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An exponential Diophatine equation on Triangular numbers

Abstract Looking to the two remarkable identities concerning triangular numbers $T_{n+1} - T_n = n+1$ and $T_{n+1}^2 - T_n^2 = (n+1)^3$, we can extend these equations to the exponential Diophantine equation $T_{n+1}^x - T_n^x = (n+1)^y$ for some positive integers x, y. In this papaer, we show that the above equation has only the solutions $(x, y) = (1, 1)$ or $(2, 3)$.

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1. Introduction. By a triangular number we call the number of the form

$$T_n = 1 + 2 + ... + n = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

when n is a natural number. The number T_n can be interpreted as a number of circles necessary to build an equilateral triangle with side of length n. Diophantine equations related to Triangular numbers has a long story. Walcaw Sierpinski in the booklet [12] and in the papers [10, 11, 12] gave many interesting results concerning the problem of solvability of diophantine equations related to triangular numbers. Hamtat and Behloul (2017) had used matrix theory to give some famillies of integer solutions to the equation

$$T_x + T_y = T_z,$$

when x,y and z are positive integers.

In triangular numbers, we have some remarkables identities such

$$T_{n+1} - T_n = n + 1,$$

and

$$T_{n+1}^2 - T_n^2 = (n+1)^3.$$

So, a natural question can be asked, for which values of positive integers (x, y), the exponential Diophantine equation

$$T_{n+1}^x - T_n^x = (n+1)^y. \qquad (1)$$

holds for all positive integers n ?

An exponential Diophantine equation is an equation of the form

$$a^x + b^y = c^z. \tag{2}$$

where a,b and c are pairewise coprime positive integers. The equation (2) has a very rich story. In 1933, K. Mehler [8] has proved that the equation (2) has only many finitly of integer solutions, under the hypothesis that $a, b, c > 1$. His method is a p-adic analogue of that given by Thue-Siegel, so it is ineffective in the sense that it gives no indication on the number of possible solutions. A few years later, Gel'fond [4] gave an effective result for solutions of (2). His method was based on Baker's theory, which uses linear forms in the logarithms of algebraic numbers. In 1956, L. Jésmanowicz [6] conjectured that if a, b, c are Pythagorean numbers, i.e., positive integers satisfying $a^2 + b^2 = c^2$, then the Diophantine equation $a^x + b^y = c^z$ has only the positive integral solution $(x, y, z) = (2, 2, 2)$. This conjecture have been solved for many special cases. Different conjectures concerning (2) were identified and discussed. Terai [13] proposed that if a, b, c, p, q, r are fixed positive integers satisfying $a^p + b^q = c^r$ with $a, b, c, p, q, r \geq 2$ and $gcd(a, b) = 1$, then equation (2) has only the trivial solution $(x, y, z) = (p, q, r)$ except for a handful of triples (a, b, c). This conjecture has been proved to be true in many special cases. However, it is still unsolved in its full generality.

The generalisation of the two above identities of triangular numbers give us an exponential Diophantine equation of the form of (2). (1) considered as an exponential Diophantine equation, we shall prove the following result

Theorem 1 *(Main theorem) Equation* (1) *has only the positive integer solutions* $(x, y) = (1, 1), (2, 3)$. □

The organisation of this paper is as follows: In Section 2, we recall a result due to W.J. LeVesque [7]. Also we prove a result by the means of Lucas sequences. These results are useful for the proof of our main theorem that will be shown in Section 3. First, for $n = 1$, LeVeque's result helps to find the solutions $(1, 1)$ and $(2, 3)$. Then we use an elementary method to prove that equation (1) has only the solutions $(1, 1)$ and $(2, 3)$, for $n \geq 2$.

2. Some Lemmas

First, we recall a simplified result due to LeVesque [7]

Lemma 2.1 *For a fixed integers* $a > 1$ *and* $b > 1$*, the Diophnatine equation*

$$a^x - b^y = 1.$$

has just the two solutions $(1, 1)$ *and* $(2, 3)$ *if* a, $b = 2$. *In all other cases, it has at most one solution.*

In 2016, Miyazaki, Togbé and Yuan [9] gave the following result in :

LEMMA 2.2 *Suppose that $a > 1$ is an odd positive integer. Then the Diophantine equation*

$$a^x + 2^y = (a+2)^z. \tag{3}$$

has only the positive solution $(x, y, z) = (1, 1, 1)$, whenever neither $a = 2^{k-1} - 1$ with an integer $k \geq 3$ nor $a = 89$. If $a = 2^{k-1} - 1$ or $a = 89$, then the additional solutions are given by $(2, k+1, 2), (1, 13, 2)$, respectively.

