary number, but later, this notation will be used to denote the generalized derivative of the multiplier $\mu(t)$ with respect to t.

Let $(x(t),u(t))$, $t \in [0,T]$ be an admissible process. Denote by $\lambda(t) : [0,T] \mapsto \mathbb{R}^{n*}$ an arbitrary function of bounded variation. By $\operatorname{Var}\lambda$ we denote the variation of λ, and by $d\lambda$ the Radon measure, defined on $[0,T]$, which corresponds to the function $\lambda(t)$. Similarly, by $\mu(t) : [0,T] \mapsto \mathbb{R}$ we denote a function of bounded variation, and by $d\mu$ the Radon measure, which corresponds to $\mu(t)$. The conditions of the minimum principle for the process $(x(t),u(t))$, $t \in [0,T]$ are as follows: there exist a constant α_0, and functions of bounded variation $\lambda(t)$ and $\mu(t)$ such that

$$\alpha_0 \geq 0, \quad d\mu(t) \geq 0, \quad t \in [0,T], \tag{10}$$

$$d\mu(t) g(x(t)) = 0, \quad t \in [0,T], \tag{11}$$

$$-d\lambda(t) = H_x(x(t),u(t),\lambda(t))\,dt + d\mu(t) g_x(x(t)), \quad t \in [0,T], \tag{12}$$

$$\min_{u \in U} H(x(t),u,\lambda(t)) = H(x(t),u(t),\lambda(t)) \quad \text{a.e. on} \quad [0,T], \tag{13}$$

$$H(x(t),u(t),\lambda(t)) = -\alpha_0 \quad \text{a.e. on} \quad [0,T], \tag{14}$$

$$\operatorname{Var}\mu + \operatorname{Var}\lambda + |\lambda(0)| > 0. \tag{15}$$

We call (10) the *non-negativity conditions*, (11) the *complementary slackness condition*, (12) the *adjoint equation*, (13) the *minimality condition of the pre-Hamiltonian*, and (15) the *non-triviality condition.* Condition (14) unites the *condition of the constancy of the Hamiltonian* (which holds for autonomous problems) and the *transversality condition.*

Let us note that from the negativity condition $d\mu(t) \geq 0$ it follows that $\mu(t)$ is a monotone nondecreasing function, and condition (11) implies that the measure $d\mu$ is concentrated on the set $\{t \in [0,T] : g(x(t)) = 0\}$ of the *boundary points* of the trajectory $x(t)$. Finally, note that the adjoint equation (12) may be considered as an equality between measures. Dividing it by dt, we get

$$-\frac{d\lambda(t)}{dt} = H_x(x(t),u(t),\lambda(t)) + \frac{d\mu(t)}{dt} g_x(x(t)), \quad t \in [0,T], \tag{16}$$

where the derivatives $\frac{d\lambda(t)}{dt}$ and $\frac{d\mu(t)}{dt}$ should be understood in the sense of the theory of generalized functions. Using the notations

$$\dot{\lambda} = \frac{d\lambda(t)}{dt}, \quad \dot{\mu} = \frac{d\mu(t)}{dt}$$

and definition (9) of the augmented Pontryagin function, we rewrite the adjoint equation (16) in a simple form

$$-\dot{\lambda}(t) = H_x^a(x(t), u(t), \lambda(t), \dot{\mu}(t)), \tag{17}$$

easy to remember. Without loss of generality we can assume that the functions $\lambda(t)$ and $\mu(t)$ are left-continuous.

The following theorem holds (see, e.g., Milyutin, Dmitruk, and Osmolovskii, 2004):

THEOREM 3.1 *Let a process $(x(t), u(t))$, $t \in [0, T]$ be a solution to the problem (3)–(6). Then there exist a constant α_0 and left-continuous functions of bounded variation $\lambda(t)$ and $\mu(t)$ such that the conditions (10)–(15) of the minimum principle hold.*

3.3. Order of the state constraint

Suppose that the functions f and g are of class C^2. We say that the state constraint has the *order one* if the function

$$g^{(1)}(x, u) := g'(x)f(x, u)$$

depends on u, i.e. $g_u^{(1)}$ is not identically zero over $\mathbb{R}^n \times \mathbb{R}^m$. Now, let $g_u^{(1)}$ be identically zero, i.e. $g^{(1)} = g^{(1)}(x)$, and let the functions f and g be of class C^3. Define the function

$$g^{(2)}(x, u) := g_x^{(1)}(x)f(x, u).$$

If this function depends on u (i.e. $g_u^{(2)}$ is not identically zero over $\mathbb{R}^n \times \mathbb{R}^m$), then we say that the state constraint has the *order two*. Similarly, we can inductively define the state constraint of any order $q \in \mathbb{N}$.

4. Minimum principle in the problem of the fastest obstacle avoidance

We represent the state constraint $|x| \geq r$ in the form

$$g(x) := \frac{1}{2}(r^2 - \langle x, x \rangle) \leq 0.$$

Since $g^{(1)} = -\langle x, y \rangle$ does not depend on u, and $g^{(2)} = -\langle y, y \rangle - \langle x, u \rangle$ depends on u explicitly, the order of the state constraint is equal to two.

Let a process $(x(t), y(t), u(t))$, $t \in [0, T]$ be a solution to problem (1)-(2). Let $\lambda_x \in (\mathbb{R}^2)^*$, $\lambda_y \in (\mathbb{R}^2)^*$, $\dot{\mu} \in \mathbb{R}$. According to (8) and (9), the pre-Hamiltonian (or the Pontryagin function) and the augmented pre-Hamiltonian (or the augmented Pontryagin function) of the problem have the form (see, for instance, Dubovitskii and Milyutin, 1981; Milyutin, Dmitruk, and Osmolovskii, 2004):

$$H = \lambda_x y + \lambda_y u, \quad H^a = H + \frac{\dot{\mu}}{2}(r^2 - \langle x, x\rangle),$$

respectively. Conditions (10)–(15) of the minimum principle for the solution $(x(t), y(t), u(t))$, $t \in [0, T]$ are as follows:

$$\begin{aligned}
&\alpha_0 \geq 0, \quad d\mu(t) \geq 0, \quad d\mu(t)(r^2 - \langle x(t), x(t)\rangle) = 0,\\
&\dot{\lambda}_x(t) = -H^a_x = \dot{\mu}(t)x(t), \quad \dot{\lambda}_y(t) = -H^a_y = -\lambda_x(t),\\
&H(t) := \lambda_x(t)y(t) + \lambda_y(t)u(t) = -\alpha_0 \quad \text{a.e. on} \quad [0, T],\\
&\min\nolimits_{|u|\leq 1} \lambda_y(t)u = \lambda_y(t)u(t) \quad \text{a.e. on} \quad [0, T],\\
&(\lambda_x, \lambda_y, d\mu) \neq (0, 0, 0),
\end{aligned} \tag{18}$$

where λ_x, λ_y, and μ are the functions of bounded variation (continuous from the left), which define the measures $d\lambda_x$, $d\lambda_y$, and $d\mu$, respectively. The derivatives $\dot{\lambda}_x$, $\dot{\lambda}_y$, and $\dot{\mu}$ are understood in the sense of the theory of generalized functions, so that the products $\dot{\lambda}_x\, dt$, $\dot{\lambda}_y\, dt$, and $\dot{\mu}\, dt$ are equal to the measures $d\lambda_x$, $d\lambda_y$, and $d\mu$, respectively. From the adjoint equation $\dot{\lambda}_y(t) = -\lambda_x(t)$ it follows that the function λ_y is Lipschitz continuous.

