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IMPROVING THE STABILITY OF DISCRETIZATION ZEROS WITH THE TAYLOR METHOD USING A GENERALIZATION OF THE FRACTIONAL–ORDER HOLD

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Remarkable improvements in the stability properties of discrete system zeros may be achieved by using a new design of the fractional-order hold (FROH) circuit. This paper first analyzes asymptotic behaviors of the limiting zeros, as the sampling period T tends to zero, of the sampled-data models on the basis of the normal form representation for continuous-time systems with a new hold proposed. Further, we also give the approximate expression of limiting zeros of the resulting sampled-data system as power series with respect to a sampling period up to the third order term when the relative degree of the continuous-time system is equal to three, and the corresponding stability of the discretization zeros is discussed for fast sampling rates. Of particular interest are the stability conditions of sampling zeros in the case of a new FROH even though the relative degree of a continuous-time system is greater than two, whereas the conventional FROH fails to do so. An insightful interpretation of the obtained sampled-data model can be made in terms of minimal intersample ripple by design, where multirate sampled systems have a poor intersample behavior. Our results provide a more accurate approximation for asymptotic zeros, and certain known results on asymptotic behavior of limiting zeros are shown to be particular cases of the ideas presented here.

Keywords: stability, discretization zeros, Taylor method, signal reconstruction, sampled-data model.

1. Introduction

Zeros, along with poles, are fundamental characteristics of linear time-invariant systems, and stability of zeros is one of the most important issues in model matching and adaptive control problems. When a continuous-time system is discretized by the use of a sampler and a hold, the mapping between the discrete-time poles and their continuous-time counterparts is very simple, namely, the stability of poles is reserved. There is unfortunately no simple transformation between the discrete-time zeros and their continuous-time counterparts because the zeros of discrete-time systems depend on the sampling period T (Åström *et al.*, 1984; Zeng *et al.*, 2013). Thus, it is generally impossible for a continuous-time system with zeros in the left-half plane to be able to be transformed to a discrete-time system with zeros inside the unit circle. In other words, the stability of zeros is not necessarily preserved except in special cases. Therefore, the limiting case when the sampling period T tends to zero has attracted considerable attention (Åström *et al.*, 1984; Hagiwara *et al.*, 1993; Ishitobi, 1996b; Liang and Ishitobi, 2004a; Kaczorek, 1987; 2010; 2013; Tokarzewski, 2009; Ugalde *et al.*, 2012; Ostalczyk, 2012).

Perhaps the first attempt to study the zeros was given by Åström *et al.* (1984), who described the asymptotic behavior of the discrete-time zeros for a fast sampling rate when the original continuous-time plant is discretized with a zero-order hold (ZOH), and further the zeros in this case are called limiting zeros, which are composed of the intrinsic zeros and the sampling zeros (Hagiwara *et al.*, 1992). The former have a counterpart in the underlying continuous-time system, and go to unity (Hagiwara, 1996), while the latter which have no continuous-time counterparts and are generated in the sampling process, go toward roots of a certain polynomial (Hagiwara *et al.*, 1993; Weller *et al.*, 2001) determined by the relative degree of the continuous-time system.

In many discussions about the properties of discrete-time zeros, the ZOH has been mainly employed as a hold circuit since it is used most commonly in practice (Åström *et al.*, 1984; Hagiwara, 1996; Błachuta, 1999; Hayakawa *et al.*, 1983; Weller, 1999; Ishitobi, 2000; Liang *et al.*, 2007; Ruzbehani, 2010; Karampetakis and Karamichalis, 2014). Taking into account the fact that the type of hold circuit used critically influences the position of zeros, it is an interesting problem to investigate the zeros in the case of various holds. Hagiwara *et al.* (1993) carried out a comparative study and demonstrated that the first-order hold (FOH) provides no advantage over the ZOH as far as the stability of zeros of the resulting discrete-time systems is concerned. Further results on the behavior of the FOH have been reported (Błachuta, 1998; Zhang *et al.*, 2011). Passino and Antsaklis (1988) considered the fractional-order hold (FROH) as an alternative to the ZOH and showed that it can locate the zeros of a discrete-time system inside the unit circle by some examples while the ZOH fails to do so.

In a very motivating work by Ishiboti (1996), the asymptotic properties of limiting zeros with a FROH have been analyzed, and corresponding stability conditions have been also derived when the continuous-time systems have a relative degree up to five for sufficiently small sampling periods. Further, Bàrcena *et al.* (2000; 2001), Liang *et al.* (2003) as well as Liang and Ishitobi (2004b) respectively extended Ishitobi's results (Ishitobi, 1996) from different angles and with methods by investigating the limiting zeros in the case of a FROH.

In addition, the results of limiting FROH zeros (Ishitobi, 1996) were also extended by Błachuta (2001), who described the accuracy of the asymptotic results for both the intrinsic and the sampling zeros in terms of Bernoulli numbers and parameters of the continuous-time transfer function for sufficiently small sampling periods. However, the FROH does yield better discretization zeros, but only within a limited margin, mainly because it has just one tuning parameter, which does not allow to place the limiting zeros as one wishes. In particular, it can be seen that the sampling zeros with a ZOH or a FROH always lie strictly outside the unit circle when the relative degree of a continuous-time system is greater than or equal to three (Åström *et al.*, 1984; Ishitobi, 1996; 2000; Liang *et al.*, 2003). In many engineering applications, fast sampling rates and the continuous-time relative degree more than two commonly occur.

These facts sparked interest in other holds such as multirate sampling control and digital control with the generalized sampled-data hold function (GSHF) (Kabamba, 1987; Chan, 1998; 2002; Liang and Ishitobi, 2004a; Yuz *et al.*, 2004; Liang *et al.*, 2010; Ugalde *et al.*, 2012). Though some deficiencies such as poor intersample behavior in the case of a GSHF cannot be avoided, the GSHF can be used to solve many more ambitious control problems for linear systems as long as it is formulated exclusively in intersample terms. Moreover, it is well known that the GSHF can be also used to shift the zeros of sampled-data models for linear continuous-time systems because intersample ripples can be suppressed by using a linear-quadratic optimization (Chan, 1998) or can be alleviated efficiently by minimizing the variation the control input (Liang and Ishitobi, 2004a).

However, in contrast with a ZOH or a FROH, rather poor intersample behavior is often unavoidable. Although this can be alleviated as mentioned above, the fact is that for a sampled-data model with a discrete integrator to be able to reject step disturbances in continuous interval, and the impulse response of the hold in question it must have continuous-time zeros where a ZOH and a FROH have theirs, while a GSHF does not (Feuer and Goodwin, 1996; Middleton and Freudenberg, 1995). Hence, we present a new design of the FROH which is composed of the polynomic function instead of simple design parametrization. Our new hold characterization merges two interesting features: conventional FROH behavior under constant input together with as many tuning parameters as desired. On the one hand, the former provides a very simple way to minimize the intersample issue; on the other, the latter allows the discretization zeros to be placed wherever desired.

The aim of this paper is first to analyze the asymptotic behaviors of the limiting zeros of discrete-time models on the basis of the normal form representation of continuous-time systems, and also derive their approximate expression in the case of a new FROH as power series with respect to a sampling period up to the third order term when the relative degree of the continuous-time system is equal to three. Besides the obvious differences in terms of the technique in researching our FROH and other hold circuits, we can deeply feel that this study is important owing to the complexity and importance of discretization zeros, especially for the sampled-data model and stability of sampling zeros.

More importantly, we also show how our new hold, irrespectively of whether the continuous-time relative degree is greater than two or not, can be designed to remove only the effects of the sampling process by placing the sampling zeros of the discrete-time system asymptotically to the stable regions at will. One of the principal contributions in this paper, in particular, would consequently propose an analytical method to obtain the limiting zeros as stable as possible for a wider class of continuous-time plants. Moreover, an insightful interpretation of the resulting sampled-data model can be made in terms of minimal intersample ripple by design, where the multirate sampled systems have usually a poor intersample behavior. Finally, we further obtain the stability condition of the sampling zeros for sufficiently small sampling periods.

