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CHARACTERISTICS AND DECOMPOSITION OF EXPRESSIONS IN THE PF-NOTATION

Abstract. The paper presents a selected aspects of classical parenthesis-free notation. With the introduction of the concepts of the pattern of expression and the characteristics were obtained convenient tools for classification and decomposition of expressions in the PF-notation. Some original results and the independent proofs of known results are presented.

1. Introduction

The parenthesis-free notation (PF-notation) is widely known in professional literature and practice as the Polish notation. It was introduced in the 1920s by the Polish logician and philosopher Jan Łukasiewicz [3, 4]. It concerned mainly the expressions which contained one or binary functors, as it was used to describe classical logical formulas [5]. Over the years it was also used in arithmetic, and its dynamical carrier was marked with the development of computer technology, which in many cases, was the basics of programming languages, as its twin form of the reverse notation. Its main advantage is the simplicity of syntax. In this paper an attempt has been made to systematize some of the concepts associated with the PF-notation. The concept of pattern of expression is introduced, on the base of which the majority of the proofs of theorem concerning the PF-notation is

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derived. Thanks to established pattern, it is easy to introduce the concept of the characteristic $\mathcal{X}$ for any expression of notation. The characteristic with the entered operation $\circ$ makes it easy to calculate the characteristic of compound expressions. On the ground of the theorem about the decomposition, it is easily to analyze any expressions of PF notation. The final part of the article is focused on the graphs interpretation of the expressions and characteristics and some additional concepts and properties of PF-notation [7]. It has been shown that if the notation symbols contain at least one more than one-arity operator symbol, then none of the classes of expressions $\langle \alpha, \beta \rangle$ is not empty. Moreover, invariant sets and normalizing sets of expression to the term form (well-formed expression) are indicated.

2. Symbols of notation

Let $\mathcal{A}$ be an arbitrary set. Let's create a set of operators

$$\mathcal{F} = \{f_{\mathcal{A}} : \mathcal{A}^n \to \mathcal{A}/n = 1, 2, \ldots\} \tag{1}$$

and introduce the mapping $arn : \mathcal{F} \to N$:

$$\forall (f_{\mathcal{A}} \in \mathcal{F}) \qquad (arn(f_{\mathcal{A}}) = n \leftrightarrow dl(f_{\mathcal{A}}) = \mathcal{A}^n). \tag{2}$$

The mapping arn assigns a number equal to the dimension of the operator domain to each operator.

In order to facilitate further discussion, we introduce the mapping $\eta : \mathcal{F} \to N_0$:

$$\forall (f_{\mathcal{A}} \in \mathcal{F}) \qquad (\eta(f_{\mathcal{A}}) = arn(f_{\mathcal{A}}) - 1). \tag{3}$$

Let's build a non-empty set $\mathbf{A}$, whose elements are the symbols of:

1. variables whose field is set $\mathcal{A}$,
2. names of elements of $\mathcal{A}$.

$\mathbf{F}$ denotes a set of the symbols of the functions that belong to $\mathcal{F}$. Assuming that $arn(f) = arn(f_{\mathcal{A}})$, where f is the symbol of function $f_{\mathcal{A}}$. A similar concurrence applies to mapping η.

Definition 1. *The set of symbols of parenthesis-free notation for $\mathcal{A}$ and for the set of operators $\mathcal{F}$ described in(1) is the set $\mathbf{F} \cup \mathbf{A}$ and is denoted as $V(\mathcal{A}, \mathcal{F})$.*

In the next parts of this paper $V(\mathcal{A}, \mathcal{F})$ shall be denoted by V.

3. Expressions. Terms

Definition 2. *An expression in V is a finite sequence of the symbols of V.*

The set of all expressions of $V(\mathcal{A},\mathcal{F})$ may be labeled as a set of expressions of the parenthesis-free notation (PFN) $\mathcal{A}$ and $\mathcal{F}$ and denoted as $\overset{*}{V}(\mathcal{A},\mathcal{F})$ (hereunder referred to as $\overset{*}{V} = \overset{*}{V}(\mathcal{A},\mathcal{F})$). One-element sequence is also an expression. Hence $V \subset \overset{*}{V}$.

We introduce the mapping $\delta : \overset{*}{V} \to N$:

$$\forall(v \in \overset{*}{V})(\delta(v) = n \leftrightarrow (\exists(v_1, v_2, \ldots, v_n) \in V^n \quad v = (v_1, v_2, \ldots, v_n))), \tag{4}$$

$\delta(v)$ shall be called the length of expression v.

On the elements of set $\overset{*}{V}$ the operation $* : \overset{*}{V} \times \overset{*}{V} \to \overset{*}{V}$ shall be described

$$\begin{aligned} &\forall(v^1 = (v^1_1, v^1_2, \ldots, v^1_n) \in \overset{*}{V}) \quad \forall(v^2 = (v^2_1, v^2_2, \ldots, v^2_m) \in \overset{*}{V}) \\ &v^1 * v^2 = (v^1_1, v^1_2, \ldots, v^1_n, v^2_1, v^2_2, \ldots, v^2_m). \end{aligned} \tag{5}$$

Operation $*$ is referred to as concatenation.

It is easy to notice that $\delta(v^1 * v^2) = \delta(v^1) + \delta(v^2)$.

Set $\overset{*}{V}$ with concatenation $*$ is a semi-group. Including empty expression it is free monoid under $\overset{*}{V}$ (see [8]).

In set of expressions $\overset{*}{V}$ we can distinguish the set of terms [1, 5, 6] (in some publications it is labeled as the set of well-formed expressions (wfe) and in some WFF [2]).