Obviously, the system (18) is equivalent to the following *dual system*:

$$\begin{aligned}
&d\mu(t) \geq 0, \quad d\mu(t)(r^2 - |x(t)|^2) = 0, \quad (\lambda_x, \lambda_y, d\mu) \neq (0, 0, 0),\\
&d\lambda_x(t) = x(t)\, d\mu(t), \quad \dot{\lambda}_y(t) = -\lambda_x(t),\\
&\lambda_x(t)y(t) + \lambda_y(t)u(t) \equiv -\alpha_0 \leq 0,\\
&u(t) = -\tfrac{\lambda_y^*(t)}{|\lambda_y(t)|} \text{ if } \lambda_y(t) \neq 0,
\end{aligned} \tag{19}$$

where λ^* is the transposed vector. Let us add the conditions of the *primal system*:

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = u(t), \quad |x(t)| \geq r, \quad |x(0)| > r, \quad |x(T)| > r. \tag{20}$$

A triple (x, y, u) such that there exists a triple $(\lambda_x, \lambda_y, d\mu)$, satisfying (19) and (20), will be called *Pontryagin's extremal*. We will seek the Pontryagin's extremals (with no account of boundary conditions).

Let us note that the last two conditions of system (19) imply

$$-\lambda_x(t)y(t) + |\lambda_y(t)| = \alpha_0 \geq 0. \tag{21}$$

This condition should be fulfilled for all $t \in [0, T]$, but, as is known, it suffices to check the sign of (21) only at one point $t \in [0, T]$.

Also note that the last condition in (19) can be obtained as follows. The control constraint, written in the form $\varphi(u) := \langle u, u\rangle - 1 \le 0$, can be included in the augmented pre-Hamiltonian with multiplier $\nu \ge 0$. Then, the stationarity condition with respect to the control $H_u + \nu\varphi_u = 0$ gives $\lambda_y + \nu u^* = 0$, whence $|\lambda_y|^2 = \nu^2|u|^2$. This relation along with complementary slackness condition $\nu(\langle u, u\rangle - 1) = 0$ implies $\nu = |\lambda_y|$. It follows that $u(t) = -\lambda_y^*(t)/|\lambda_y(t)|$ if $\lambda_y(t) \ne 0$.

5. Analysis of the interior arc

5.1. Analysis of conditions of the minimum principle

Let us consider an interval $\Delta = (t', t'')$ such that, on this interval, the trajectory $x(t)$ belongs to the interior of the set of admissible positions: $|x(t)| > r$ for all $t \in (t', t'')$. Such interval necessarily exists, at least at the beginning and at the end of the motion, since we assume that $|x(0)| > r$ and $|x(T)| > r$. Then the complementary slackness condition $d\mu(r^2 - \langle x, x\rangle) = 0$ implies that the measure $d\mu$ vanishes on the interval Δ. Additionally we assume that $\lambda_y(\cdot)$ does not vanish on Δ. Then the extremality conditions on Δ become

$$\begin{aligned} &\dot\lambda_x = 0, \quad \dot\lambda_y = -\lambda_x, \quad \lambda_y(\cdot) \ne 0,\\ &-\lambda_x y + |\lambda_y| = \alpha_0 \ge 0,\\ &u(t) = -\tfrac{\lambda_y^*(t)}{|\lambda_y(t)|} \quad \text{if} \quad \lambda_y(t) \ne 0,\\ &\dot x = y, \quad \dot y = u, \quad |x| > r. \end{aligned} \tag{22}$$

This system was analyzed in Osmolovskii, Figura and Końska (2013). The first two conditions imply that λ_x is a constant vector and λ_y is a linear function, not equal to zero identically, i.e.,

$$-\lambda_y^* = kt + b, \quad \lambda_x = k, \quad k^2 + b^2 > 0, \tag{23}$$

where $k, b \in \mathbb{R}^2$. The following three cases are possible.

(i) The vectors k and b are linearly independent. In this case the function λ_y does not vanish. Consequently,

$$u(t) = \frac{kt + b}{|kt + b|} \tag{24}$$

is a continuous function, and the motion on Δ is uniquely defined by the conditions:

$$\begin{aligned} &\dot x(t) = y(t), \quad \dot y(t) = u(t) = \tfrac{kt+b}{|kt+b|}, \quad |x(t)| > r, \quad t \in (t', t''),\\ &x(t_0) = x_0, \quad y(t_0) = y_0,\\ &-\langle k, y_0\rangle + |kt_0 + b| = \alpha_0 \ge 0, \end{aligned} \tag{25}$$

where $t_0 \in (t', t'')$, $k, b, x_0, y_0 \in \mathbb{R}^2$ are given point and vectors, respectively. In order to find any of such extremals, one has to choose k, b, x_0, y_0 and t_0 such that the following conditions hold

$$-\langle k, y_0 \rangle + |kt_0 + b| \geq 0, \quad k, b \text{ are linearly independent}, \quad |x_0| > r, \quad (26)$$

and then to solve the system

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = u(t) = \frac{kt + b}{|kt + b|}, \quad x(t_0) = x_0, \quad y(t_0) = y_0, \qquad (27)$$

to the left and to the right from the point t_0 until the following condition holds

$$|x(t)| > r. \qquad (28)$$

The interval Δ can be chosen as the maximal open interval such that the condition (28) is fulfilled. In the next section, we will give formulas for $x(t)$ and $y(t)$ in the case (i).