2. Sampled-data model with a new FROH

Consider an n-th order continuous-time system with relative degree $r = n - m$ described by the transfer function

$$G(s) = K\frac{N(s)}{D(s)}, \quad K \neq 0, \tag{1}$$

where

$$N(s) = s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \cdots + b_0, \tag{2}$$
$$D(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_0. \tag{3}$$

The normal form of (1) with the relative degree $r = n - m$ is represented with an input u and an output y (Isidori, 1995; Khalil, 2002) as

$$\begin{cases} \dot{\boldsymbol{\xi}} = \begin{bmatrix} \mathbf{0}_{r-1} & I_{r-1} \\ 0 & \mathbf{0}_{r-1}^T \end{bmatrix} \boldsymbol{\xi} + \begin{bmatrix} \mathbf{0}_{r-1} \\ 1 \end{bmatrix} \\ \qquad \times (Ku - \omega - d_0\xi_1 - d_1\xi_2 - \cdots - d_{r-1}\xi_r), \\ \dot{\boldsymbol{\eta}} = P\boldsymbol{\eta} + \mathbf{q}\xi_1, \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{\xi}, \end{cases} \tag{4}$$

where

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_r \end{bmatrix}^T,$$
$$\boldsymbol{\eta} = \begin{bmatrix} \eta_1 & \cdots & \eta_{n-r} \end{bmatrix}^T,$$
$$\omega = \mathbf{c}^T\boldsymbol{\eta}, \quad \mathbf{c} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-r-1} \end{bmatrix}^T,$$
$$P = \begin{bmatrix} 0 & 1 & & O \\ & & \ddots & \\ & O & & 1 \\ -b_0 & \cdots & -b_{n-r-2} & -b_{n-r-1} \end{bmatrix},$$
$$\mathbf{q} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T,$$

and the scalars d_i $(i = 0, \ldots, r-1)$ and c_i $(i = 0, \cdots, n-r-1)$ are obtained from

$$D(s) = Q(s)N(s) + R(s), \tag{5}$$
$$Q(s) = s^r + d_{r-1}s^{r-1} + \cdots + d_1 s + d_0, \tag{6}$$
$$R(s) = c_{n-r-1}s^{n-r-1} + \cdots + c_0, \tag{7}$$

where

$$\begin{aligned} d_{r-1} &= a_{n-1} - b_{n-r-1}, \\ d_{r-2} &= a_{n-2} - b_{n-r-2} - b_{n-r-1}d_{r-1}, \\ d_{r-3} &= a_{n-3} - b_{n-r-3} - b_{n-r-2}d_{r-1}, \\ &\vdots \\ d_0 &= a_{n-r} - b_{n-2r} - b_{n-2r+1}d_{r-1}, \\ c_i &= a_i - b_{i-r} - b_{i-r+1}d_{r-1} - \cdots - b_{i-1}d_1 - b_i d_0 \\ & \qquad i = 0, \ldots, n-r-1. \end{aligned}$$

We are interested in the sampled-data model for the linear system (4) when the input is a piecewise continuous signal generated by a new FROH reconstruction, i.e.,

$$\begin{aligned} u(t) &= u(kT) + \sum_{\ell=0}^{N-1} \beta_\ell (t - kT)^\ell \\ &\quad \times \left[\frac{u(kT) - u((k-1)T)}{T}\right], \\ &\beta_\ell \in \mathbb{R}, \quad N > 1, \\ &kT \leq t < (k+1)T, \quad k = 0, 1, \ldots, \end{aligned} \tag{8}$$

where β_ℓ is a real coefficient and T is a sampling period. In particular, our new hold (8) with a polynomial can be regarded as a generalization of the conventional FROH. In contrast to a simple linear pattern, the polynomial approach not only turns out to provide minimal intersample ripple issues, but also places limiting zeros of the discretized model at will with as many tuning parameters as desired.

Given the complexity of calculation, we assume the condition $N = 2$ while guaranteeing the desired control performance for our new hold (8). Moreover, a new FROH is used and the relations

$$\begin{aligned} \dot{u}(t) &= \beta_1 \left[\frac{u(kT) - u((k-1)T)}{T}\right], \\ \ddot{u}(t) &= \cdots = 0 \end{aligned} \tag{9}$$

are noticed. Furthermore, the normal form (4) leads to the derivatives of the output

$$y = \xi_1, \quad \dot{y} = \xi_2, \quad \ldots, \quad y^{(r-1)} = \xi_r \tag{10}$$

$$y^{(r)} = Ku - c^T\boldsymbol{\eta} - d_0\xi_1 - \cdots - d_{r-1}\xi_r, \tag{11}$$

$$\begin{aligned} y^{(r+1)} &= (d_0 d_{r-1} - \mathbf{c}^T\mathbf{q})\xi_1 + (d_1 d_{r-1} - d_0)\xi_2 \\ &\quad + \cdots + (d_{r-2}d_{r-1} - d_{r-3})\xi_{r-1} \\ &\quad + (d_{r-1}^2 - d_{r-2})\xi_r + (d_{r-1}\mathbf{c}^T - \mathbf{c}^T P)\boldsymbol{\eta} \\ &\quad - d_{r-1}Ku + K\dot{u}, \end{aligned} \tag{12}$$

$$\begin{aligned} y^{(r+2)} &= (d_0 d_{r-2} - d_0 d_{r-1}^2 + d_{r-1}\mathbf{c}^T\mathbf{q} - \mathbf{c}^T P\mathbf{q})\xi_1 \\ &\quad + (d_1 d_{r-2} - d_1 d_{r-1}^2 + d_0 d_{r-1} - \mathbf{c}^T\mathbf{q})\xi_2 \\ &\quad + \cdots + (2d_{r-1}d_{r-2} - d_{r-1}^3 - d_{r-3})\xi_r \\ &\quad + (d_{r-1}^2 - d_{r-2})Ku - d_{r-1}K\dot{u} \\ &\quad + \Big\{(d_{r-2} - d_{r-1}^2)\mathbf{c}^T \\ &\quad + d_{r-1}\mathbf{c}^T P - \mathbf{c}^T P^2\Big\}\boldsymbol{\eta}. \end{aligned} \tag{13}$$

Further, the derivatives of $\boldsymbol{\eta}$ are also represented as

$$\dot{\boldsymbol{\eta}} = P\boldsymbol{\eta} + \mathbf{q}\xi_1, \tag{14}$$

$$\begin{aligned} \ddot{\boldsymbol{\eta}} &= P(P\boldsymbol{\eta} + \mathbf{q}\xi_1) + \mathbf{q}\dot{\xi}_1 \\ &= P^{(2)}\boldsymbol{\eta} + P\mathbf{q}\xi_1 + \mathbf{q}\dot{\xi}_1, \end{aligned} \tag{15}$$

$$\begin{aligned} \boldsymbol{\eta}^{(3)} &= P^{(2)}(P\boldsymbol{\eta} + \mathbf{q}\xi_1) + P\mathbf{q}\dot{\xi}_1 + \mathbf{q}\ddot{\xi}_1 \\ &= P^{(3)}\boldsymbol{\eta} + P^{(2)}\mathbf{q}\xi_1 + P\mathbf{q}\dot{\xi}_1 + \mathbf{q}\ddot{\xi}_1, \end{aligned} \tag{16}$$

$$\vdots$$

$$\begin{aligned} \boldsymbol{\eta}^{(r)} &= P^{(r-1)}(P\boldsymbol{\eta} + \mathbf{q}\xi_1) + P^{(r-2)}\mathbf{q}\dot{\xi}_1 + \mathbf{q}\xi_1^{(r-1)} \\ &= P^{(r)}\boldsymbol{\eta} + P^{(r-1)}\mathbf{q}\xi_1 + P^{(r-2)}\mathbf{q}\dot{\xi}_1 + \mathbf{q}\xi_1^{(r-1)}, \end{aligned} \tag{17}$$