Definition 3. *The set of the terms of notation $\overset{*}{V}(\mathcal{A},\mathcal{F})$ where $\mathbf{V} = \mathbf{F} \cup \mathbf{A}$ is the smallest set $T(\mathcal{A},\mathcal{F})$, such that*

$$T(\mathcal{A},\mathcal{F}) \supset \bigcup_{n=0}^{\infty} T_n, \qquad \textit{where} \tag{6}$$

1. *$T_0 = \mathbf{A}$ (one-element sequence of $\mathbf{A}$),*
2. *$\forall(f \in \mathbf{F}) \quad (arn(f) = k \leftrightarrow \forall(t_1, \ldots, t_k) \in (\bigcup_{i=0}^{n-1} T_i)^k \quad f * t_1 * \ldots * t_k \in T_n)$ for $\quad n = 1, 2, \ldots$*

Surely $T(\mathcal{A},\mathcal{F}) \subset \overset{*}{V}(\mathcal{A},\mathcal{F})$.

4. Patterns of expressions

Now we define the main notion used for the investigation and classification of the parenthesis-free notation. It is the concept of the pattern of the expression which enables the replacement leading from the study of expressions to the study of sequences of the integers.

Let us assign the mapping

$$\omega : \overset{*}{V} \to \overset{*}{N}_{-1}, \qquad \text{where} \quad N_{-1} = \{-1, 0, 1, 2, \ldots\} \quad \text{and} \quad \mathbf{V} = \mathbf{F} \cup \mathbf{A}, \tag{7}$$

1. $\forall (v \in \mathbf{A}) \quad \omega(v) = -1,$
2. $\forall (v \in \mathbf{F}) \quad \omega(v) = \eta(v),$
3. $\forall (v = (v_1, \ldots, v_n) \in \overset{*}{V}) \qquad \omega(v) = (\omega(v_1), \ldots, \omega(v_n))$ for $n = 2, 3, \ldots$

Definition 4. *The pattern of expression $v \in \overset{*}{V}$ is a sequence $\omega(v)$.*

$\omega(\overset{*}{V})$ is the set of all patterns of notation $\overset{*}{V}$. Note that any pattern from $\omega(\overset{*}{V})$ can be a pattern of more than one expression belonging to the $\overset{*}{V}$.

Let mx be the next relation

$$\forall (v \in \overset{*}{V}) \quad \forall (w \in \overset{*}{V}) \quad (v \quad mx \quad w) \leftrightarrow \omega(v) = \omega(w). \tag{8}$$

Accordingly, mx is the equivalence relation.

Class $[v]_{mx}$ represents its pattern. Let m pattern of expression belonging to $\omega(\overset{*}{V})$. Class $[v]_{mx}$ that $\omega(v) = m$ we name the representation of pattern m of notation $\overset{*}{V}$ and denote $(m)_{\overset{*}{V}}$.

5. Main theorem of the parenthesis-free notation

Now we construct two mappings W_1 and W_2. Let Z be a set of the integers,

$$\begin{gathered} W_1 : \overset{*}{Z} \to \{0, 1\}, \\ \forall (k \in N) \quad \forall z = (z_1, \ldots, z_k) \in \overset{*}{Z} \end{gathered} \tag{9}$$

$$(W_1(z) = 1 \Leftrightarrow \sum_{i=1}^{k} z_i = -1) \quad \wedge \quad (W_1(z) = 0 \Leftrightarrow \sum_{i=1}^{k} z_i \neq -1) \tag{10}$$

and

$$\begin{gathered} W_2 : \overset{*}{Z} \to \{0,1\}, \\ (\forall(z \in Z) \quad W_2(z) = 1) \quad \wedge \quad \Big(\forall(k \geqslant 2) \quad \forall z = (z_1, \ldots, z_k) \in \overset{*}{Z} \end{gathered} \tag{11}$$

$$\begin{aligned} (W_2(z) = 1 \quad &\Leftrightarrow \quad \forall l \quad (1 \leqslant l < k) \sum_{i=1}^{l} z_i \geqslant 0) \quad \wedge \\ (W_2(z) = 0 \quad &\Leftrightarrow \quad \exists l \quad (1 \leqslant l < k) \sum_{i=1}^{l} z_i < 0)\Big). \end{aligned} \tag{12}$$

If for $a \in \overset{*}{Z}$ $W_1(a) = 1$ ($W_2(a) = 1$; $W_1(a) = 0$; $W_2(a) = 0$) we may say that sequence a satisfies the first condition (respectively: it satisfied the second condition; not satisfied the first condition; not satisfied the second condition).

Theorem 5 (main). *Let v be an expression of notation $\overset{*}{V}$. The next two expressions are equivalent:*

1. *pattern $\omega(v)$ satisfied the first and second condition;*
2. *expression v is a term of notation $\overset{*}{V}$.*

Proof. ($1 \Rightarrow 2$) The proof of induction due to the length of the expression $\delta(v)$. Let $\delta(v) = 1$. Hence $\omega(v) = a \in Z$. If $W_1(a) = 1$ and $W_2(a) = 1$ then $a = -1$. And, hence $v \in A = T_0$ that v is a term.