(ii) $k \neq 0$ and $b \parallel k$, i.e. b is collinear to k. Then there is such τ that $b = -k\tau$ and hence λ_y has the form $\lambda_y = -k^*(t - \tau)$. In this case

$$u(t) = k^0 \operatorname{sgn}(t - \tau), \text{ where } k^0 = \frac{k}{|k|}, \qquad (29)$$

i.e., the function $u(t)$ is piecewise constant, taking only two values, k^0 and $-k^0$, and having one possible switching at the point τ if $\tau \in (t', t'')$. Thus, in the case (ii), we have the so-called *generalized bang-bang control*. In this case, the condition $H(t) := \lambda_x(t)y(t) + \lambda_y(t)u(t) = -\alpha_0 \leq 0$ on $[0, T]$ can be checked at the point τ: $H(\tau) := \lambda_x(\tau)y(\tau) = -\alpha_0 \leq 0$ or $ky_\tau \leq 0$, where $y_\tau = y(\tau)$. In order to obtain an extremal corresponding to case (ii), one has to choose arbitrary $k^0, x_\tau, y_\tau \in \mathbb{R}^2$ and τ such that the conditions

$$|k^0| = 1, \quad \langle k^0, y_\tau \rangle \leq 0, \quad |x_\tau| > r, \qquad (30)$$

hold, and then to solve the system

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = u(t) = k^0 \operatorname{sgn}(t - \tau), \quad x(\tau) = x_\tau, \quad y(\tau) = y_\tau \quad (31)$$

finding an interval Δ such that $|x(t)| > r$ on this interval.

Consider a system of a more general form, with one switching of the control at a point τ and with the initial data given at a point t_0 (which may be different from the point τ):

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = u(t) = k^0 \operatorname{sgn}(t - \tau), \quad x(t_0) = x_0, \quad y(t_0) = y_0. (32)$$

System (32) is easily integrable. Consider any interval (t_1, t_2) such that $\tau \notin (t_1, t_2)$. Obviously $\operatorname{sgn}(t - \tau)$ is a constant on (t_1, t_2), and let it be equal to σ, where $\sigma = \pm 1$. Let $t_0 \in (t_1, t_2)$. Conditions

$$\dot{y} = u = k^0 \sigma, \quad y(t_0) = y_0$$

imply

$$y = k^0\sigma(t - t_0) + y_0, \tag{33}$$

and then the conditions

$$\dot{x} = y, \quad x(t_0) = x_0$$

yield

$$x = \frac{\sigma k_0}{2}(t - t_0)^2 + y_0(t - t_0) + x_0. \tag{34}$$

(iii) Let $k = 0$, $b \neq 0$. In this case $u = b^0$, where $b^0 = b/|b|$, and for given x_0, y_0, t_0 we get a system

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = u(t) = b^0, \quad x(t_0) = x_0, \quad y(t_0) = y_0, \tag{35}$$

which is also easily integrable.

So, the cases (ii) and (iii), where k and b are linearly dependent, are quite simple. Case (i) is not so simple, but again (as it was shown in Osmolovskii, Figura and Kośka, 2013) $x(t)$ and $y(t)$ can be expressed using elementary functions. We will show this below.

5.2. Integration of equations of motion in the Case of linearly independent k and b.

Consider system of equations (27):

$$\begin{aligned} &\dot{x}(t) = y(t), \quad \dot{y}(t) = u(t) = -\tfrac{\lambda_y^*(t)}{|\lambda_y(t)|}, \quad -\lambda_y^*(t) = kt + b, \quad t \in (t', t''), \\ &x(t_0) = x_0, \quad y(t_0) = y_0, \end{aligned}$$

where $t_0 \in (t', t'')$. We assume that k and b are linearly independent and then (taking into account the possibility of multiplication of the pair (λ_x, λ_y) by a positive constant) we can represent the function $-\lambda_y^* = kt + b$ in the form

$$-\lambda_y^* = \beta(t - t_0)k_0 + b_0,$$

where

$$\beta \in \mathbb{R}, \quad \beta \neq 0, \quad b_0 \in \mathbb{R}^2, \quad |b_0| = 1, \quad k_0 = Ab_0,$$

and A is the rotation matrix by the angle $\pi/2$ counterclockwise:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that

$$-\lambda_y^*(t_0) = b_0 = u(t_0).$$

Denote $u_0 := u(t_0)$. It is convenient to use the complex plane, so that the real axis is u_1 and the imaginary axis is u_2. Then

$$u = u_1 + iu_2.$$

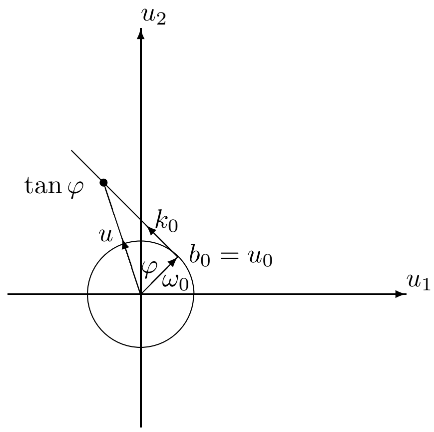

Figure 1. The control inside the set of admissible positions

We hope that the same notation, used for the two-dimensional real vectors and their representation by complex numbers, will not lead to confusion. In this regard, we want to emphasize that the notation $\langle x, y\rangle$ is always used to denote the scalar product of vectors x and y, while the product of corresponding complex numbers is denoted by xy.

Let $\omega(t)$ be the angle between the real axis and the unit vector $u(t)$, and $\varphi(t)$ be the angle between the unit vector $b_0 = u(t_0)$ and the vector $u(t)$. Finally, let ω_0 be the angle between the real axis and the unit vector b_0. Then

$$\omega(t) = \varphi(t) + \omega_0, \quad -\frac{\pi}{2} < \varphi(t) < \frac{\pi}{2} \quad \forall\, t, \quad \varphi(t_0) = 0,$$

$$u_0 = u(t_0) = e^{i\omega_0} = b_0, \quad u(t) = e^{i\omega(t)} = e^{i(\varphi(t)+\omega_0)} = u_0 e^{i\varphi(t)}.$$

It is clear (see Fig.1) that

$$\tan\varphi(t) = \beta(t - t_0), \qquad \text{and hence} \qquad \varphi(t) = \arctan\beta(t - t_0). \tag{36}$$

Since

$$\dot{y} = \frac{dy}{d\varphi}\frac{d\varphi}{dt} = u = u_0 e^{i\varphi},$$

we get from here

$$\frac{dy}{d\varphi} = \frac{dt}{d\varphi} u_0 e^{i\varphi}.$$

Moreover, the condition $\tan\varphi = \beta(t - t_0)$ implies that

$$\frac{dt}{d\varphi} = \frac{1}{\beta\cos^2\varphi}.$$

Thus,

$$\frac{dy}{d\varphi} = \frac{u_0}{\beta\cos^2\varphi}(\cos\varphi + i\sin\varphi).$$

Consequently,

$$dy = \frac{u_0}{\beta}\left(\frac{1}{\cos\varphi} + i\frac{\sin\varphi}{\cos^2\varphi}\right)d\varphi,$$

whence

$$y = \frac{u_0}{\beta}\left(\int\frac{d\varphi}{\cos\varphi} + i\int\frac{\sin\varphi}{\cos^2\varphi}\,d\varphi\right).$$

From the inequalities $-\pi/4 < \varphi/2 < \pi/2$ it follows that that

$$\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} > 0.$$

Therefore,

$$\int\frac{d\varphi}{\cos\varphi} = \ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} + C.$$