$$\begin{aligned} \boldsymbol{\eta}^{(r+1)} &= P^{(r)}(P\boldsymbol{\eta} + \mathbf{q}\xi_1) + P^{(r-1)}\mathbf{q}\dot{\xi}_1 + \cdots + \mathbf{q}\xi_1^{(r)} \\ &= \Big[P^{(r)}\mathbf{q} - \mathbf{q}d_0\Big]\xi_1 + \Big[P^{(r-1)}\mathbf{q} - \mathbf{q}d_1\Big]\xi_2 + \cdots \\ &\quad + (P\mathbf{q} - \mathbf{q}d_{r-1})\xi_r + \Big[P^{(r+1)} - \mathbf{q}\mathbf{c}^T\Big]\boldsymbol{\eta} + \mathbf{q}ku, \end{aligned} \tag{18}$$

$$\begin{aligned} \boldsymbol{\eta}^{(r+2)} &= P^{(r+1)}(P\boldsymbol{\eta} + \mathbf{q}\xi_1) + P^{(r)}\mathbf{q}\dot{\xi}_1 + P^{(r-1)}\mathbf{q}\dot{\xi}_1 \\ &\quad + \cdots + \mathbf{q}\xi_1^{(r+1)} \\ &= \Big[\mathbf{q}d_0 d_{r-1} - P\mathbf{q}d_0 + P^{(r+1)}\mathbf{q} - \mathbf{q}\mathbf{c}^T\mathbf{q}\Big]\xi_1 \\ &\quad + \Big[P^{(r)}\mathbf{q} - d_0\mathbf{q} - P\mathbf{q}d_1 + \mathbf{q}d_1 d_{r-1}\Big]\xi_2 + \cdots \\ &\quad + \Big[\mathbf{q}d_{r-1}^2 - P\mathbf{q}d_{r-1} + P^2\mathbf{q} - \mathbf{q}d_{r-2}\Big]\xi_r \\ &\quad + (P\mathbf{q} - \mathbf{q}d_{r-1})ku + qk\dot{u} \\ &\quad + \Big[\mathbf{q}\mathbf{c}^T d_{r-1} - P\mathbf{q}\mathbf{c}^T + P^{(r+2)} - \mathbf{q}\mathbf{c}^T P\Big]\boldsymbol{\eta}. \end{aligned} \tag{19}$$

Hence, substituting (10)–(19) into the right-hand side of

$$y_{k+1} = \sum_{i=0}^{\infty} \frac{T^i}{i!} y_k^{(i)}, \tag{20}$$

$$\dot{y}_{k+1} = \sum_{i=0}^{\infty} \frac{T^i}{i!} y_k^{(i+1)}, \tag{21}$$

$$\vdots$$

$$y_{k+1}^{(r-1)} = \sum_{i=0}^{\infty} \frac{T^i}{i!} y_k^{(i+r-1)}, \tag{22}$$

$$\boldsymbol{\eta}_{k+1} = \sum_{i=0}^{\infty} \frac{T^i}{i!} \boldsymbol{\eta}_k^{(i)}, \tag{23}$$

and defining the state variables

$$x_k = \begin{bmatrix} y_k & \dot{y}_k & \cdots & y_k^{(r-1)} & \boldsymbol{\eta}_k^T \end{bmatrix}^T, \tag{24}$$

the discrete-time state equations in the case of a new FROH are definitely obtained.

Now, the zeros of the discrete-time system (20)–(23) are analyzed using the explicit expressions of $y_k, \dot{y}_k, \cdots, y_k^{(r+2)}$ and $\boldsymbol{\eta}_k, \dot{\boldsymbol{\eta}}_k, \cdots, \boldsymbol{\eta}_k^{(r+2)}$ as follows:

$$\begin{aligned} & y_{k+1} \\ &= \sum_{i=0}^{r+2} \frac{T^i}{i!} y_k^{(i)} + O\left(T^{r+3}\right) \\ &= \Bigg\{1 - \frac{d_0}{r!}T^r + \frac{d_0 d_{r-1} - \mathbf{c}^T\mathbf{q}}{(r+1)!}T^{r+1} \\ &\quad + \frac{d_{r-1}\mathbf{c}^T\mathbf{q} + d_0 d_{r-2} - d_0 d_{r-1}^2 - \mathbf{c}^T P\mathbf{q}}{(r+2)!}T^{r+2}\Bigg\} y_k \\ &\quad + \Bigg\{T - \frac{d_1}{r!}T^r + \frac{d_1 d_{r-1} - d_0}{(r+1)!}T^{r+1} \\ &\quad \frac{d_1 d_{r-2} - d_1 d_{r-1}^2 + d_0 d_{r-1} - \mathbf{c}^T\mathbf{q}}{(r+2)!}T^{r+2}\Bigg\}\dot{y}_k \\ &\quad + \cdots + \Bigg\{\frac{T^{r-1}}{(r-1)!} + \frac{d_{r-1}}{r!}T^r \\ &\quad + \frac{d_{r-1}^2 - d_{r-2}}{(r+1)!}T^{r+1} \\ &\quad + \frac{2d_{r-1}d_{r-2} - d_{r-1}^3 - d_{r-3}}{(r+2)!}T^{r+2}\Bigg\} y_k^{(r-1)} \\ &\quad + \Bigg\{\left[\frac{1 + \beta_0 + \beta_1}{r!} + \frac{\beta_1}{(r+1)!}\right]T^r \\ &\quad - \left[\frac{d_{r-1}\beta_1}{(r+2)!} + \frac{(1 + \beta_0 + \beta_1)d_{r-1}}{(r+1)!}\right]T^{r+1} \\ &\quad + \Bigg[\frac{\beta_1(d_{r-1}^2 - d_{r-2})}{(r+3)!} \\ &\quad + \frac{(1 + \beta_0 + \beta_1)(d_{r-1}^2 - d_{r-2})}{(r+2)!}\Bigg]T^{r+2}\Bigg\} Ku_k \end{aligned}$$

$$
\begin{aligned}
&-\left\{\left[\frac{\beta_0+\beta_1}{r!}+\frac{\beta_1}{(r+1)!}\right]T^r\right.\\
&-\left.\left[\frac{(\beta_0+\beta_1)d_{r-1}}{(r+1)!}+\frac{\beta_1 d_{r-1}}{(r+2)!}\right]T^{r+1}\right\}Ku_{k-1}\\
&+\left\{-\frac{\mathbf{c}^T}{r!}T^r+\frac{d_{r-1}\mathbf{c}^T-\mathbf{c}^TP}{(r+1)!}T^{r+1}\right.\\
&\left.\frac{d_{r-1}\mathbf{c}^TP-\mathbf{c}^TP^2+(d_{r-2}-d_{r-1}^2)\mathbf{c}^T}{(r+2)!}T^{r+2}\right\}\boldsymbol{\eta}_k\\
&+O\left(T^{r+3}\right),
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
&\dot{y}_{k+1}\\
&=\sum_{i=0}^{r+1}\frac{T^i}{i!}y_k^{(i+1)}+O\left(T^{r+2}\right)\\
&=\left\{-\frac{d_0}{(r-1)!}T^{r-1}+\frac{d_0d_{r-1}-\mathbf{c}^T\mathbf{q}}{r!}T^r\right.\\
&\left.+\frac{d_0d_{r-2}+d_{r-1}\mathbf{c}^T\mathbf{q}-d_0d_{r-1}^2-\mathbf{c}^TP\mathbf{q}}{(r+1)!}T^{r+1}\right\}y_k\\
&+\left\{1-\frac{d_1}{(r-1)!}T^{r-1}+\frac{d_1d_{r-1}-d_0}{r!}T^r\right.\\
&\left.+\frac{d_1d_{r-2}-d_1d_{r-1}^2+d_0d_{r-1}-\mathbf{c}^T\mathbf{q}}{(r+1)!}T^{r+1}\right\}\dot{y}_k\\
&+\cdots+\left\{\frac{T^{r-2}}{(r-2)!}\right.\\
&+\frac{d_{r-1}}{(r-1)!}T^{r-1}+\frac{d_{r-1}^2-d_{r-2}}{r!}T^r\\
&\left.+\frac{2d_{r-1}d_{r-2}-d_{r-1}^3-d_{r-3}}{(r+1)!}T^{r+1}\right\}y_k^{(r-1)}\\
&+\left\{\left[\frac{1+\beta_0+\beta_1}{r-1!}+\frac{\beta_1}{r!}\right]T^{r-1}-\left[\frac{d_{r-1}\beta_1}{(r+1)!}\right.\right.\\
&\left.+\frac{(1+\beta_0+\beta_1)d_{r-1}}{r!}\right]T^r+\left[\frac{\beta_1(d_{r-1}^2-d_{r-2})}{(r+2)!}\right.\\
&\left.\left.+\frac{(1+\beta_0+\beta_1)(d_{r-1}^2-d_{r-2})}{(r+1)!}\right]T^{r+1}\right\}Ku_k\\
&-\left\{\left[\frac{\beta_0+\beta_1}{(r-1)!}+\frac{\beta_1}{r!}\right]T^{r-1}-\left[\frac{(\beta_0+\beta_1)d_{r-1}}{r!}\right.\right.\\
&\left.\left.+\frac{\beta_1d_{r-1}}{(r+1)!}\right]T^r\right\}Ku_{k-1}\\
&+\left\{-\frac{\mathbf{c}^T}{(r-1)!}T^{r-1}+\frac{d_{r-1}\mathbf{c}^T-\mathbf{c}^TP}{r!}T^r\right.\\
&+\frac{d_{r-1}\mathbf{c}^TP+(d_{r-2}-d_{r-1}^2)\mathbf{c}^T-\mathbf{c}^TP^2}{(r+1)!}\\
&\left.\times T^{r+1}\right\}\boldsymbol{\eta}_k+O\left(T^{r+2}\right),
\end{aligned} \tag{26}
$$