Let us assume that

$$\forall(v \in \overset{*}{V}) \quad ((\delta(v) \leqslant n-1 \quad \wedge \quad W_1(\omega(v)) = 1 \quad \wedge \quad W_2(\omega(v)) = 1) \to v \in T).$$

Let us take any expression $v \in \overset{*}{V}$, such that $\delta(v) = n$ and $v = (v_1, \ldots, v_n) \in V^n$. We shall show that if $W_1(\omega(v)) = 1 \ \wedge \ W_2(\omega(v)) = 1$ then

$$\begin{gathered} v = f * u_1 * \ldots u_k, \quad \text{where} \quad f \in F, \quad arn(f) = k \quad \text{and} \quad u_i \in T \\ \text{for} \quad i = 1, 2, \ldots, k. \end{gathered} \tag{13}$$

Let $\omega(v) = a = (a_1, \ldots, a_n)$. If $W_2(a) = 1$ and $\delta(a) > 1$ then $a_1 \geqslant 0$, and hence $v_1 \in F$. Let us assume that $k = arn(v_1)$. In the next step, we shall find indexes $l_0, l_1, \ldots, l_k$, so that $1 = l_0 < l_1 < \ldots < l_k = n$ and $\forall i \quad (1 \leqslant i \leqslant k) \quad W_1(a_{l_{i-1}+1}, \ldots, a_{l_i}) = 1 \ \wedge \ W_2(a_{l_{i-1}+1}, \ldots, a_{l_i}) = 1$. If

$$l_1 = \min(l : (2 \leqslant l \leqslant n) \quad \wedge \quad (\sum_{i=2}^{l} a_i = -1)), \tag{14}$$

then $W_2((a_2, \ldots, a_{l_1})) = 1$. Supposing that this condition is not met. Let m be the minimal index $(2 \leqslant m < l_1)$, for which $\sum_{i=2}^{m} a_i < 0$. Hence:

1. $\sum_{i=2}^{m} a_i = -1$ (contradiction to (14)),

   or

2. $\sum_{i=2}^{m} a_i < -1$; $\sum_{i=2}^{m-1} a_i + a_m < -1$; because $a_m \geqslant -1$ then $\sum_{i=2}^{m-1} a_i \leqslant -1$

   which is contrary to the assumption of minimality of m.

Because $\sum_{i=1}^{n} a_i = -1 = \eta(v_1) + \sum_{i=2}^{n} a_i$ that $\sum_{i=2}^{l_1} a_i + \sum_{i=l_1+1}^{n} a_i = -arn(v_1)$ and $\sum_{i=l_1+1}^{n} a_i = -arn(v_1) + 1$.

Similarly we find l_i $(i = 2, \ldots, k)$ where $l_i = \min\{l : (l_{i-1} + 1 \leqslant l \leqslant n) \wedge W_1(a_{l_{i-1}+1}, \ldots, a_l) = 1\}$. Hence $v = f * u_1 * \ldots * u_k$ where $f = v_1$ and $arn(f) = k$ and $u_i = (v_{l_{i-1}-1}, \ldots, v_{l_i})$ $(i = 1, 2, \ldots, k)$.

Since $W_1(\omega(u_i)) = 1$, $W_2(\omega(u_i)) = 1$ and $\delta(u_i) < n$ then $u_i \in T \quad (i = 1, 2, \ldots, k)$, so from the definition of the term $v \in T$.

$(2 \Rightarrow 1)$ Proof of induction based on definition of the term. Let $v \in T_0$. Because $T_0 = A$ then $\omega(v) = -1$ and thus $W_1(\omega(v)) = 1$ and $W_2(\omega(v)) = 1$. If $v \in T_1$ then $v = f * t_1 * \ldots * t_k$; $f \in F$ and $arn(f) = k$, $t_i \in T_0$ $(i = 1, \ldots, k)$, that $\omega(v) = (k-1, \underbrace{-1, \ldots, -1}_{\text{k times}})$. It's easy to see that $W_1(\omega(v)) = 1$ and $W_2(\omega(v)) = 1$. Let $n \geqslant 2$. Let's assume that:

$$\forall v \in \bigcup_{i=0}^{n-1} T_i \qquad W_1(\omega(v)) = 1 \quad \wedge \quad W_2(\omega(v)) = 1.$$

Let's take any term $v \in T$. Then $v = f * t_1 * \ldots * t_k \quad f \in F$, $arn(f) = k$, $t_i \in \bigcup_{j=0}^{n-1} T_j$ $(i = 1, \ldots, k)$. For facilitate we introduce v in the form

$$v = (v_1, v_{a_1}, \ldots, v_{b_1}, v_{a_2}, \ldots, v_{b_{k-1}}, v_{a_k}, \ldots, v_{b_k}), \tag{15}$$

where in $v_1 = f$, $(v_{a_i}, \ldots, v_{b_i}) = t_i$ $(i = 1, \ldots, k)$. Hence $\sum_{i=1}^{b_k} \omega(v_i) = k - 1 + \sum_{i=1}^{k} \sum_{j=a_i}^{b_i} \omega(v_j) = k - 1 + \sum_{i=1}^{k} (-1) = -1$ because $W_1(\omega(t_i)) = 1$ for $t_i \in \bigcup_{j=0}^{n-1} T_j$, and hence $W_1(\omega(v)) = 1$.

Note that $\forall p$ $(1 \leqslant p < k)$ $\forall l$ $(a_p \leqslant l \leqslant b_p)$ $\sum_{j=1}^{l} \omega(v_j) = k - 1 + (p-1)(-1) + \sum_{s=a_p}^{l} \omega(v_s) \geqslant k - 1 + p(-1) \geqslant 0$ and for l $(a_k \leqslant l < b_k)$ $\sum_{j=1}^{l} \omega(v_j) = k - 1 + (k-1)(-1) + \sum_{i=a_{k-1}}^{l} \omega(v_i) \geqslant 0$ because $W_2(\omega(t_i)) = 1$ for $t_i \in \bigcup_{j=0}^{n-1} T_j$. Hence $W_2(\omega(v)) = 1$. □

6. Examples

Let $\mathcal{A}$ – be the set of two-elements and $\mathcal{F}$ – the set of operators (see (1)).
Let's create $\mathbf{A}$ as the smallest set satisfying conditions