Moreover,

$$\int\frac{\sin\varphi}{\cos^2\varphi}\,d\varphi = \frac{1}{\cos\varphi} + C.$$

Consequently,

$$y = \frac{u_0}{\beta}\left(\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} + \frac{i}{\cos\varphi}\right) + C,$$

where $C = C_1 + iC_2$. Using the condition $\varphi(t_0) = 0$, we obtain

$$y_0 := y(t_0) = \frac{iu_0}{\beta} + C,$$

whence

$$C = y_0 - \frac{iu_0}{\beta}.$$

Thus, we get

$$y = \frac{u_0}{\beta}\left(\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} + i\left(\frac{1}{\cos\varphi} - 1\right)\right) + y_0.$$

Using the formula

$$\frac{1}{\cos\varphi} - 1 = \frac{2\tan^2\frac{\varphi}{2}}{1-\tan^2\frac{\varphi}{2}} = \tan\frac{\varphi}{2}\tan\varphi, \tag{37}$$

we obtain

$$y = \frac{u_0}{\beta}\left(\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} + i\tan\frac{\varphi}{2}\tan\varphi\right) + y_0. \tag{38}$$

Taking into account the fact that $y = y_1 + iy_2$, $u_0 = \cos\omega_0 + i\sin\omega_0$, $y_0 = y_{10}+iy_{20}$, we get the following expressions for y_1 and y_2 in Cartesian coordinates

$$y_1 = \frac{1}{\beta}\left(\cos\omega_0\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} - \sin\omega_0\tan\frac{\varphi}{2}\tan\varphi\right) + y_{10},$$

$$y_2 = \frac{1}{\beta}\left(\sin\omega_0\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} + \cos\omega_0\tan\frac{\varphi}{2}\tan\varphi\right) + y_{20}.$$

Furthermore, the relation $\dot{x} = y$ gives

$$\frac{dx}{d\varphi} = \frac{dt}{d\varphi}y = \frac{1}{\beta\cos^2\varphi}y.$$

This, combined with (38), implies

$$dx = \frac{u_0}{\beta^2\cos^2\varphi}\left(\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} + \frac{i}{\cos\varphi}\right)d\varphi + \frac{\beta y_0 - iu_0}{\beta^2\cos^2\varphi}d\varphi.$$

Consequently,

$$x = \frac{u_0}{\beta^2}\left\{\int\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\frac{d\varphi}{\cos^2\varphi} + i\int\frac{d\varphi}{\cos^3\varphi}\right\} + \frac{\beta y_0 - iu_0}{\beta^2}\int\frac{d\varphi}{\cos^2\varphi}. \tag{39}$$

For the integrals in (39), the following formulas hold (see Section. 8):

$$\int\frac{d\varphi}{\cos^3\varphi} = \frac{1}{2}\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} + \frac{1}{2}\cdot\frac{\sin\varphi}{\cos^2\varphi} + C, \tag{40}$$

$$\int\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\frac{d\varphi}{\cos^2\varphi} = \ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\tan\varphi - \frac{2}{1-\tan^2\frac{\varphi}{2}} + C, \tag{41}$$

Relations (39), (40), and (41) imply

$$x = \frac{u_0}{\beta^2}\left\{\tan\varphi\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} - \frac{2}{1-\tan^2\frac{\varphi}{2}} + \frac{i}{2}\left(\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} + \frac{\sin\varphi}{\cos^2\varphi}\right)\right\} +$$

$$+\frac{\beta y_0 - iu_0}{\beta^2}\tan\varphi + C,$$

where $C = C_1 + iC_2$. Now, we use the initial condition $x(t_0) = x_0$. Taking into account the fact that $\varphi(t_0) = 0$, we obtain

$$x_0 = \frac{u_0}{\beta^2}(-2) + C,$$

whence
$$C = x_0 + \frac{2u_0}{\beta^2}.$$
Consequently,
$$x = \frac{u_0}{\beta^2}\left\{\tan\varphi \ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} + 2 - \frac{2}{1-\tan^2\frac{\varphi}{2}} + \frac{i}{2}\left(\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} + \frac{\sin\varphi}{\cos^2\varphi}\right)\right\} +$$
$$+\frac{\beta y_0 - iu_0}{\beta^2}\tan\varphi + x_0.$$
Using formula (37), we obtain
$$x = \frac{u_0}{\beta^2}\left\{\tan\varphi\left(\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} - \tan\frac{\varphi}{2}\right) + \frac{i}{2}\left(\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} + \frac{\tan\varphi}{\cos\varphi} - 2\tan\varphi\right)\right\}$$
$$+\frac{y_0}{\beta}\tan\varphi + x_0, \quad \varphi = \arctan\beta(t-t_0). \quad (42)$$
Finally, taking into account the fact that $x = x_1 + ix_2$, $u_0 = \cos\omega_0 + i\sin\omega_0$, $x_0 = x_{10} + ix_{20}$, $y_0 = y_{10} + iy_{20}$, we get the following expressions for the Cartesian coordinates x_1 and x_2:
$$x_1 = \frac{1}{\beta^2}\cos\omega_0\tan\varphi\left(\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} - \tan\frac{\varphi}{2}\right) -$$
$$-\frac{\sin\omega_0}{2\beta^2}\left(\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} + \frac{\tan\varphi}{\cos\varphi} - 2\tan\varphi\right) + \frac{y_{10}}{\beta}\tan\varphi + x_{10},$$
$$x_2 = \frac{1}{\beta^2}\sin\omega_0\tan\varphi\left(\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} - \tan\frac{\varphi}{2}\right) +$$
$$+\frac{\cos\omega_0}{2\beta^2}\left(\ln\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} + \frac{\tan\varphi}{\cos\varphi} - 2\tan\varphi\right) + \frac{y_{20}}{\beta}\tan\varphi + x_{20},$$
where $\varphi = \arctan\beta(t-t_0)$.

It is interesting to express the dependence of coordinates on time in a more direct way. To this end let us use (36) along with the following formulas:
$$\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}} = \tan\varphi + \sqrt{1+\tan^2\varphi} = \beta(t-t_0) + \sqrt{1+\beta^2(t-t_0)^2},$$
$$\tan\varphi\tan\frac{\varphi}{2} = \sqrt{1+\tan^2\varphi} - 1 = \sqrt{1+\beta^2(t-t_0)^2} - 1,$$
$$\frac{1}{\cos\varphi} = \sqrt{1+\tan^2\varphi} = \sqrt{1+\beta^2(t-t_0)^2}.$$
Then, formula (38) gives
$$y = \frac{u_0}{\beta}\left\{\ln\left(\beta(t-t_0) + \sqrt{1+\beta^2(t-t_0)^2}\right) + i\left(\sqrt{1+\beta^2(t-t_0)^2} - 1\right)\right\} + y_0. \quad (43)$$