$\vdots$

$$
y_{k+1}^{(r-1)}=\sum_{i=0}^{3}\frac{T^i}{i!}y_k^{(i+r-1)}+O\left(T^4\right)
$$

$$
\begin{aligned}
&=\left\{-d_0T+\frac{d_0d_{r-1}-\mathbf{c}^T\mathbf{q}}{2!}T^2\right.\\
&\left.+\frac{d_{r-1}\mathbf{c}^T\mathbf{q}+d_0d_{r-2}-d_0d_{r-1}^2-\mathbf{c}^TP\mathbf{q}}{3!}T^3\right\}y_k\\
&+\left\{-d_1T+\frac{d_1d_{r-1}-d_0}{2!}T^2\right.\\
&\left.+\frac{d_1d_{r-2}-d_1d_{r-1}^2+d_0d_{r-1}-\mathbf{c}^T\mathbf{q}}{3!}T^3\right\}\dot{y}_k\\
&+\cdots+\left\{d_{r-1}T+\frac{d_{r-1}^2-d_{r-2}}{2!}T^2\right.\\
&\left.+\frac{2d_{r-1}d_{r-2}-d_{r-1}^3-d_{r-3}}{3!}T^3\right\}y_k^{(r-1)}\\
&+\left\{\left[1+\beta_0+\frac{3\beta_1}{2}\right]T-\left[\frac{d_{r-1}\beta_1}{3!}\right.\right.\\
&\left.+\frac{1+\beta_0+\beta_1)d_{r-1}}{2!}\right]T^2+\left[\frac{\beta_1(d_{r-1}^2-d_{r-2})}{4!}\right.\\
&\left.\left.+\frac{(1+\beta_0+\beta_1)(d_{r-1}^2-d_{r-2})}{3!}\right]T^3\right\}Ku_k\\
&-\left\{\left[\beta_0+\frac{3\beta_1}{2}\right]T-\left[\frac{(\beta_0+\beta_1)d_{r-1}}{2!}\right.\right.\\
&\left.\left.+\frac{\beta_1d_{r-1}}{3!}\right]T^2\right\}Ku_{k-1}+\left\{\mathbf{c}^TT\right.\\
&+\frac{d_{r-1}\mathbf{c}^T-\mathbf{c}^TP}{2!}T^2\\
&\left.+\frac{d_{r-1}\mathbf{c}^TP-\mathbf{c}^TP^2+(d_{r-2}-d_{r-1}^2)\mathbf{c}^T}{3!}T^3\right\}\boldsymbol{\eta}_k\\
&+O\left(T^4\right),
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&\boldsymbol{\eta}_{k+1}\\
&=\sum_{i=0}^{r+2}\frac{T^i}{i!}\boldsymbol{\eta}_k^{(i)}+O\left(T^{r+3}\right)\\
&=\left(\mathbf{q}T+\frac{P\mathbf{q}}{2!}T^2+\frac{P^2\mathbf{q}}{3!}T^3+\cdots+\frac{P^{r-1}\mathbf{q}}{r!}T^r\right.\\
&+\frac{P^r\mathbf{q}-\mathbf{q}d_0}{(r+1)!}T^{r+1}\\
&\left.+\frac{P^{r+1}\mathbf{q}-\mathbf{q}\mathbf{c}^T\mathbf{q}+\mathbf{q}d_0d_{r-1}-P\mathbf{q}d_0}{(r+2)!}T^{r+2}\right)y_k\\
&+\left(\frac{\mathbf{q}}{2!}T^2+\frac{P\mathbf{q}}{3!}T^3+\cdots+\frac{P^{r-2}\mathbf{q}}{r!}T^r\right.\\
&+\frac{P^{r-1}\mathbf{q}-\mathbf{q}d_1}{(r+1)!}T^{r+1}\\
&\left.+\frac{P^r\mathbf{q}-P\mathbf{q}d_1+\mathbf{q}d_1d_{r-1}-\mathbf{q}d_0}{(r+2)!}T^{r+2}\right)\dot{y}_k\\
&+\cdots+\left(\frac{\mathbf{q}}{r!}T^r+\frac{P\mathbf{q}-\mathbf{q}d_{r-1}}{(r+1)!}T^{r+1}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{P^2\mathbf{q}-P\mathbf{q}d_{r-1}+\mathbf{q}d_{r-1}^2-\mathbf{q}d_{r-2}}{(r+2)!}T^{r+2}\Big)y_k^{(r-1)}\\
&+\Big\{\big(\frac{(1+\beta_0+\beta_1)\mathbf{q}}{(r+1)!}+\frac{\mathbf{q}\beta_1}{(r+2)!}\big)T^{r+1}\\
&\frac{1+\beta_0+\beta_1)(P\mathbf{q}-\mathbf{q}d_{r-1})}{(r+2)!}T^{r+2}\Big\}Ku_k\\
&-\Big\{\Big[\frac{\mathbf{q}\beta_1}{(r+2)!}+\frac{(1+\beta_0+\beta_1)\mathbf{q}}{(r+1)!}\Big]T^{r+1}\Big\}Ku_{k-1}\\
&+\Big(I+PT+\frac{P^2}{2!}T^2+\frac{P^3}{3!}T^3+\cdots+\frac{P^r}{r!}T^r\\
&+\frac{P^{r+1}-\mathbf{q}\mathbf{c}^T}{(r+1)!}T^{r+1}\\
&+\frac{P^{r+2}-P\mathbf{q}\mathbf{c}^Td_{r-1}\mathbf{q}\mathbf{c}^T-\mathbf{q}\mathbf{c}^TP}{(r+2)!}T^{r+2}\Big)\\
&\boldsymbol{\eta}_k+O\left(T^5\right). \qquad (28)
\end{aligned}
$$

Similarly, the reason why the explicit expressions of $y_k, \dot{y}_k, \cdots, y_k^{(r+2)}$ and $\boldsymbol{\eta}_k, \dot{\boldsymbol{\eta}}_k, \cdots, \boldsymbol{\eta}_k^{(r+2)}$ are used is to obtain the approximate expansion of the limiting zeros of the discrete-time system with the order T^3 when the relative degree of continuous-time systems is $r = n - m$.

3. Main results

In the following, a more accurate approximate model of the sampled-data system is considered by neglecting the higher order terms, and the approximate expression of the limiting zeros is further calculated in this section. When a continuous-time system is discretized, unstable zeros may appear in the discrete-time model due to the existence of unstable sampling zeros even though the continuous-time system is of minimum phase (Åström *et al.*, 1984; Ishitobi *et al.*, 2013).