1. $\{0, 1\} \subset \mathbf{A}$ where 0, 1 names of elements of $\mathcal{A}$,

2. $\{p, q, r\} \subset \mathbf{A}$ where p, q, r names of variables whose fields is set $\mathcal{A}$,

and $\mathbf{F}$ the set of the symbols of the operators that belong to $\mathcal{F}$ where $\{N, C, I\} \subset \mathbf{F}$ are symbols of operator

$$\begin{aligned} N_{\mathcal{A}} &: \mathcal{A} \to \mathcal{A} & N_{\mathcal{A}}(x) &= 1 - x, \\ C_{\mathcal{A}} &: \mathcal{A}^2 \to \mathcal{A} & C_{\mathcal{A}}(x, y) &= x \cdot y, \\ I_{\mathcal{A}} &: \mathcal{A}^2 \to \mathcal{A} & I_{\mathcal{A}}(x, y) &= 1 + x - x \cdot y. \end{aligned}$$

That $\{N, C, I, 0, 1, p, q, r\} \subset \mathbf{F} \cup \mathbf{A} = \mathbf{V}(\mathcal{A}, \mathcal{F})$. For example the expression of notation $\overset{*}{V}(\mathcal{A}, \mathcal{F})$ is

$$v = (N, 0, p, C, 1, r, q, I, 0), \qquad \text{and its pattern is}$$
$$\omega(v) = (0, -1, -1, 1, -1, -1, -1, 1, -1).$$

The next expression

$$t = (C, N, C, p, q, I, 1, r), \text{ is a term with pattern}$$
$$\omega(t) = (1, 0, 1, -1, -1, 1, -1, -1).$$

It's easy to notice that $\omega(t)$ satisfying first and second condition.

7. Characteristic $\mathcal{X}$

Let us define on the parenthesis-free notation $\overset{*}{V}(\mathcal{A}, \mathcal{F})$ the mapping which assigns pair $(\alpha, \beta) \in N \times N_0$ to each expression. This mapping shall be labeled as the characteristic $\mathcal{X}$. It is the basic tool for the decomposition of expressions. The characteristic $\mathcal{X}$ could be improved in many different ways. The following two definitions are equivalence.

Definition 6 (using pattern). $\mathcal{X} : \overset{*}{V} \to N \times N_0$

$\forall v = (v_1, \ldots, v_n) \in \overset{*}{V} \quad \mathcal{X}(v) = (\alpha, \beta)$ *where*

$$\alpha = 1 + \min \Big\{ k \in N_0 \mid k + \sum_{i=1}^{n} \omega(v_i) \geqslant -1 \quad \wedge$$
$$\wedge \quad \forall j \ (1 \leqslant j < n) \quad k + \sum_{i=1}^{j} \omega(v_i) \geqslant 0 \Big\}, \tag{16}$$

$$\beta = \alpha + \sum_{i=1}^{n} \omega(v_i). \tag{17}$$

Definition 7 (using term). $\mathcal{X} : \overset{*}{V} \to N \times N_0$

$\forall v = (v_1, \ldots, v_n) \in \overset{*}{V} \quad \mathcal{X}(v) = (\alpha, \beta)$ *where*

$$\alpha = \min \Big\{ k \in N \mid \exists f \in F \quad \big(arn(f) = k \wedge \exists l \in N_0 \forall (t_1, \ldots, t_l) \in T^l \\ f * v * t_1 * \ldots * t_l \in T\big) \Big\}, \tag{16'}$$

$$\beta = l \leftrightarrow \forall f \in F \quad \forall \big((t_1, \ldots, t_l) \in T^l\big) \big((arn(f) = \alpha) \to \\ (f * v * t_1 * \ldots * t_l \in T)\big). \tag{17'}$$

Furthermore, there is also interesting graph definition of $\mathcal{X}$ which will be presented on the end of this article. On the grounds of the main theorem of PFN we shall now proof the equivalence of Definitions 6 and 7.

Proof. 1° Let us assume that $v = (v_1, \ldots, v_n)$ and, accordingly Def. 7, $\mathcal{X}(v) = (\alpha, \beta)$. Then for any $f \in F : arn(f) = \alpha$ and for any sequence $(t_1, \ldots, t_\beta) \in T^\beta$ $u = f * v * t_1 * \ldots * t_\beta \in T$. Because the pattern of expression u satisfied the first condition so

$$\alpha - 1 + \sum_{i=1}^{n} \omega(v_i) - \beta = -1 \text{ and thus } \beta = \alpha + \sum_{i=1}^{n} \omega(v_i).$$

Because $\alpha - 1 + \sum_{i=1}^{n} \omega(v_i) = \beta - 1 \geqslant -1$ and $W_2(\omega(u)) = 1$, that $\alpha - 1 + \sum_{i=1}^{j} \omega(v_i) \geqslant 0$ for $j = 1, 2, \ldots, n-1$. It should be notice that α is minimal also according to a Definition 6.

2° Let's assume that $v = (v_1, \ldots, v_n)$ and accordingly Def.6 $\mathcal{X}(v) = (\alpha, \beta)$. Let's create $u = f * v * t_1 * \ldots t_\beta$ where $f \in F, \quad arn(f) = k$ and $t_i \in T (i = 1, 2, \ldots, \beta)$. It should be noticed that $\alpha - 1 + \sum_{i=1}^{n} \omega(v_i) + (-\beta) = -1$ because $\beta = \alpha + \sum_{i=1}^{n} v_i$.

Thus $W_1(\omega(u)) = 1$, and for $j = 1, 2, \ldots, n-1 \quad \alpha - 1 + \sum_{i=1}^{j} \omega(v_i) \geqslant 0$ it meets the second condition. Again, α is a minimal but now in accordance with Definition 7. □

The definitions lead to the following conclusions:

$(W1) \qquad \forall (v \in \overset{*}{V})(\mathcal{X}(v) = (1, 0) \leftrightarrow v \in T).$

Proof. 1° Let $v = (v_1, \ldots, v_n) \in \overset{*}{V}$ and $\mathcal{X}(v) = (1, 0)$. Hence and from (17) $0 = 1 + \sum_{i=1}^{n} \omega(v_i)$ that $W_1(\omega(v)) = 1$. From (16) $1 - 1 + \sum_{i=1}^{j} \omega(v_i) \geqslant 0$ which implies $W_2(\omega(v)) = 1$. More from the main PFN theorem $v \in T$.