Consequently,

$$y_1 = \frac{\cos\omega_0}{\beta}\ln\left(\beta(t-t_0)+\sqrt{1+\beta^2(t-t_0)^2}\right) - \frac{\sin\omega_0}{\beta}\left(\sqrt{1+\beta^2(t-t_0)^2}-1\right)+y_{10}, \tag{44}$$

$$y_2 = \frac{\sin\omega_0}{\beta}\ln\left(\beta(t-t_0)+\sqrt{1+\beta^2(t-t_0)^2}\right) + \frac{\cos\omega_0}{\beta}\left(\sqrt{1+\beta^2(t-t_0)^2}-1\right)+y_{20}. \tag{45}$$

And from (42) we obtain

$$x = \frac{u_0}{\beta^2}\Big\{\Big[\beta(t-t_0)\ln\left(\beta(t-t_0)+\sqrt{1+\beta^2(t-t_0)^2}\right) - \sqrt{1+\beta^2(t-t_0)^2}+1\Big] + \frac{i}{2}\Big[\ln\left((\beta(t-t_0)+\sqrt{1+\beta^2(t-t_0)^2}\right) + \beta(t-t_0)\left(\sqrt{1+\beta^2(t-t_0)^2}-2\right)\Big]\Big\} + y_0(t-t_0)+x_0. \tag{46}$$

It follows, therefore, that

$$x_1 = \frac{\cos\omega_0}{\beta^2}\Big[\beta(t-t_0)\ln\left(\beta(t-t_0)+\sqrt{1+\beta^2(t-t_0)^2}\right) - \sqrt{1+\beta^2(t-t_0)^2}+1\Big] - \frac{\sin\omega_0}{2\beta^2}\Big[\ln\left((\beta(t-t_0)+\sqrt{1+\beta^2(t-t_0)^2}\right) + \beta(t-t_0)\left(\sqrt{1+\beta^2(t-t_0)^2}-2\right)\Big] + y_{10}(t-t_0)+x_{10}; \tag{47}$$

$$x_2 = \frac{\sin\omega_0}{\beta^2}\Big[\beta(t-t_0)\ln\left(\beta(t-t_0)+\sqrt{1+\beta^2(t-t_0)^2}\right) - \sqrt{1+\beta^2(t-t_0)^2}+1\Big] + \frac{\cos\omega_0}{2\beta^2}\Big[\ln\left((\beta(t-t_0)+\sqrt{1+\beta^2(t-t_0)^2}\right) + \beta(t-t_0)\left(\sqrt{1+\beta^2(t-t_0)^2}-2\right)\Big] + y_{20}(t-t_0)+x_{20}. \tag{48}$$

6. Analysis of the boundary arc

6.1. General equations

Now, let $\Delta = (t', t'')$ denote an interval such that the trajectory belongs to the boundary of the set of admissible positions on this interval:

$$|x(t)| = r, \quad t \in (t', t'').$$

We assume that the measure $d\mu$ is absolutely continuous on the interval Δ. Let $\dot{\mu}(\cdot)$ denote the density of the measure $d\mu$ on Δ. In this case, the system defining an extremal on Δ has the form:

$$\begin{aligned} &\dot{\lambda}_x^* = \dot{\mu}x, \quad -\dot{\lambda}_y = \lambda_x, \\ &\dot{\mu} \geq 0, \quad -\lambda_x y + |\lambda_y| = \alpha_0 \geq 0, \\ &|x| = r, \quad \dot{x} = y, \quad \dot{y} = u = -\tfrac{\lambda_y^*}{|\lambda_y|} \text{ if } \lambda_y \neq 0. \end{aligned} \tag{49}$$

Let $\lambda_y(t) \neq 0$ on Δ . Then the latter condition implies

$$|u| = 1.$$

Let us analyze the conditions

$$|x| = r, \quad |u| = 1.$$

Again, it is convenient to use the complex plane:

$$x(t) = re^{i\theta(t)}, \tag{50}$$

where $\theta(t)$ is the angle of rotation of vector $x(t)$ at time t. We have

$$y = \dot{x} = ire^{i\theta}\dot{\theta}, \tag{51}$$

$$u = \ddot{x} = -re^{i\theta}(\dot{\theta})^2 + ire^{i\theta}\ddot{\theta}. \tag{52}$$

These conditions, together with the condition $\langle u, u\rangle = 1$, imply

$$(\dot{\theta})^4 + (\ddot{\theta})^2 = \frac{1}{r^2}. \tag{53}$$

The latter equation has obvious solutions such that

$$\ddot{\theta} = 0, \quad (\dot{\theta})^2 = \frac{1}{r}. \tag{54}$$

It follows that

$$\dot{\theta} = \frac{1}{\sqrt{r}} \quad \text{or} \quad \dot{\theta} = -\frac{1}{\sqrt{r}}, \tag{55}$$

whence we obtain two families of solutions

$$\theta(t) = \frac{1}{\sqrt{r}}(t - t_0) \quad \text{and} \quad \theta(t) = -\frac{1}{\sqrt{r}}(t - t_0).$$

But of course, there are other solutions to equation (53), which correspond to the case where the angular velocity $\dot{\theta}$ is not constant.

6.2. Case of a constant angular velocity

Consider the case $\ddot{\theta} = 0$. In this case, conditions (50), (52), and (54) imply

$$\ddot{x} = u = -e^{i\theta} = -\frac{1}{r}x.$$

In other words, $x(t)$ is a solution to the Cauchy problem

$$\ddot{x} = -\frac{1}{r}x, \quad x(t_0) = x_0, \quad \dot{x}(t_0) = y_0,$$

where

$$|x_0| = r, \quad y_0 = \pm\frac{1}{\sqrt{r}}Ax_0,$$

and A is the rotation matrix by the angle $\pi/2$ counter-clockwise. Let us show that this motion is a "part" of Pontryagin's extremal. Set

$$\lambda_x = -y^*, \quad \lambda_y = x^*, \quad \dot{\mu} = \frac{1}{r}. \tag{56}$$

Then we obtain

$$\dot{\lambda}_y = \dot{x}^* = y^* = -\lambda_x, \quad \dot{\lambda}_x^* = -\dot{y} = -u = \frac{1}{r}x = \dot{\mu}x, \tag{57}$$

$$|\lambda_y| = |x| = r, \quad u = -\frac{1}{r}x = -\frac{1}{r}\lambda_y^* = -\frac{\lambda_y^*}{|\lambda_y|}, \tag{58}$$

$$\alpha_0 = -\lambda_x y + |\lambda_y| = |y|^2 + |\lambda_y| = r + r = 2r > 0, \tag{59}$$

since in view of (51) and (55) we have

$$|y| = r|\dot{\theta}| = r\frac{1}{\sqrt{r}} = \sqrt{r}$$

and hence $|y|^2 = r$. Thus, all conditions of the minimum principle are satisfied.