For example, it is noticed that unstable discretization zeros may be generated by a ZOH or a FROH when we sample continuous-time systems having relative degree greater than or equal to three (Åström *et al.*, 1984; Hayakawa *et al.*, 1983; Ishitobi, 1996; Liang *et al.*, 2003). In this work we propose to use a new kind of FROH to place sampling zeros at will. To avoid the complexity of its calculation, we mainly consider the case when the relative degree of a continuous-time system is three without loss of generality.

When a continuous-time system with relative degree three is sampled at a fast rate, the corresponding discrete-time model arising from a ZOH or a FROH may have unstable zeros. On the other hand, though such multirate sampling control and digital control schemes have the clear advantages over the conventional control systems, several authors have pointed out that the unexpected drawbacks occur, such as intersample ripples. In particular, our new hold can alleviate intersample issues, and well exhibit minimal intersample ripple by design. An approximate expression of limiting zeros of a discrete-time model for a continuous-time system with relative degree three is derived from (25)–(28). The first result is given by the following theorem.

Theorem 1. *The zeros of a discrete-time system corresponding to a continuous-time transfer function (4) with a new FROH are given for $T \ll 1$ approximately by the roots of*

$$Q_1Q_2Q_3Q_4 = 0, \qquad (29)$$

where

$$
\begin{aligned}
Q_1 = \Big[&-z-1+\frac{12\beta_0+16\beta_1}{4\beta_0+5\beta_1}+\Big(1+\beta_0+\frac{\beta_1}{2}\Big)T\\
&-\frac{3d_1-3-3\beta_0-4\beta_1}{6}T^2\\
&+\frac{d_1d_2-d_0+5d_2\beta_1+4d_2\beta_0+4d_2}{24}T^3\Big],\\
Q_2 = \Big[&-z-1+\frac{12\beta_0+18\beta_1}{4\beta_0+5\beta_1}-\Big(d_2-1-\beta_0-\frac{3\beta_1}{2}\Big)T\\
&+\frac{3d_2^2-3d_1+4d_2\beta_1+3d_2\beta_0+3d_2}{6}T^2\\
&+\frac{-d_2^3+2d_1d_2-d_0+(4+4\beta_0+5\beta_1)(d_2^2-d_1)}{24}T^3\Big],\\
Q_3 = \Big[&-z-1+\frac{4+4\beta_0+5\beta_1}{4\beta_0+5\beta_1}+(1-d_1)T\\
&+\frac{d_1d_2-d_0+d_2}{2}T^2\\
&+\frac{d_1^2-d_1d_2^2+d_0d_2-\mathbf{c}^T\mathbf{q}+d_2^2-d_1}{6}T^3\Big],\\
Q_4 = \Big|&(1-z)I+PT+\frac{P^2}{2}T^2+\frac{P^3}{6}T^3\Big|.
\end{aligned}
$$

Proof. The limiting zeros of a discrete-time system (20)–(23) are equivalent to zeros in (25)–(28), which are given by substituting $y_k = y_{k+1} = 0$ into (25)–(28) as follows:

$$M\begin{bmatrix} Y_{d1}\\ Y_{d2}\\ KU_k\\ KU_{k-1}\\ H\end{bmatrix} = \mathbf{0}_n, \qquad (30)$$

where $Y_{d1}, Y_{d2}, U_k, U_{k-1}$ and H are the $\mathcal{Z}$-transforms of $\dot{y}_k$, $\ddot{y}_k$, u_k, u_{k-1} and $\boldsymbol{\eta}_k$, respectively, and the matrix M is defined by

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & \mathbf{m}_{15}^T\\ m_{21} & m_{22} & m_{23} & m_{24} & \mathbf{m}_{25}^T\\ m_{31} & m_{32} & m_{33} & m_{34} & \mathbf{m}_{35}^T\\ 0 & 0 & -z & 1 & \mathbf{0}^T\\ \mathbf{m}_{51} & \mathbf{m}_{52} & \mathbf{m}_{53} & \mathbf{m}_{54} & M_{55}\end{bmatrix}, \qquad (31)$$

with

$$
\begin{aligned}
m_{11} &= T\overline{m}_{11} + O(T^6),\\
m_{12} &= T\overline{m}_{12} + O(T^6),\\
m_{13} &= T^2\overline{m}_{13} + O(T^6),\\
m_{14} &= T^2\overline{m}_{14} + O(T^5),\\
\mathbf{m_{15}^T} &= T\overline{\mathbf{m}}_{15}^T + O(T^6),\\
\overline{m}_{11} &= 1 - \frac{d_1}{6}T^2 + \frac{d_1d_2 - d_0}{24}T^3\\
&\quad + \frac{d_1^2 - d_1d_2^2 + d_0d_2 - c_{n-4}}{120}T^4,\\
\overline{m}_{12} &= \frac{T}{2} + \frac{d_2}{6}T^2 + \frac{d_2^2 - d_1}{24}T^3\\
&\quad + \frac{2d_1d_2 - d_2^3 - d_0}{120}T^4,\\
\overline{m}_{13} &= \left(\frac{1+\beta_0+\beta_1}{6} + \frac{\beta_1}{24}\right)T - \left[\frac{(1+\beta_0+\beta_1)d_2}{24}\right.\\
&\quad \left. + \frac{d_2\beta_1}{120}\right]T^2 + \left[\frac{\beta_1\left(d_2^2 - d_1\right)}{720}\right.\\
&\quad \left. + \frac{(1+\beta_0+\beta_1)(d_2^2 - d_1)}{120}\right]T^3,\\
\overline{m}_{14} &= -\left(\frac{\beta_0+\beta_1}{6} + \frac{\beta_1}{24}\right)T + \left[\frac{d_2\beta_1}{120}\right.\\
&\quad \left. + \frac{(\beta_0+\beta_1)d_2}{24}\right]T^2,\\
\overline{\mathbf{m}}_{15}^T &= -\frac{\mathbf{c}^T}{6}T^2 + \frac{d_2\mathbf{c}^T - \mathbf{c}^TP}{24}T^3\\
&\quad + \frac{-(d_2^2 - d_1)\mathbf{c}^T + d_2\mathbf{c}^TP - \mathbf{c}^TP^2}{120}T^4,\\
m_{21} &= -z + 1 - \frac{d_1}{2}T^2 + \frac{d_1d_2 - d_0}{6}T^3\\
&\quad + \frac{d_1^2 - d_1d_2^2 + d_0d_2 - c_{n-4}}{24}T^4 + O(T^5),\\
m_{22} &= T + \frac{d_2}{2}T^2 + \frac{d_2^2 - d_1}{6}T^3\\
&\quad + \frac{2d_1d_2 - d_2^3 - d_0}{24}T^4 + O(T^5),\\
m_{23} &= \left(\frac{1+\beta_0+\beta_1}{2} + \frac{\beta_1}{6}\right)T^2 - \left[\frac{(1+\beta_0+\beta_1)d_2}{6}\right.\\
&\quad \left. + \frac{d_2\beta_1}{24}\right]T^3 + \left[\frac{\beta_1\left(d_2^2 - d_1\right)}{120}\right.\\
&\quad \left. + \frac{(1+\beta_0+\beta_1)(d_2^2 - d_1)}{24}\right]T^4 + O(T^5),\\
m_{24} &= -\left(\frac{\beta_0+\beta_1}{2} + \frac{\beta_1}{6}\right)T^2 + \left[\frac{d_2\beta_1}{24}\right.\\
&\quad \left. + \frac{(\beta_0+\beta_1)d_2}{6}\right]T^3 + O(T^4),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{m}_{25}^T &= -\frac{\mathbf{c}^T}{2}T^2 + \frac{d_2\mathbf{c}^T - \mathbf{c}^TP}{6}T^3 + \frac{-\mathbf{c}^TP^2}{}\\
&\quad \frac{+(d_1 - d_2^2)\mathbf{c}^T + d_2\mathbf{c}^TP}{24}T^4 + O(T^5),\\
m_{31} &= -d_1T + \frac{d_1d_2 - d_0}{2}T^2\\
&\quad + \frac{d_1^2 - d_1d_2^2 + d_0d_2 - c_{n-4}}{6}T^3 + O(T^4),\\
m_{32} &= -z + 1 - d_2T + \frac{d_2^2 - d_1}{2}T^2\\
&\quad + \frac{2d_1d_2 - d_2^3 - d_0}{6}T^3 + O(T^4),\\
m_{33} &= \left(1 + \beta_0 + \frac{3}{2}\beta_1\right)T - \left[\frac{(1+\beta_0+\beta_1)d_2}{2}\right.\\
&\quad \left. + \frac{d_2\beta_1}{6}\right]T^2 + \left[\frac{\beta_1\left(d_2^2 - d_1\right)}{24}\right.\\
&\quad \left. + \frac{(1+\beta_0+\beta_1)(d_2^2 - d_1)}{6}\right]T^3 + O(T^4),\\
m_{34} &= -\left(\beta_0 + \frac{3}{2}\beta_1\right)T + \left[\frac{d_2\beta_1}{6}\right.\\
&\quad \left. + \frac{(\beta_0+\beta_1)d_2}{2}\right]T^2 + O(T^3),\\
\mathbf{m}_{35}^T &= -\mathbf{c}^TT + \frac{d_2\mathbf{c}^T - \mathbf{c}^TP}{2}T^2 + \frac{-\mathbf{c}^TP^2}{}\\
&\quad \frac{+(d_1 - d_2^2)\mathbf{c}^T + d_2\mathbf{c}^TP}{6}T^3 + O(T^4),\\
\mathbf{m}_{51} &= \frac{\mathbf{q}}{2}T^2 + \frac{P\mathbf{q}}{6}T^3 + \frac{P^2\mathbf{q} - \mathbf{q}d_1}{24}T^4 + O(T^5),\\
\mathbf{m}_{52} &= \frac{\mathbf{q}}{6}T^3 + \frac{P\mathbf{q} - \mathbf{q}d_2}{24}T^4 + O(T^5),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{m}_{53} &= \left[\frac{(1+\beta_0+\beta_1)\mathbf{q}}{24} + \frac{q\beta_1}{120}\right]T^4 + O(T^5),\\
\mathbf{m}_{54} &= -\left[\frac{(1+\beta_0+\beta_1)\mathbf{q}}{24} + \frac{q\beta_1}{120}\right]T^4 + O(T^5),\\
M_{55} &= (-z+1)I + PT + \frac{P^2}{2}T^2 + \frac{P^3}{6}T^3\\
&\quad + \frac{P^4 - \mathbf{q}\mathbf{c}^T}{24}T^4 + O(T^5).
\end{aligned}
$$