2° Let $v \in T$, then for any $f \in F(arn(f) = 1 \Rightarrow f * v \in T)$ and from the Definition 7. $\mathcal{X}(v) = (1, 0)$. □

$$(W2) \qquad \forall(f \in F) \quad \mathcal{X}(f) = (1, arn(f));$$

$$(W3) \qquad \forall(n \in N) \quad \forall(t_1, \ldots, t_n) \in T^n \quad \mathcal{X}(t_1 * \ldots * t_n) = (n, 0).$$

Proof. Let $n \in N$ and $(t_1, \ldots, t_n) \in T^n$ and $f \in F$ and $arn(f) = k$. If $k < n$ then $f * t_1 * \ldots * t_k \notin T$ and for any $l > 0$ and for any $(v_l, \ldots, v_l) \in T^l f * t_1 * \ldots t_n * v_1 * \ldots * v_l \notin T$ which follows from the definition of term. When $k = n$ then $f * t_1 * \ldots t_n \in T$ and from the Definition 7, $\mathcal{X}(t_1 * \ldots * t_n) = (n, 0)$. □

$$(W4) \qquad \forall(v \in \overset{*}{V}) \quad \forall(u \in \overset{*}{V})(\omega(v) = \omega(u) \rightarrow \mathcal{X}(u) = \mathcal{X}(v)).$$

Based on the definition of characteristic we also obtain the next theorem.

Theorem 8. *Let us assume that* $v = (v_1, \ldots, v_n) \in V$ *and* $\mathcal{X}(v) = (\alpha, \beta)$ *and* $u = (v_{i_1}, \ldots, v_{i_n})$, *where*

1° $(i_1, \ldots, i_n)$ *is a permutation of* $(1, 2, \ldots, n)$,

2° $\mathcal{X}(u) = (\gamma, \delta)$,

then $\alpha - \beta = \gamma - \delta$.

Proof. Let us observe that $\sum_{j=1}^{n} \omega(v_j) = \sum_{j=1}^{n} \omega(v_{i_j})$. Because from (17) $\sum_{i=1}^{n} \omega(v_i) = \beta - \alpha$ and $\sum_{j=1}^{n} \omega(v_{i_j}) = \delta - \gamma$ so $\alpha - \beta = \gamma - \delta$. □

8. Concatenation of expressions

In the second part of this paper concatenation of expressions will be defined. However, a question arises: If for $v \in \overset{*}{V}$ $\mathcal{X}(v) = (\alpha, \beta)$ and for $u \in \overset{*}{V}$ $\mathcal{X}(u) = (\gamma, \delta)$

what is the characteristic of $\mathcal{X}(v*u)$ and how does it depend on the characteristics (α,β), (γ,δ). To address this question let us introduce in set $N\times N_0$ the following operation $\circ$:

$$\circ:(N\times N_0)^2\to N\times N_0$$
$$\forall(\alpha,\beta)\in N\times N_0 \qquad \forall(\gamma,\delta)\in N\times N_0,$$

$$(\alpha,\beta)\circ(\gamma,\delta)=(\alpha+H(\gamma-\beta),\delta+H(\beta-\gamma)), \qquad \text{where} \tag{18}$$
$$H(x)=\max\{0,x\} \qquad \text{for} \quad x\in Z. \tag{19}$$

Theorem 9. *For any two expressions v,u from $\overset{*}{V}$ the following equation is true*

$$\mathcal{X}(v*u)=\mathcal{X}(v)\circ\mathcal{X}(u). \tag{20}$$

Proof. Let us assume that $v=(v_1,\ldots,v_n)\in\overset{*}{V}$, $u=(u_1,\ldots,u_m)\in\overset{*}{V}$ and $\mathcal{X}(v)=(\alpha,\beta)$, $\mathcal{X}(u)=(\gamma,\delta)$. Two cases should be considered: 1°) for $\beta\geqslant\gamma$ and 2°) for $\beta<\gamma$.

1°) $\beta\geqslant\gamma$. Let's create $w=v*u=(w_1,\ldots,w_n,w_{n+1},\ldots,w_{n+m})$. Moreover, let us assume that $\mathcal{X}(w)=(x,y)$. Let $k=\alpha-1$. Then $k+\sum_{i=1}^{n}\omega(w_i)\geqslant-1$ and $k+\sum_{i=1}^{j}\omega(w_i)\geqslant 0$ for $j=1,\ldots,n-1$. Because $k+\sum_{i=1}^{n}\omega(w_i)=\beta-1$, and $\beta\geqslant\gamma$ that $k+\sum_{i=1}^{n}\omega(w_i)\geqslant\gamma-1$. Notice that $k+\sum_{i=1}^{n+m}\omega(w_i)=k+\sum_{i=1}^{n}\omega(w_i)+\sum_{i=n+1}^{n+m}\omega(w_i)\geqslant \gamma-1+\sum_{i=n+1}^{n+m}\omega(w_i)=\gamma-1+\sum_{i=1}^{m}\omega(u_i)\geqslant-1$ because $\mathcal{X}(u)=(\gamma,\delta)$. Similarly $k+\sum_{i=1}^{j}\omega(w_i)\geqslant 0$ for $j(1\leqslant j<n+m)$. That $x\leqslant\alpha$. If $x<\alpha$ then $\mathcal{X}(v)\neq(\alpha,\beta)$. Therefore $x=\alpha$. Let us now find $y:y=\alpha+\sum_{i=1}^{n+m}\omega(w_i)=\alpha+\sum_{i=1}^{n}\omega(v_i)+\sum_{i=1}^{n}\omega(u_i)=\beta+\delta-\gamma=\delta+(\beta-\gamma)$ and, likewise $\mathcal{X}(w)=(\alpha,\delta+(\beta-\gamma))$.