In the sequel, we consider only extremals moving along the boundary of the set of admissible positions with constant angular velocity. Moreover, it will be convenient to assume that

$$\lambda_x = -\frac{1}{r}y^*, \quad \lambda_y = \frac{1}{r}x^* = -u^*, \quad \dot{\mu} = \frac{1}{r^2}. \tag{60}$$

These multipliers are obtained from multipliers (56) by dividing them by r.

6.3. Junction analysis

Let a point t'' satisfy the following condition: to the left of this point there is an interval $\Delta_0 = [t', t'']$ $(t' < t'')$ such that $|x(t)| = r$ on this interval, and to the right of this point there is an open interval $\Delta_1 = (t'', t'' + \varepsilon)$ $(\varepsilon > 0)$ such that $|x(t)| > r$ on Δ_1. We call such a point the *exit point from the boundary of the set of admissible positions.*

So, at this point, two regimes meet: the regime of motion along the boundary of the set of admissible positions, considered in Section 6.2, and the regime of motion inside the set of admissible positions, considered in Section 5. Define the function

$$\rho(t) = \langle x(t), x(t) \rangle.$$

In the left half-neighborhood of the point t'', this function is a constant equal to r^2, and hence all left derivatives of this function at the point t'' are equal to zero. In the next section we will calculate several right derivatives of the function ρ at the point t''. Here we make some preparations for these calculations.

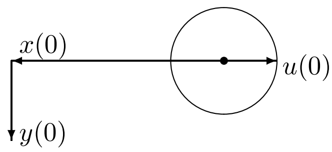

Figure 2. The exit point from the boundary of the set of admissible positions

Without loss of generality, we assume that

$$t'' = 0, \quad u(t'') = (1, 0), \quad y(t'') = -\sqrt{r}(0, 1), \quad x(t'') = -r(1, 0)$$

(see Fig. 2). These conditions correspond to the case

$$x = re^{i\theta(t)}, \quad \theta(t) = \frac{t}{\sqrt{r}} + \pi. \tag{61}$$

Hence

$$\dot{\theta} = \frac{1}{\sqrt{r}}, \quad x(t) = -re^{i\frac{t}{\sqrt{r}}}, \quad y(t) = -i\sqrt{r}e^{i\frac{t}{\sqrt{r}}}, \quad u(t) = e^{i\frac{t}{\sqrt{r}}}, \quad t \in (t', t''). \tag{62}$$

The rotation of the vector $u(t)$ is counter-clockwise. Recall that in a left half-neighbourhood of the point $t'' = 0$ the adjoint variables are defined by conditions (60). Set $e_1 = (1, 0)$, $e_2 = (0, 1)$. Then we have

$$-\lambda_y^*(t'') = u(t'') = (1, 0) = e_1,$$

$$\dot{\lambda}_y^*(t''-) = -\lambda_x^*(t''-) = \frac{1}{r}y(t'') = -\frac{1}{\sqrt{r}}(0,1) = -\frac{1}{\sqrt{r}}e_2.$$

We set

$$[\lambda_x] = \lambda_x(0+) - \lambda_x(0-), \quad [\mu] = \mu(0+) - \mu(0-).$$

Then the condition $d\lambda_x^* = xd\mu$ (see (19)) implies

$$[\lambda_x^*] = [\mu]x(0) = -[\mu]ru(0). \tag{63}$$

Recall that in view of (19)

$$[\lambda_y] = 0, \quad [u] = 0.$$

Let, for $t > t''$, the control be defined by the condition

$$u(t) = -\frac{\lambda_y^*}{|\lambda_y|}, \quad -\lambda_y^* = kt + b, \tag{64}$$

where k and b are linearly independent. In view of (60)

$$-\lambda_y^*(0) = u(0) = b = e_1.$$

Since, for $t < 0$,

$$\lambda_x^* = -\frac{1}{r}y = \frac{1}{\sqrt{r}}ie^{\frac{it}{\sqrt{r}}},$$

we obtain

$$\lambda_x^*(0-) = \frac{1}{\sqrt{r}}e_2.$$

Furthermore, in view of (63) we get

$$k = -\dot{\lambda}_y^*(0+) = \lambda_x^*(0+) = \lambda_x^*(0-) + [\lambda_x^*] = \frac{1}{\sqrt{r}}e_2 - [\mu]ru(0).$$

Consequently, for $t > 0$,

$$-\lambda_y^* = \left(\frac{1}{\sqrt{r}}e_2 - [\mu]re_1\right)t + e_1. \tag{65}$$

As in Section 5, set

$$u(t) = e^{i\omega(t)}, \quad t > 0. \tag{66}$$

From condition (64) it follows that

$$\tan\omega = \frac{(\lambda_y)_2}{(\lambda_y)_1},$$

where in view of (65)

$$-(\lambda_y)_2 = \frac{t}{\sqrt{r}}, \quad -(\lambda_y)_1 = 1 - [\mu]rt.$$

Consequently,

$$\tan\omega = \frac{t}{\sqrt{r}(1-[\mu]rt)}, \quad t>0, \tag{67}$$

whence

$$\omega = \arctan\frac{t}{\sqrt{r}(1-[\mu]rt)}, \quad t>0. \tag{68}$$

6.4. Right derivatives of the function $\rho = \langle x, x\rangle$ at the exit point of the set of admissible positions

Let us start the calculation of right derivatives of the function $\rho(t) = \langle x(t), x(t)\rangle$ at the point $t'' = 0$.

(i) Differentiating (formally) this function with respect to t and taking into account that $\dot{x} = y$, $\dot{y} = u$, we obtain:

$$\frac{d\rho}{dt} = 2\langle x, \dot{x}\rangle = 2\langle x, y\rangle, \tag{69}$$

$$\frac{d^2\rho}{dt^2} = 2\big(\langle \dot{x}, y\rangle + \langle x, \dot{y}\rangle\big) = 2\big(\langle y, y\rangle + \langle x, u\rangle\big), \tag{70}$$

$$\begin{aligned}\frac{d^3\rho}{dt^3} &= 2\big(2\langle y, \dot{y}\rangle + \langle \dot{x}, u\rangle + \langle x, \dot{u}\rangle\big)\\ &= 2\big(2\langle y, u\rangle + \langle y, u\rangle + \langle x, \dot{u}\rangle\big) = 6\langle y, u\rangle + 2\langle x, \dot{u}\rangle.\end{aligned} \tag{71}$$