Thus, the zeros are derived from

$$|M| = 0. \tag{32}$$

The matrix (31) is divided into several submatrices by using the partitioning technique as described below:

$$M = \begin{bmatrix} \overline{M}_{11} & \overline{M}_{12} \\ \overline{M}_{21} & \overline{M}_{22} \end{bmatrix}, \tag{33}$$

where

$$\overline{M}_{11} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & -z & 1 \end{bmatrix},$$

$$\overline{M}_{12} = \begin{bmatrix} \mathbf{m}_{15}^T & \mathbf{m}_{25}^T & \mathbf{m}_{35}^T & \mathbf{0}^T \end{bmatrix}^T,$$

$$\overline{M}_{21} = \begin{bmatrix} \mathbf{m}_{51} & \mathbf{m}_{52} & \mathbf{m}_{53} & \mathbf{m}_{54} \end{bmatrix},$$

$$\overline{M}_{22} = M_{55}.$$

Simple calculation yields

$$|M| = |\overline{M}_{22}||\overline{M}_{11} - \overline{M}_{12}\overline{M}_{22}^{-1}\overline{M}_{21}| \tag{34}$$

and

$$\begin{aligned}
&\overline{M}_{12}\overline{M}_{22}^{-1}\overline{M}_{21} \\
&\approx [\mathbf{m}_{15}^T \quad \mathbf{m}_{25}^T \quad \mathbf{m}_{35}^T \quad \mathbf{0}^T]^T \frac{1}{1-z} \\
&\quad \times \left[I - \frac{1}{(1-z)} \left(PT + \frac{P^2}{2}T^2 + \frac{P^3}{6}T^3 \right) \right] \\
&\quad \times \begin{bmatrix} \mathbf{m}_{51} & \mathbf{m}_{52} & \mathbf{m}_{53} & \mathbf{m}_{54} \end{bmatrix} \\
&= \begin{bmatrix} O(T^{10}) & O(T^{10}) & O(T^{10}) & O(T^{10}) \\ O(T^{10}) & O(T^{10}) & O(T^{10}) & O(T^{10}) \\ O(T^9) & O(T^9) & O(T^9) & O(T^9) \\ & & O & \end{bmatrix}.
\end{aligned}$$

Note here that the order of each block matrix of the first three lines in $\overline{M}_{11}$ is lower than that in $\overline{M}_{12}\overline{M}_{22}^{-1}\overline{M}_{21}$, so we have

$$|\overline{M}_{11} - \overline{M}_{12}\overline{M}_{22}^{-1}\overline{M}_{21}| \approx |\overline{M}_{11}|. \tag{35}$$

Further, consider a matrix $\overline{M}_{11,\alpha}$ which is defined by neglecting the higher order terms $O(.)$ with respect to T in the matrix $\overline{M}_{11}$ because the interests lie in the case of $T \ll 1$.

Postmultiplying M by

$$R = \mathrm{diag}\left(1,\ 1,\ \frac{1}{T},\ \frac{1}{T}\right) \tag{36}$$

and premultiplying the result by

$$L = \begin{bmatrix} \frac{1}{T} & 0 & 0 & 0 \\ \ell_1 & 1 & 0 & 0 \\ \ell_2 & 0 & 1 & 0 \\ \ell_3 & 0 & 0 & 1 \end{bmatrix}, \tag{37}$$

where

$$\begin{aligned}
\ell_1 &= -\frac{1}{m_0}\left\{ \left(\frac{\beta_0+\beta_1}{2} + \frac{\beta_1}{6} \right) T^2 \right. \\
&\quad \left. + \left[\frac{d_2\beta_1}{24} + \frac{(\beta_0+\beta_1)d_2}{6} \right] T^3 \right\}, \\
\ell_2 &= -\frac{1}{m_0}\left\{ \left(\beta_0 + \frac{3}{2}\beta_1 \right) T \right. \\
&\quad \left. + \left[\frac{d_2\beta_1}{6} + \frac{(\beta_0+\beta_1)d_2}{2} \right] T^2 \right\}, \\
\ell_3 &= -\frac{1}{m_0}, \\
m_0 &= -\left(\frac{\beta_0+\beta_1}{6} + \frac{\beta_1}{24} \right) T \\
&\quad + \left[\frac{d_2\beta_1}{120} + \frac{(\beta_0+\beta_1)d_2}{24} \right] T^2,
\end{aligned}$$

and further premultiplying the result by

$$\overline{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \tilde{\ell}_1 \\ 0 & 0 & 1 & \tilde{\ell}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{38}$$

where

$$\begin{aligned}
\tilde{\ell}_1 = -\frac{1}{\overline{m}_0} & \left\{ \left(\frac{1+\beta_0+\beta_1}{2} + \frac{\beta_1}{6} \right) T^2 - \left[\frac{d_2\beta_1}{24} \right.\right. \\
&+ \left. \frac{(1+\beta_0+\beta_1)d_2}{6} \right] T^3 + \left[\frac{\beta_1\left(d_2^2 - d_1\right)}{120} \right. \\
&+ \left.\left. \frac{(1+\beta_0+\beta_1)(d_2^2-d_1)}{24} \right] T^4 \right\}, \\
\tilde{\ell}_2 = -\frac{1}{\overline{m}_0} & \left\{ \left(1+\beta_0 + \frac{3}{2}\beta_1 \right) T - \left[\frac{(1+\beta_0+\beta_1)d_2}{2} \right.\right. \\
&+ \left. \frac{d_2\beta_1}{6} \right] T^2 + \left[\frac{\beta_1\left(d_2^2-d_1\right)}{24} \right. \\
&+ \left.\left. \frac{(1+\beta_0+\beta_1)(d_2^2-d_1)}{6} \right] T^3 \right\}, \\
\overline{m}_0 = -z + 1 & - \frac{4+4\beta_0+5\beta_1}{4\beta_0+5\beta_1} + (1-d_1)T \\
&+ \frac{d_1d_2 - d_0 + d_2}{2} T^2 \\
&+ \frac{d_1^2 - d_1d_2^2 + d_0d_2 - \mathbf{c}^T\mathbf{q} + d_2^2 - d_1}{6} T^3
\end{aligned}$$