Let α, β be algebraic integers. If $\alpha + \beta$ and $\alpha\beta$ are non-zero coprime rational integers and $\frac{\alpha}{\beta}$ is not a root of unity, then (α, β) is called a Lucas pair. Given a Lucas pair (α, β), one defines the corresponding sequence of Lucas numbers by

$$U_n(\alpha, \beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

A prime p is called a primitive divisor of $U_n(\alpha, \beta)$ if $p|U_n$ and

$$p \nmid (\alpha^2, \beta^2) U_1 U_2 ... U_{n-1}.$$

An important problem is the existence of primitive divisor of Lucas numbers. Bilu et al. (2001) solved the problem. The remaining cases were solved by Abouzaid (2006). The case when α, β are integers was solved by Birkhoff–Vandiver [3] and Zsigmondy [14] in 1904 and 1892, independently. They proved that $U_n(\alpha, \beta)$ has a primitive divisor if $n > 6$. Following Lemma is an early version of the primitive divisor theorem for integers which is known as Zsigmondy's theorem [14].

LEMMA 2.3 *Let $a > b \geq 1$ be a be relatively prime integers and let $(U_n)_{n\geq 1}$ be a sequence defined as*

$$U_n(a, b) = a^n - b^n$$

If $n > 1$ then U_n has a primitive divisor, except for $(a, b, n) = (2, 1, 6)$ or $n = 2$ and $a + b = 2^k$ for some positive integer k.

3. Proof of our Theorem

In this section, we prove our theorem, we distinguish two cases from the parity of n.

When $n = 1$, the equation (1) becomes

$$3^x - 1 = 2^y$$

it has only two positive solutions $(x, y) = (1, 1)$, $(2, 3)$ by Lemma 2.1 . Now we suppose that $n \geq 2$.

Case 1: n **is even**.

Lets put $n = 2m$ with $m \geq 1$ We rewrite the equation (1) into the form

$$((2m+1)(m+1))^x - (m(2m+1))^x = (2m+1)^y, \tag{4}$$

1- If $x = y$, the the above equation (4) becomes

$$(m+1)^x - m^x = 1,$$

if $x \geq 2$, then we get

$$1 = (m+1)^x - m^x \geq (m+1)^2 - m^2 = 2m+1,$$

and this is a contradiction since $m \geq 1$, then we get $x = y = 1$. Therefore we obtain the first solution of (1).

2- If $x > y$, then the equation (4) becomes

$$(2m+1)^{x-y}((m+1)^x - m^x) = 1,$$

one can see that is equivalent to

$$(2m+1)^{x-y} = 1,$$

and this is impossile since $x > y$.

3- If $x < y$, in this case the equation (4) becomes

$$(m+1)^x - m^x = (2m+1)^z, z = y - x \geq 1 \tag{5}$$

Let assume that $z = 1$, in this case the equation (5) is equivalent to

$$(m+1)^x - m^x = (2m+1),$$

for $x \geq 3$, we have

$$2m+1 = (m+1)^x - m^x \geq (m+1)^3 - m^3 = 3m^2 + 3m + 1$$

and this is impossible since $m \geq 1$. Then we get $x = 1$ or $x = 2$. For $x = 1$, we get $z = 0$ which is impossible. So $x = 2$ and $y = 3$. Therefore we obtain the second solution of (1). Now we assume that $z > 1$, we rewrite the equation (5) as

$$(2m+1)^z = (m+1)^x - m^x = U_x(m+1, m).$$

Clearly $m+1$ and m are relatively prime . Since $m > 1$, from (5), we see that the sequence U_x has no primitive divisor since $x \geq 1$. From Lemma 2.3, we have two possibilities: either $m+1 = 2$, which is impossible since $m > 1$, or $x = 2$ and $2m+1 = 2^k$ for some positive integer k. But if $x = 2$, we get $z = 1$, which is impossible since $z > 1$.

Case 2: n **is odd**.

Assuming now, that n is odd, let $n = 2m - 1$, with $m \geq 2$. Equation (1) becomes

$$(m(2m+1))^x - (m(2m-1))^x = (2m)^y, \tag{6}$$

and we distinguish as always three subcases :

- If $x = y$, then equation (6) is equivalent to

$$(2m+1)^x - (2m-1)^x = 2^y, \tag{7}$$

Using Lemma 2.2, equation (7) has as first solution $x = y = 1$. For the additional solutions, we have $(x, y) = (2, k+1)$ when $a = 2m - 1 = 2^{k-1} - 1$, so $m = 2^{k-2}$ for some integer $k \geq 3$. In this case, we must have $2 = k + 1$ because $x = y$, so we get $k = 1$, which is impossible since $k \geq 3$. The second possibility is absurde. It follows that the equation (7) does not have a solution with $x \geq 2$.

- If $x > y$, equation (6) is equivalent to

$$m^{x-y}((2m+1)^x - (2m-1)^x) = 2^y, .$$

it is easy to see that the case when $y = 1$ leads us to the first solution of (1). Then we take $y \geq 2$. Using the factorisation method, since 2 is prime then, we get

$$m^{x-y} = 2^s,$$

and

$$(2m+1)^x - (2m-1)^x = 2^t, \tag{8}$$

for some positive integers $s > 0$, $t > 0$ such that $s + t = y$. Similary as in case of equation (6), the equation (8) has a solution in positive integers only if $x = t = 1$. Contradiction since $y \geq 2$, so (8) does not have solutions.