2°) $\beta < \gamma$. Let $k = \alpha + (\gamma - \beta) - 1$. Because $\gamma > \beta$ then $k > \alpha - 1$ and the same holds for $k + \sum_{i=1}^{n} \omega(w_i) \geqslant -1$ and for $j(1 \leqslant j < n) \quad k + \sum_{i=1}^{j} \omega(w_i) \geqslant 0$. Hence $k + \sum_{i=1}^{n} \omega(v_i) = \alpha + \gamma - \beta - 1 + \beta - \alpha = \gamma - 1$ so that $k + \sum_{i=1}^{n+m} \omega(w_i) = \gamma - 1 + \sum_{i=1}^{m} \omega(u_i) \geqslant -1$ because $\mathcal{X}(u) = (\gamma, \delta)$. Similarly, for $j(1 \leqslant j < n+m) \quad k + \sum_{i=1}^{j} \omega(w_i) \geqslant 0$ which leads to $x \leqslant \alpha + (\gamma - \beta)$. If $x < \alpha + (\gamma - \beta)$ then $\mathcal{X}(v) \neq (\alpha, \beta)$ or $\mathcal{X}(u) \neq (\gamma, \delta)$. Thus $x = \alpha + (\gamma - \beta)$ and $y = \alpha + \gamma - \beta + \sum_{i=1}^{n+m} \omega(w_i) = \alpha + \gamma - \beta + \sum_{i=1}^{n} \omega(v_i) + \sum_{i=1}^{m} \omega(u_i) = \delta$ and, consequently $\mathcal{X}(w) = (\alpha + (\gamma - \beta), \delta)$.

By combining cases 1° and 2° $\mathcal{X}(w) = (\alpha + H(\gamma - \beta), \delta + H(\beta - \gamma))$. □

On the ground of the proved theorem, operation $\circ$ on set $(N \times N_0)^2$ makes it possible to use characteristic $\mathcal{X}$ for composing and decomposing any expressions of the PF notation.

9. Decomposition theorem

Decomposition theorem is an illustration of the importance of the characteristic $\mathcal{X}$. In addition, it shows us that any expression of PF notation $\overset{*}{V}$ is either an expression with characteristic $\mathcal{X}$ equal to $(1, \beta)$, $(\beta = 0, 1, 2, \ldots)$ or a concatenation of finite number of the expressions of described above.

Theorem 10 (about decomposition). *Let $v \in \overset{*}{V}$. If this is assumed $\mathcal{X}(v) = (\alpha, \beta)$ and $\alpha \geqslant 2$ there is exactly one sequence $(t_1, \ldots, t_{\alpha-1}) \in T^{\alpha-1}$ and there exists only one expression $u \in \overset{*}{V}$ with characteristic $\mathcal{X}(u) = (1, \beta)$ such that*

$$v = t_1 * \ldots * t_{\alpha-1} * u. \tag{21}$$

At first, we proof the following lemmas:

Lemma 11. *Let $v \in \overset{*}{V}$. If $\mathcal{X}(v) = (\alpha, 0)$ there is exactly one sequence $(t_1, \ldots, t_\alpha)$ $\in T^\alpha$ such that*

$$v = t_1 * \ldots * t_\alpha. \tag{22}$$

Proof. (by induction)

1° For $\alpha = 1$ the lemma is obvious, see $(W1)$.

2° Let's assume that lemma is satisfied for any $v \in \overset{*}{V}$ for which $\mathcal{X}(v) = (\alpha - 1, 0)$ it meaning that there is an unambiguous decomposition of expression v into $\alpha - 1$ terms ($\alpha \geqslant 2$). Let's consider the expression $v = (v_1, \dots, v_n) \in \overset{*}{V}$ for which $\mathcal{X}(v) = (\alpha, 0)$. Because $\sum_{i=1}^{n} \omega(v_i) = -\alpha$ there exists $l (1 \leqslant l < n)$ such that

$$W_1(\omega(v_1, \dots, v_l)) = 1 \quad \text{and} \quad W_2(\omega(v_1, \dots, v_l)) = 1. \tag{23}$$

Let l_0 be the smallest l that satisfies equation (23). Therefore $t = (v_1, \dots, v_{l_0}) \in T$ and $\mathcal{X}(t) = (1, 0)$. Let $u = (v_{l_0+1}, \dots, v_n)$. Hence $v = t * u$. By solving the equation: $\mathcal{X}(v) = \mathcal{X}(t) \circ \mathcal{X}(u)$ ie. $(\alpha, 0) = (1, 0) \circ (x, y)$ we derive $x = \alpha - 1, \quad y = 0$ ie. $\mathcal{X}(u) = (\alpha - 1, 0)$.