yields

$$\overline{L}L\overline{M}_{11,\alpha}R = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ \overline{m}_{21} & \overline{m}_{22} & 0 & 0 \\ \overline{m}_{31} & \overline{m}_{32} & 0 & 0 \\ \# & \# & \overline{m}_0 & 0 \end{bmatrix}, \tag{39}$$

where # denotes an appropriate vector which does not affect the result and

$$\overline{m}_{21} = -z + 1 - \frac{12\beta_0 + 16\beta_1}{4\beta_0 + 5\beta_1} - \frac{d_1}{2}T^2 + \frac{d_1 d_2 - d_0}{6}T^3 + O(T^4),$$

$$\overline{m}_{22} = \left(1 - \frac{6\beta_0 + 8\beta_1}{4\beta_0 + 5\beta_1}\right)T + \frac{d_2}{2}T^2 + \frac{d_2^2 - d_1}{6}T^3 + O(T^4),$$

$$\overline{m}_{31} = 2d_2 - d_1 T + \frac{d_1 d_2 - d_0}{2}T^2 + \frac{d_1^2 - d_1 d_2^2 + d_0 d_2 - c_{n-4}}{6}T^3 + O(T^4),$$

$$\overline{m}_{32} = -z + 1 - \frac{12\beta_0 + 18\beta_1}{4\beta_0 + 5\beta_1} - d_2 T + \frac{d_2^2 - d_1}{2}T^2 + \frac{2d_1 d_2 - d_2^3 - d_0}{6}T^3 + O(T^4).$$

Noting here that

$$|R| = \frac{1}{T^2}, \quad |L| = \frac{1}{T}, \quad |\overline{L}| = 1$$

leads to

$$|\overline{M}_{11}| = -T^3 |\overline{L} L \overline{M}_{11,\alpha} R| = -T^3 m_{14} \overline{m}_0 (\overline{m}_{21}\overline{m}_{32} - \overline{m}_{22}\overline{m}_{31}), \tag{40}$$

where

$$\begin{aligned}\Delta &= \overline{m}_{21}\overline{m}_{32} - \overline{m}_{22}\overline{m}_{31} \\ &= \Bigg[-z - 1 + \frac{12\beta_0 + 16\beta_1}{4\beta_0 + 5\beta_1} + \left(1 + \beta_0 + \frac{\beta_1}{2}\right)T - \frac{3d_1 - 3 - 3\beta_0 - 4\beta_1}{6}T^2 + \frac{d_1 d_2 - d_0 + 5d_2\beta_1 + 4d_2\beta_0 + 4d_2}{24}T^3\Bigg] \\ &\times \Bigg[-z - 1 + \frac{12\beta_0 + 18\beta_1}{4\beta_0 + 5\beta_1} - \left(d_2 - 1 - \beta_0 - \frac{3\beta_1}{2}\right)T + \frac{3d_2^2 - 3d_1 + 4d_2\beta_1 + 3d_2\beta_0 + 3d_2}{6}T^2 \\ &+ \frac{-d_2^3 + 2d_1 d_2 - d_0 + (4 + 4\beta_0 + 5\beta_1)(d_2^2 - d_1)}{24}T^3\Bigg].\end{aligned} \tag{41}$$

Hence, the approximate values of the zeros of the discrete-time system are obtained as the roots of (29).

As a result, the proof is complete. ■

Remark 1. Equation (29) implies that an approximation of the sampling zeros is expressed as

$$z_1 = -1 + \frac{12\beta_0 + 16\beta_1}{4\beta_0 + 5\beta_1} + \left(1 + \beta_0 + \frac{\beta_1}{2}\right)T - \frac{3d_1 - 3 - 3\beta_0 - 4\beta_1}{6}T^2 + \frac{d_1 d_2 - d_0 + 5d_2\beta_1 + 4d_2\beta_0 + 4d_2}{24}T^3, \tag{42}$$

$$z_2 = -1 + \frac{12\beta_0 + 18\beta_1}{4\beta_0 + 5\beta_1} - \left(d_2 - 1 - \beta_0 - \frac{3\beta_1}{2}\right)T + \frac{3d_2^2 - 3d_1 + 4d_2\beta_1 + 3d_2\beta_0 + 3d_2}{6}T^2 + \frac{-d_2^3 + 2d_1 d_2 - d_0 + (4 + 4\beta_0 + 5\beta_1)(d_2^2 - d_1)}{24}T^3, \tag{43}$$

$$z_3 = -1 + \frac{4 + 4\beta_0 + 5\beta_1}{4\beta_0 + 5\beta_1} + (1 - d_1)T + \frac{d_1 d_2 - d_0 + d_2}{2}T^2 + \frac{d_1^2 - d_1 d_2^2 + d_0 d_2 - \mathbf{c}^T\mathbf{q} + d_2^2 - d_1}{6}T^3, \tag{44}$$

and the approximate values of the intrinsic zeros are derived from

$$z = \left|I + PT + \frac{P^2}{2}T^2 + \frac{P^3}{6}T^3\right|. \tag{45}$$

Remark 2. When the relative degree of a continuous-time system is greater than two, at least one of the limiting zeros of the resulting sampled-data model is unstable for sufficiently small sampling periods in the case of a ZOH or a FROH. Nevertheless, our contribution of the discretization zeros (29) shows that the discrete system zeros can be arbitrarily assigned inside the unit circle by choosing design parameters β_0 and β_1 so that the sampling zero asymptotic polynomial (42)–(44) is identical to a desired stable region.

Remark 3. An insightful observation in Theorem 1 is that it has the form of a correction to the asymptotic result of the previous results (Åström *et al.*, 1984; Hagiwara *et al.*, 1993; Ishitobi, 1996; Liang and Ishitobi, 2004a) in the form of a power term of T. The reason is that our new FROH design is built as a generalization of well-known hold devices. Moreover, our achievements of both the intrinsic zeros, and sampling zeros as shown in Theorem 1, are also clarified in a more precise manner and a higher-order of accuracy than the corresponding results.

Remark 4. Generally speaking, notice here that the relative degree of many linear or nonlinear mechanical systems in the practical field is two. In the case of the relative degree two, the asymptotic expression of discretization zeros can be simply derived owing to the special choices of the following scalars and vectors in our equation (29) in Theorem 1, and it can be also obtained using a similar idea.

Remark 5. Based on the similarity method in the proof of Theorem 1, the asymptotic expansion expression of discretization zeros can also be represented in the case of a continuous-time relative degree greater than or equal to four for sufficiently small sampling periods, though it seems difficult to derive directly.

Remark 6. Explicit asymptotic characterization of these discrete-time zeros plays an important role in the design and analysis of controlled systems. The reason is that the explicit asymptotic behavior of the limiting zeros is an interesting issue because the limiting zeros are stable for sufficiently small sampling periods if they approach the unit circle from inside as the sampling periods goes to zero. More importantly, many techniques for design of control systems are based on the cancellation of process zeros. Such methods will not work when the process has unstable zeros. For example, several adaptive algorithms that are currently investigated belong to this category.

Equation (45) demonstrates that the limiting zeros corresponding to continuous-time zeros, i.e., intrinsic zeros, are located in the unit circle. In particular, if the corresponding continuous system zeros are stable, the intrinsic zeros approach unity from inside the unit circle as the sampling period tends to zero. But for sampling zeros we can find that their stability is related to the design parameters β_0 and β_1 of the new FROH hold circuit. From Theorem 1, it is easy to obtain the following corollary which shows a stability condition of the sampling zeros with a new FROH when the relative degree of a continuous-time system is three.