Since at the point t'' we have $\langle x, y\rangle = 0$, $\langle y, y\rangle + \langle x, u\rangle = \langle y, y\rangle - r\langle u, u\rangle = r - r = 0$, and $\langle y, u\rangle = 0$, for the right derivatives at this point we get:

$$\frac{d\rho}{dt} = 0, \quad \frac{d^2\rho}{dt^2} = 0, \quad \frac{d^3\rho}{dt^3} = 2\langle x, \dot{u}\rangle, \tag{72}$$

$$\frac{d^4\rho}{dt^4} = 6\langle u, u\rangle + 8\langle y, \dot{u}\rangle + 2\langle x, \ddot{u}\rangle, \tag{73}$$

$$\frac{d^5\rho}{dt^5} = 20\langle u, \dot{u}\rangle + 10\langle y, \ddot{u}\rangle + 2\langle x, u^{(3)}\rangle, \tag{74}$$

$$\frac{d^6\rho}{dt^6} = 20\langle \dot{u}, \dot{u}\rangle + 30\langle u, \ddot{u}\rangle + 12\langle y, u^{(3)}\rangle + 2\langle x, u^{(4)}\rangle. \tag{75}$$

(ii) Recall that, according to (68), the following formulas hold for $t > t'' = 0$:

$$u = e^{i\omega}, \quad \omega = \arctan\frac{t}{\sqrt{r} - st}, \tag{76}$$

where

$$s = r^{\frac{3}{2}}[\mu] \geq 0 \tag{77}$$

(again we use the complex plane). Consequently,

$$\dot{u} = ie^{i\omega}\dot{\omega},$$

and then, for $t > 0$, we obtain

$$\langle x, \dot{u} \rangle = 0, \tag{78}$$

i.e.,

$$\frac{d^3\rho}{dt^3}(0+) = 0.$$

(iii) Furthermore,

$$\ddot{u} = e^{i\omega}\left(-\dot{\omega}^2 + i\ddot{\omega}\right), \quad u^{(3)} = e^{i\omega}\left(-i\dot{\omega}^3 - 3\dot{\omega}\ddot{\omega} + i\omega^{(3)}\right),$$

$$u^{(4)} = e^{i\omega}\left(\dot{\omega}^4 - 6i\dot{\omega}^2\ddot{\omega} - 4\dot{\omega}\omega^{(3)} - 3\ddot{\omega}^2 + i\omega^{(4)}\right).$$

Moreover,

$$\dot{\omega} = \frac{\sqrt{r}}{(\sqrt{r} - st)^2 + t^2}, \quad \ddot{\omega} = \frac{2\sqrt{r}\left(s\sqrt{r} - (s^2+1)t\right)}{\left((\sqrt{r} - st)^2 + t^2\right)^2}.$$

It follows, therefore, that

$$\dot{\omega}(0+) = \frac{1}{\sqrt{r}}, \quad \ddot{\omega}(0+) = \frac{2s}{r} = 2\sqrt{r}[\mu].$$

Let us calculate the fourth right derivative of the function ρ at the point $t'' = 0$. Since

$$\langle y, \dot{u} \rangle = -\sqrt{r}\langle e_2, e_2 \rangle \dot{\omega}(+0) = -1,$$
$$\langle x, \ddot{u} \rangle = r\langle -e_1, -e_1\dot{\omega}^2 + e_2\ddot{\omega} \rangle = r\dot{\omega}^2,$$

we obtain

$$\frac{d^4\rho}{dt^4} = 6\langle u, u \rangle + 8\langle y, \dot{u} \rangle + 2\langle x, \ddot{u} \rangle = 6 - 8 + 2 = 0. \tag{79}$$

Thus, first four right derivatives of the function ρ vanish at zero.

(iv) Let us proceed with calculation of the fifth right derivative of ρ at $t'' = 0$. According to (74), we have

$$\frac{d^5\rho}{dt^5} = 20\langle u, \dot{u} \rangle + 10\langle y, \ddot{u} \rangle + 2\langle x, u^{(3)} \rangle.$$

Since $u = e^{i\omega}$ and $\dot u = ie^{i\omega}\dot\omega$, we get $\langle u, \dot u\rangle = 0$. Since at $t'' = 0$ the following is true: $y = -\sqrt{r}e_2$, $\ddot u = -\dot\omega^2 e_1 + \ddot\omega e_2$, and $\ddot\omega(0+) = 2\sqrt{r}[\mu]$, we deduce from here that $\langle y, \ddot u\rangle = -2r[\mu]$. Let us calculate $\langle x, u^{(3)}\rangle$. For $t > t'' = 0$ we have

$$\langle x, u^{(3)}\rangle = (-r)\langle e_1, -\dot\omega^3 e_2 - 3\dot\omega\ddot\omega e_1 + \omega^{(3)} e_2\rangle = 3r\dot\omega\ddot\omega = 6r[\mu].$$

Consequently,

$$\frac{d^5\rho}{dt^5} = -20r[\mu] + 12r[\mu] = -8r[\mu] \le 0. \tag{80}$$

If $[\mu] > 0$, then $d^5\rho/dt^5 < 0$. The latter means that the extremal violates the state constraint. Hence the following condition is nesessary for quitting the set of admissible positions at $t'' = 0$:

$$[\mu] = 0,$$

i.e., the jump of μ at $t'' = 0$ should be equal to zero. In what follows we assume that this condition is fulfilled.

(v) Finally, let us calculate the sixth right derivative of ρ at $t'' = 0$. To this end, let us calculate the right derivative $\omega^{(3)}$ at $t'' = 0$. We have:

$$\omega^{(3)} = 2\sqrt{r}\frac{d}{dt}\left\{\frac{s\sqrt{r} - (s^2+1)t}{\left((\sqrt{r} - st)^2 + t^2\right)^2}\right\} =$$

$$= 2\sqrt{r}\left\{\frac{-(s^2+1)\left((\sqrt{r} - st)^2 + t^2\right) + 4\left(s\sqrt{r} - (s^2+1)t\right)^2}{\left((\sqrt{r} - st)^2 + t^2\right)^3}\right\}.$$

This implies that

$$\omega^{(3)}(+0) = 2\sqrt{r}\left\{\frac{-(s^2+1)r + 4s^2 r}{r^3}\right\} = 2\sqrt{r}\left\{\frac{-r + 3s^2 r}{r^3}\right\} = 2\frac{3s^2 - 1}{r\sqrt{r}},$$

i.e.,

$$\omega^{(3)}(0+) = 2\frac{3r^3[\mu]^2 - 1}{r\sqrt{r}}.$$

Since $[\mu] = 0$, we get

$$\omega^{(3)}(0+) = \frac{-2}{r\sqrt{r}}.$$

Let us find $\langle x, u^{(4)}(0+)\rangle$. We have

$$u^{(4)} = e_1(\dot\omega^4 - 4\dot\omega\omega^{(3)} - 3\ddot\omega^2) + e_2(\omega^{(4)} - 6\dot\omega^2\ddot\omega), \quad x = -re_1, \quad [\mu] = 0.$$

Consequently, for $t > t'' = 0$ we get

$$\langle x, u^{(4)}\rangle = (-r)\left(\dot\omega^4 - 4\dot\omega\omega^{(3)} - 3\ddot\omega^2\right) = -\frac{9}{r}.$$