So, under the assumption 2° there exists exactly one sequence $(t_1, \dots, t_{\alpha-1}) \in T^{\alpha-1}$ such that $u = t_1 * \dots * t_{\alpha-1}$ and accordingly

$$v = t * t_1 * \dots * t_{\alpha-1} \quad \text{where} \quad (t, t_1, \dots, t_{\alpha-1}) \in T^{\alpha}. \tag{24}$$

Now we shall show that this decomposition is unambiguous. If $t = (v_1, \dots, v_l)$ and $l < l_0$ on the ground of the minimality $l_0 \quad t \notin T$. Otherwise, if $l > l_0$ it is $W_2(\omega(t)) = 0$. So, assuming the minimality l_0 and unambiguous decomposition of the expression u, we obtain that (24) is an unambiguous decomposition of v, which proves (22). □

Lemma 12. *Let v be any expression of the PF notation $\overset{*}{V}$. If $\mathcal{X}(v) = (\alpha, \beta)$ and $(\alpha \geqslant 2)$ there is exactly one expression $u_1 \in \overset{*}{V}$ of characteristic $\mathcal{X}(u_1) = (\alpha - 1, 0)$ and exactly one expression u_2 of characteristic $\mathcal{X}(u_2) = (1, \beta)$ such that $v = u_1 * u_2$.*

Proof. Let $v = (v_1, \dots, v_n)$ and $\mathcal{X}(v) = (\alpha, \beta)$. Therefore $(\alpha - 1) + \sum_{i=1}^{n} \omega(v_i) \geqslant -1$ and $(\alpha - 1) + \sum_{i=1}^{j} \omega(v_i) \geqslant 0$ for $j = 1, 2, \dots, n-1$ so there exist $l (1 \leqslant l < n)$ such that $(\alpha - 1) + \sum_{i=1}^{l} \omega(v_i) = 0$ basing on definition 5. and the minimality of α. Let l_0 will be the smallest l which satisfied this condition. It should be noticed, that for $w_1 = (v_1, \ldots, v_{l_0})$ $\mathcal{X}(w_1) = (\alpha - 1, 0)$ because $\alpha - 2 + \sum_{i=1}^{l_0} \omega(v_i) = -1$ and $\alpha - 2 + \sum_{i=1}^{j} \omega(v_i) \geqslant 0$ for $j (1 \leqslant j < l_0)$. Assuming that $w_2 = (v_{l_0+1}, \ldots, v_n)$ on the ground of based on Theorem 3. $(\alpha, \beta) = (\alpha - 1, 0) \circ (x, y)$ where $(x, y) = \mathcal{X}(w_2)$, after the solution of the equation, $(x, y) = (1, \beta)$. Thus $v = w_1 * w_2$ where $\mathcal{X}(w_1) = (\alpha - 1, 0)$ and $\mathcal{X}(w_2) = (1, \beta)$. Similarly just like in Lemma 11 the unambiguity of the decomposition is proved. □

Proof of Theorem 10 directly follows from Lemmas 11 and 12.

Elements $(t_1, \ldots, t_{\alpha-1})$ are called the term components of expression v, element u – the degenerated component, if $\mathcal{X} = (1, \beta)$ and $\beta > 0$; but if $\beta = 0$ then element $t_\alpha = u$ is also the term component of expression v.

10. Supplements

In the preceding paragraphs we defined the characteristic $\mathcal{X}$ and decomposition theorem by means of the definition based on the pattern of expression. To study PF-notation we may also use graph theory by applying this notation to objects of the graph theory [7].

Definition 13. *Let $v = (v_1, \ldots, v_n) \in \overset{*}{V}$. The graph of expression v is referred to as pair (X_v, U_v) where*

1° $X_v = \{1, 2, \ldots, n\}$,

2° $U_v \subset X_v \times X_v$,

3° pair $(j, i) \in X_v \times X_v$ belongs to U_v if and only if the following conditions are met:

a) $j > i$,

b) $v_i \in F$,

c) $W_2(\omega(v_i, \ldots, v_j)) = 1$,

d) for $m (i < m < j) \quad W_2(\omega(v_m, \ldots, v_j)) = 0$.

Definition 14. *The tree graph shall be labeled as the connectivity graph without loops and cycles.*

Definition 15. *The forest graph shall be labeled as no connectivity graph without loops and cycles.*

Theorem 16. *Let $v \in \overset{*}{V}$ and $G_v(X_v, U_v)$ be a graphs of expression v.*
1° Graph $G_v(X_v, U_v)$ is a tree or a forest.
2° $W_2(\omega(v)) = 1$ if and only if $G_v(X_v, U_v)$ is connective.
3° Graph $G_v(X_v, U_v)$ has $(|X_v| - |U_v|)$ connective elements.

Proof. 1° From definition of the graph of expression, point 3a eliminates the cycles and loops. Because from definition it also follows that for any j there is only one pair $(j, i) \in X_v \times X_v$, which belongs to U_v, which definitely eliminates any cycles.

2° Let $v = (v_1, \ldots, v_n)$ and $G_v(X_v, U_v)$ be the graph of expression v. If $W_2(\omega(v)) = 1$, then for each $j \in X_v$ and $j \neq 1$ there exists $i \in X_v$ such that $(j, i) \in U_v$ and $j > i$ and, in consequence, there is a path to node 1. This property is equivalent to the connectivity of graph $G_v(X_v, U_v)$.

If graph $G_v(X_v, U_v)$ is connective, there exist a path from any vertex $j \in X_v \quad j \neq 1$ to vertex 1. Let $1 = j_0 < j_1 < \ldots < j_{k-1} < j_k = n$ vertexes of the path connecting n with 1. Then, for any l $(1 \leqslant l \leqslant k)$ $W_2(\omega(v_{j_{l-1}}, \ldots, v_{j_l})) = 1$ and for any m $(j_{l-1} < m < j_l)$ $W_2(\omega(v_m, \ldots, v_{j_l}) = 0$, which implies that $W_2(\omega(v)) = 1$.

3° If graph $G_v(X_v, U_v)$ is a tree, there $|X_v| - |U_v| = 1$. If graph $G_v(X_v, U_v)$ is a forest, that it is a disjoint union of trees. Let any graph of expression v is a forest and has q connective elements. Let $D_i(X_i, U_i)$ be the ith element. Because for any trees $|X_i| - |U_i| = 1$, so $|X_v| - |U_v| = |X_1| + \ldots + |X_q| - |U_1| + \ldots + |U_q| = q$, which concludes the proof. □

Let $v = (v_1, \ldots, v_n) \in \overset{*}{V}$. Number

$$\beta(v) = |X_v| - |U_v| + \sum_{i=1}^{n} \omega(v_i) \tag{25}$$

shall be referred to as the number of the degeneration of graph $G_v(X_v, U_v)$.