Corollary 1. *Assume that the relative degree of a continuous-time system is three. For a sufficiently small sampling period T all the limiting zeros of the discrete-time model (25)–(28) are stable if all the zeros of the original continuous-time system (4) are stable and*

$$\begin{cases} \beta_1 > 0, \\ \beta_0 < -\frac{5}{4}\beta_1, \end{cases} \tag{46}$$

or

$$\begin{cases} \beta_1 < 0, \\ \beta_0 > -\frac{3}{2}\beta_1. \end{cases} \tag{47}$$

Remark 7. When the new FROH signal reconstruction device is used, we propose to place sampling zeros of the discretized model to the stable region with as many tuning parameters as desired. Meanwhile, we propose a constructive solution to the intersample ripples with a polynomial instead of a simple linear pattern. In other words, the appropriate β_0 and β_1 are determined to obtain our hold that provides sampling zeros as stable as possible, or with improved stability properties even when unstable, for a wider class of continuous-time plants.

Remark 8. If the relative degree of a continuous-time transfer function is three, the corresponding discrete-time model must have unstable zeros in the case of a ZOH or a conventional FROH, at least one. Thus, the limiting zeros of the discrete-time system with a new FROH of the conditions (47) stay definitely inside the unit circle while those with a ZOH or a FROH may lie outside or on the unit circle. In other words, the new FROH with the conditions (47) will produce all stable sampling zeros for a wider class of continuous-time plants than a ZOH or a conventional FROH. Further, on a controlled system which has a relative degree three, the closed-loop system becomes unstable when we obtain the discrete-time model with a ZOH or a FROH , and design a feedback controller which requires the stability of the zeros. However, from Corollary 2, in such a case, we can get a stable feedback control system when a new FROH with the conditions (47) is used for a hold.

4. Numerical example

This section presents an interesting example to show the stability of sampling zeros for discretized systems with a new FROH. It shows that the stability of limiting zeros can be improved by using a new FROH instead of other conventional holds. Both kinds of the limiting zeros are determined with the use of MATLAB, and in the simulation figures below, the solid and dotted lines indicate the exact and approximate values, respectively.

Example 1. Consider a transfer function of a vertical-takeoff airplane for roll angle control with relative degree three (Filatov *et al.*, 1996),

$$G(s) = \frac{6.84}{s^2(s+3.02)}. \tag{48}$$

It is clear that the corresponding discretized system has an unstable zero when a ZOH or a FROH is used for the sampling period $T = 0.01$ s. Now we use our new FROH to place limiting zeros of the discrete-time system proposed in the case of the relative degree three. The approximate values (29) and the exact values of sampling zeros of the sampled-data system for the transfer function (48) are shown in Tables 1–3 and Figs. 1–3.

When the transfer function (48) of a continuous-time system has relative degree three, the corresponding discrete-time system has no intrinsic zeros and three sampling zeros in the case of a new FROH. In particular, the stability of a sampling zero with our FROH depends on the design parameters β_0 and β_1. When these two parameters satisfy one of the conditions (47), then the sampling zeros of the discrete-time model are stable in the case of a new FROH for small sampling periods T, and vice versa. There exists a set of solutions $\beta_1 = 1.2938035$ and $\beta_2 = -1.6041123$ such that the discretization zeros of our new FROH at the stable region.

Table 1. Absolute values of the sampling zero of a discrete-time model with relative degree three.

T	Approximate values (29)	Exact values
0.01	0.633507563	0.636319933
0.02	0.634339729	0.639258964
0.05	0.636624858	0.641146285
0.1	0.638193888	0.643047943
0.2	0.640146658	0.646131263

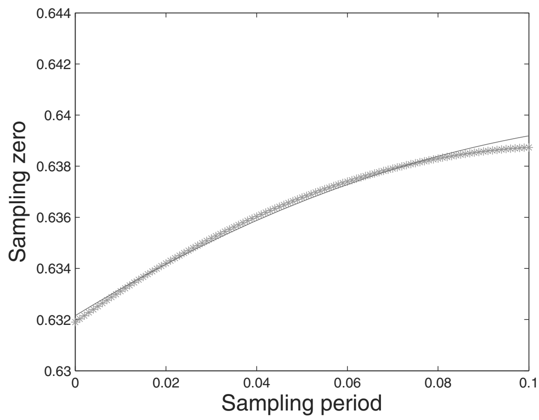

Fig. 1. Absolute values of the sampling zero of a discrete-time model with relative degree three.

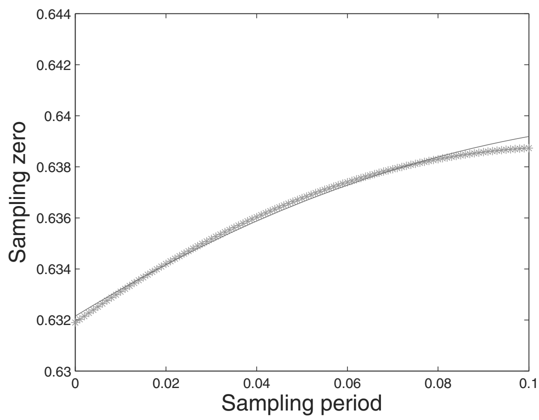

Fig. 2. Absolute values of the sampling zero of discrete-time model with relative degree three.

From Tables 1–3 and Figs. 1–3, Eqn. (29) yields a good approximation also for the case of a continuous-time transfer function (48) with the new FROH. ♦

5. Conclusions

This paper analyzes the asymptotic behavior of limiting zeros of a discrete-time system when, on the basis of the normal form representation of the continuous-time system with relative degree three, it is discretized using our new FROH circuit hold. It also proposes an approximate asymptotic expression of limiting zeros as power series expansions with respect to the sampling periods up to the third-order term. Moreover, an insightful interpretation is given in terms of an explicit characterization of the linear sampling zeros for the obtained model.

Further, the stability of the sampling zeros is discussed when the sampling periods tend to zero, while giving a constructive solution to the intersample issue. As a result of this work, it has been shown that a new FROH offers an advantage over a ZOH or a conventional FROH with stability of the limiting zeros of sampled-data systems. For a future study, an extension of the approach to multivariable systems is planned.

Table 2. Absolute values of the sampling zero of a discrete-time model with relative degree three.

T	Approximate values (29)	Exact values
0.01	0.633015522	0.636029257
0.02	0.634962901	0.639977015
0.05	0.636635742	0.641876959
0.1	0.638115478	0.643513566
0.2	0.640215369	0.646015125

Table 3. Absolute values of the sampling zero of a discrete-time model with relative degree three.

T	Approximate values (29)	Exact values
0.01	0.329717854	0.329667743
0.02	0.327136589	0.327023114
0.05	0.315367825	0.315131842
0.1	0.302515546	0.294528929
0.2	0.283673871	0.248067657

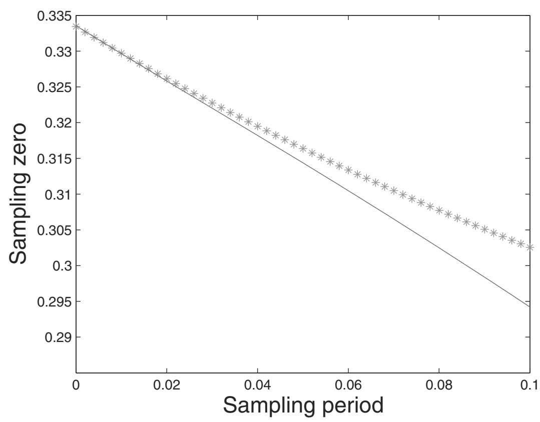

Fig. 3. Absolute values of the sampling zero of a discrete-time model with relative degree three.

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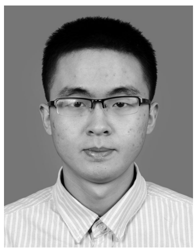

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