This and the relations

$$\frac{d^6\rho}{dt^6} = 20\langle \dot{u}, \dot{u}\rangle + 30\langle u, \ddot{u}\rangle + 12\langle y, u^{(3)}\rangle + 2\langle x, u^{(4)}\rangle,$$

$$\dot{u} = e_2\dot{\omega}, \quad \dot{\omega} = \frac{1}{\sqrt{r}},$$

$$\langle u, \ddot{u}\rangle = \langle e_1,\ -e_1\dot{\omega}^2 + e_2\ddot{\omega}\rangle = -\dot{\omega}^2 = -\frac{1}{r},$$

$$\langle y, u^{(3)}\rangle = -\sqrt{r}\langle e_2,\ -e_2\dot{\omega}^3 - 3e_1\dot{\omega}\ddot{\omega} + e_2\omega^{(3)}\rangle$$
$$= -\sqrt{r}(-\dot{\omega}^3 + \omega^{(3)}) = \frac{3}{r}$$

imply

$$\frac{d^6\rho}{dt^6} = \frac{20}{r} - \frac{30}{r} + 12\frac{3}{r} + 2(-\frac{9}{r}) = \frac{8}{r} > 0. \tag{81}$$

Hence, the extremal leaves the boundary of the set of admissible positions.

7. Conclusion

We have shown the following. According to (72), (78), and (79), the first four right derivatives of $\rho(t) = \langle x(t), x(t)\rangle$ at t'' are equal to zero:

$$\frac{d\rho}{dt}(t''+) = \frac{d^2\rho}{dt^2}(t''+) = \frac{d^3\rho}{dt^3}(t''+) = \frac{d^4\rho}{dt^4}(t''+) = 0, \tag{82}$$

while, according to (80), the fifth right derivative is nonpositive:

$$\frac{d^5\rho}{dt^5}(t''+) = -8r[\mu] \leq 0. \tag{83}$$

Hence, the condition $[\mu] = 0$ (i.e. the absence of jump of μ at t'') is necessary for the existence of the extremal to the right of t''. If this condition is fulfilled, then the fifth right derivative is also equal to zero:

$$\frac{d^5\rho}{dt^5}(t''+) = 0. \tag{84}$$

Finally, according to (81), the sixth right derivative at the exit point is positive:

$$\frac{d^6\rho}{dt^6}(t''+) = \frac{8}{r} > 0. \tag{85}$$

The latter means that the extremal with a constant angular velocity of $x(t)$ on the boundary of the set of admissible positions and with zero jump of μ at the exit point really leaves the boundary of the set of admissible positions. Similarly, one can prove the possibility of "landing" of the extremal on the boundary of the set of admissible positions, i.e. the existence of *entry point*.

So, we have shown that there exists an extremal with the following properties: 1) it has one entry point and one exit point and hence it has one *boundary interval*; 2) it moves along the boundary of the set of admissible positions with constant angular velocity; 3) the Lagrange multiplier $d\mu$, which corresponds to the state constraint, is an absolutely continuous measure (hence, it has no atoms) with a constant density $\dot{\mu}$ on the boundary.

Of course, there is a possibility that an interval of motion along the boundary is absent, i.e. the extremal has no contacts with the boundary at all. Another possibility is that this interval is a singleton, i.e. an extremal has exactly one *touch point* on the boundary. It may happen that the measure has no atom at the touch point, and then this case is quite similar to the case of extremal having no boundary points. An interesting question is the following: are there extremals with isolated touch point having a jump of the multiplier at this point? This question needs careful investigation.

There is a more general question concerning a full description of all types of extremals in the problem. Obviously, there are extremals with a richer structure than the one discussed above (here we take into account the opinions of the three anonymous referees of the paper). In particular, there are extremals with a non constant angular velocity on the boundary of the set of admissible positions. Probably, there are extremals with two touch points on the boundary. Are there extremals with one touch point and one boundary subarc? Are there extremals with two boundary subarcs? All these questions require further investigation. Numerical experiments would be extremely useful in order to get some intuition about the behavior of extremals.

8. Appendix

Here we prove formulas (40) and (41). We have

$$\int \frac{d\varphi}{\cos^3\varphi} = \int \frac{\cos\varphi\, d\varphi}{\cos^4\varphi} = \int \frac{d\sin\varphi}{(1-\sin^2\varphi)^2}. \tag{86}$$

In order to calculate the latter integral, let us find the integral $\int \frac{dz}{(z^2-1)^2}$. Using the method of of undetermined coefficients, we obtain

$$\frac{1}{(z^2-1)^2} = \frac{1}{4}\left(-\frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right).$$

By integrating this function we get

$$\int \frac{dz}{(z^2-1)^2} = \frac{1}{4}\ln\left|\frac{z+1}{z-1}\right| + \frac{1}{2}\cdot\frac{z}{1-z^2} + C.$$

Consequently,

$$\int \frac{d\sin\varphi}{(1-\sin^2\varphi)^2} = \frac{1}{4}\ln\left|\frac{1+\sin\varphi}{1-\sin\varphi}\right| + \frac{1}{2}\cdot\frac{\sin\varphi}{\cos^2\varphi} + C.$$

This, combined with (86), implies

$$\int \frac{d\varphi}{\cos^3\varphi} = \frac{1}{4}\ln\left|\frac{1+\sin\varphi}{1-\sin\varphi}\right| + \frac{1}{2}\cdot\frac{\sin\varphi}{\cos^2\varphi} + C.$$

Since $1+\sin\varphi = 1+2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2} = (\sin\frac{\varphi}{2}+\cos\frac{\varphi}{2})^2$ and similarly $1-\sin\varphi = (\sin\frac{\varphi}{2}-\cos\frac{\varphi}{2})^2$, formula (40) follows.

Formula (41) can be obtained by integrating by parts. Assume that $1-\tan^2\frac{\varphi}{2} > 0$. Then

$$\int \ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\frac{d\varphi}{\cos^2\varphi} = \int \ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right) d\tan\varphi =$$

$$= \ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\tan\varphi - \int \tan\varphi\, d\left\{\ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\right\} =$$

$$= \ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\tan\varphi - \int \tan\varphi\left(\frac{1-\tan\frac{\varphi}{2}}{1+\tan\frac{\varphi}{2}}\right)\frac{d\varphi}{\cos^2\frac{\varphi}{2}\left(1-\tan\frac{\varphi}{2}\right)^2}$$

$$= \ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\tan\varphi - \int \frac{2\tan\frac{\varphi}{2}}{\left(1-\tan^2\frac{\varphi}{2}\right)^2}\cdot\frac{d\varphi}{\cos^2\frac{\varphi}{2}} + C$$

$$= \ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\tan\varphi - \int \frac{4\tan\frac{\varphi}{2}}{\left(1-\tan^2\frac{\varphi}{2}\right)^2}\, d\tan\frac{\varphi}{2}$$

$$= \ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\tan\varphi - \frac{2}{1-\tan^2\frac{\varphi}{2}} + C. \qquad \square$$

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