The followings properties are true:

$(W1) \qquad \beta(v) \geqslant 0 \qquad \text{for } v \in \overset{*}{V},$

$(W2) \qquad \beta(v) = 0 \qquad$ if and only if, where exists a natural k that $v \in T^k$.

It should be noticed that if $v = (v_1, \ldots, v_n)$ $\sum_{i=1}^{n} \omega(v_i) = -k$, where $k \in N$ graph $G_v(X_v, U_v)$ has minimally k connective elements, because there exists a sequence $l_0, l_1, \ldots, l_k$ such that $1 = l_0 < l_1 < \ldots < l_{k-1} < l_k \leqslant n$ and for $i = 1, \ldots, k$ $W_1(\omega(v_{l_{i-1}}, \ldots, v_{l_i})) = 1$, $W_2(\omega(v_{l_{i_1}}, \ldots, v_{l_i})) = 1$ and $(\{l_{i-1}, \ldots, l_i\} \times \{l_{j-1}, \ldots, l_j\}) \cap U_v = \varnothing$ which follows from the definition of the graph of expression, and, accordingly $(W1)$ and $(W2)$.

If $v \in \overset{*}{V}$ and $W_2(\omega(v)) = 1$ graph $G_v(X_v, U_v)$ is called

1° the tree of term $\beta(v)$ – degenerated when, $\beta(v) > 0$,

2° the tree of term, when $\beta(v) = 0$.

Definition 17 (by graph).

$$\mathcal{X} : \overset{*}{V} \to N \times N_0$$

For any $v \in \overset{*}{V}$ $\qquad \mathcal{X}(v) = (\alpha, \beta)$ *where*

(1") α – is the number of the connected components of graph $G_v(X_v, U_v)$;

(2") β – is the number of the degeneration of graph $G_v(X_v, U_v)$.

$\mathcal{X}$ is labeled as the characteristic of expression v.

Theorem 18 (decomposition graph). *Let $v \in \overset{*}{V}$ and $G_v(X_v, U_v)$ be the graph of expression v. If $\mathcal{X}(v) = (\alpha, \beta)$ then graph of expression v has α connected components, where*

1° α components are the trees of term, if $\beta = 0$;

2° $\alpha - 1$ components are the trees of term and exactly one component is the $\beta(v)$-degenerated tree of term, where $\beta > 0$.

Let $\langle \alpha, \beta \rangle$ is a set of all expressions $v \in \overset{*}{V}$ of notation $\mathbf{V}(\mathcal{F}, \mathcal{A}) = \mathbf{F} \cup \mathbf{A}$, such that $\mathcal{X}(v) = (\alpha, \beta)$.

Theorem 19. *Let $H \subset \mathbf{F}$ and $H \neq \varnothing$. If $f \in H$ and $arn(f) \geqslant 2$ then for any $(\alpha, \beta) \in N \times N_0$:*

$$\overset{*}{V}_H \ \cap\ \langle \alpha, \beta \rangle \neq \varnothing. \tag{26}$$

Proof. To prove this theorem the properties of the characteristic $\mathcal{X}$ and operation $\circ$ are used. Let $f \in H$ and $arn(f) = 2$. Then $\mathcal{X}(f) = (1, 2)$. Let's create expressions $x_1, \ldots$ as follows

$$x_1 = f * v \text{ when } v \in A,$$
$$x_2 = f \text{ and for } \beta > 2 \qquad x_\beta = \underbrace{f * f * \ldots * f}_{\beta - 1}.$$

It easy to notice that $\mathcal{X}(x_i) = (1, i)$ for $i = 1, 2, \ldots$. Let $(\alpha, \beta) \in N \times N_0$ and $\alpha > 1$ and $x = \underbrace{v * v * \ldots * v}_{\alpha-1} * x_\beta$ Hence

$$\mathcal{X}(x) = \underbrace{(1, 0) \circ \ldots \circ (1, 0)}_{\alpha - 1} \circ (1, \beta) = (\alpha, \beta)$$

which proves the theorem for $f \in H$ and $arn(f) = 2$. If $arn(f) = k > 2$ then

$$y = f * \underbrace{v * \ldots * v}_{k-2}$$

has characteristic $(1, 2)$ and we the same logic can be applied substituting the expression y for f, which proves the correctness of the theorem. $\square$

Let $v \in \overset{*}{V}$:

$$E_L(v) = \left\{ w \in \overset{*}{V} \mid \mathcal{X}(w * v) = \mathcal{X}(v) \right\} \tag{27}$$

— a set of left-sided invariants for v,

$$E_P(v) = \left\{ w \in \overset{*}{V} \mid \mathcal{X}(v * w) = \mathcal{X}(v) \right\} \tag{28}$$

— a set of right-sided invariants for v,

$$E_T(v) = \left\{ (u, w) \in (\overset{*}{V})^2 \mid \mathcal{X}(u * v * w) = (1, 0) \right\} \tag{29}$$

— a set of the pair of expressions normalizing v in respect of the term.

It's easy to prove for above-mentioned sets the next properties.

Let $v \in \overset{*}{V}$ and $\mathcal{X}(v) = (\alpha, \beta)$. Then

$$E_L(v) = \bigcup_{1 \leqslant x \leqslant \alpha} \langle x, x \rangle, \tag{30}$$

$$E_P(v) = \bigcup_{1 \leqslant x \leqslant \beta} \langle x, x \rangle, \tag{31}$$

$$E_T(v) = \bigcup_{\alpha \leqslant x} (\langle 1, x \rangle \times \langle x - (\alpha - \beta), 0 \rangle). \tag{32}$$

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