- If $x < y$, equation (6) becomes

$$(2m+1)^x - (2m-1)^x = 2^y m^{y-x} \tag{9}$$

Abviously, $x = 1$ implies $y = 1$. Then we take $y > x \geq 2$, consider the equation (9) modulo 4, we get

$$1 - (-1)^x \equiv (0 \mod 4)$$

it follows that x is even, let put $x = 2X$, with $X \geq 1$, equation (9) is equivalent to

$$(2m+1)^{2X} - (2m-1)^{2X} = 2^y m^{y-2X} \tag{10}$$

we factor the above expression to obtain

$$((2m+1)^X + (2m-1)^X)((2m+1)^X - (2m-1)^X) = 2^y m^{y-2X} \tag{11}$$

From equation (11) , we introduce two even positive integers P and Q as follows:

$$P \cdot Q = 2^y m^{y-2X},$$

where

$$P = (2m+1)^X + (2m-1)^X, \tag{12}$$

and

$$Q = (2m+1)^X - (2m-1)^X. \tag{13}$$

We note that

$$P \equiv 0, 2 \pmod 4$$

and

$$Q \equiv 0, 2 \pmod 4$$

By expressions (12) and (13), we see that $pgcd(P, Q) = 2$ and $P + Q \equiv 0, 2 \pmod 4$, so $P/2$ and $Q/2$ are integers of different parities, we distinguish two cases :

1. m is odd: in this case ,we have $pgcd(Q/2, m) = 1$. Since

$$2^{y-2} \cdot m^{y-2X} = (P/2)(Q/2),$$

and $Q/2$ is coprime to m, it follows that only $Q/2 = 1$ or $Q/2 = 2^{y-2}$ are possible, and we have

$$(2m+1)^X - (2m-1)^X = 2,$$

it is follows that $X = 1$, so $x = 2$, substuting this value in (11), we get

$$4m = 2^y m^{y-2},$$

but m is odd, so $m^{y-2} = 1$ which implies that $y = 2$. Contradiction since $x > y$. The second possibility give

$$(2m+1)^X - (2m-1)^X = 2^{y-1}.$$

However, we have already shown that the above equation does not have a solutions in positive integers only if $X = 1$ and $y - 1 = 1$, it follows that $x = y = 2$. Contradiction.

2. m is even: We will prove this case in two steps :

a. $y - x \geq 2$. From the equation (10) one can see that

$$1 + 2mX + (-1)^X - 2mX \equiv 0 \pmod{4m^2},$$

it follows that

$$1 + (-1)^X \equiv 0 \pmod{4m^2}. \tag{14}$$

Suppose that X is even, we get from (14), that

$$2 \equiv 0 \pmod{4m^2},$$

contradiction sicnce $m \geq 2$. So, X must be odd, it follows that

$$Q/2 \equiv 0 \pmod{4m^2},$$

only the case when $Q = 2$ is possible. Hence, one has

$$(2m+1)^X - (2m-1)^X = 2,$$

and

$$(2m+1)^X - (2m-1)^X = 2^{y-1}m^{y-2X}.$$

from the first equation, we get $X = 1$, so $x = 2$, and

$$4m = 2^{y-1}m^{y-2},$$

which implies that $y = 3$. Therefore we obtain the second solution of our equation (1).

b. Now , we consider the equation (9) for $y - x = 1$, then we get

$$(2m+1)^x - (2m-1)^x = 2^y m,$$

by the same argument, one can see that

$$Q/2 \equiv 1(\mod 2m),$$

only the case when $Q = 2$ is possible . Hence, we can deduce that

$$(2m+1)^X - (2m-1)^X = 2,$$

and

$$(2m+1)^X - (2m-1)^X = 2^{y-1}m.$$

but the first equation gives us $X = 1$, so $x = 2$, it follows that $y = 3$. This completes the proof of our main theorem.

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Wykładnicze równanie diofantyczne na liczbach trójkątnych.

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Streszczenie Liczby trójkątne mają tę właściwość, że różnica dwóch kolejnych wyrazów ciągu liczb trójkątnych jest równa indeksowi pierwszego. Mają też tę właściwość, że różnica kwadratów kolejnych liczb jest równa trzeciej potędze indeksu pierwszej liczby. Celem pracy jest zbadanie analogicznego difantycznego równania wykładniczego. Rozważmy równanie: różnica x-tych potęg ciągu liczb trójkątnych jest równa potędze y indeksu pierwszej, dla pewnych dodatnich liczb całkowitych x, y. Pokazujemy, że powyższe równanie ma tylko rozwiązania $(x, y) = (1, 1)$ i $(2, 3)$.

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