ECONTECHMOD. AN INTERNATIONAL QUARTERLY JOURNAL – 2013, Vol. 2, No. 1, 15–22

Oligopolistic market: stability conditions of the equilibrium point of the generalized Cournot-Puu model

N. IWASZCZUK[1], I. KAVALETS[2]

[1] AGH University of Science and Technology, Krakow, Faculty of Management;
[2] Lviv Polytechnic National University, Lviv, Institute of Applied Mathematics and Fundamental Sciences, Applied Mathematics Department

Received July 08.2012: accepted July 30.2012

Abstract. The paper presents a model describing the behavior of participants of the oligopolistic market. Economic model of the oligopoly – a generalized Cournot-Puu model is constructed. Notion of Cournot equilibrium is introduced. Study on the stability of the equilibrium point of the constructed model is described. As an example, the model of duopoly is considered in detail.

Key words: oligopoly, duopoly, a generalized Cournot-Puu model, Cournot equilibrium, linearization of the system, stability, Routh-Hurwitz procedure.

INTRODUCTION

Many scientific papers have been devoted to investigations of the enterprises stability in different economic conditions. In particular, Chukhray N. studied the competition as a strategy of enterprise functioning in the ecosystem of innovations [16]. Feshchur R., Samulyak V., Shyshkovskyi S. and Yavorska N. analyzed different analytical instruments of management development of industrial enterprises [21]. Moroz O., Karachyna N. and Filatova L. studied economic behavior of machine-building enterprises in analytic and managerial aspects [26]. In turn, Petrovich J.P. and Nowakiwskii I.I. analyzed the modern concept of a model design of an organizational system of enterprise management [28]. In this article we examine the behavior of enterprises in oligopolistic market.

Two main types of market structure without high competition are described in the scientific literature. This is an oligopoly and oligopsony. *Oligopoly* is a such market structure in which a few large manufacturing firms dominates. *Oligopsony* is a market structure in which a few large customers dominates. When they say “big three”, “big four” or “big six”, then we are talking about oligopoly.

Oligopoly is a market structure in which the small number of rival firms dominates in the same sector. One or two of them produce a significant share of production in this industry. The emergence of new vendors is difficult or impossible. Typically, there are from two to ten firms in oligopolistic markets. They account for half or more of total product sales. In such markets all or some of the firms obtain substantial profit in the long time interval, because entry barriers make it difficult or impossible to input of firms-newcomers to the market of this product. A product may be *homogeneous* (standardized) and *heterogeneous* (differentiated) on the oligopolistic market. If the market sells a homogeneous product (i.e., buyers have no choice), we are dealing with *homogeneous oligopoly*, and if the various product (i.e., buyers can choose according to their preferences), we are dealing with a *heterogeneous oligopoly* (differentiated oligopoly) [19].

Oligopoly is the predominant form of market structure. Automotive industry, steelmaking industry, petrochemical industry, electrical industry, energy industry, computer industry and others belong to oligopolistic industries. In the oligopolistic markets, some of the firms can exert influence on the product price because they cover a significant share of its products in total manufactured product. Sellers are aware of their interdependence in this market. It is assumed that each firm in the industry recognizes that a change in its price or output provokes a reaction with other firms. The reaction, which is one of the oligopolist firms expects from competing firms in response to changes in prices established by it, output of production or changes in the marketing strategy is the main factor that determines its decision. Such reactions can influence the equilibrium of oligopolistic markets.

Sufficiently large number of models describing the behavior of firms in oligopolistic market is known today.

Oligopolistic markets are distinguished after this sign, or their members-oligopolists operate completely independently of each other, at their own risk, or, alternatively, enter into a conspiracy that may be obvious, open or secret (closed). In the first case, we usually say about *noncooperative oligopoly*, and in the second case we say about *cooperative oligopoly*, one of the forms of which is a cartel.

Obviously, when we analyze the behavior of oligopolists operating completely independently of each other, i.e. in the case of noncooperative oligopoly, differences in assumptions regarding the reaction of competitors are crucial. Depending on what oligopolist chooses control variables – the value of output or price – we distinguish oligopoly of firms that set the value of output, called *quantitative oligopoly,* and oligopoly of firms that set price, called *price oligopoly.*

There are models of quantitative oligopoly: Cournot model (*Antoine-Augustin Cournot, 1838*), Chamberlain model (*Edward Hastings Chamberlin*), and Stackelberg model (*Heinrich von Stackelberg*), which offers an asymmetric behavior of oligopolists; and models of pricing oligopoly: Bertrand model (*Joseph Louis François Bertrand, 1883*), Edgeworth model (*Francis Ysidro Edgeworth*) and Sweezy model (*Paul Sweezy*).

Let us consider Cournot model, from which the modern theory of oligopoly began. Basic model of oligopoly was proposed in 1838 by the French mathematician and economist Antoine Augustin Cournot. In the work [17, 30] he posed the problem of oligopolistic interdependence and the need for each firm in determining its market strategy to take into account the behavior of competitors.

Cournot considered duopoly, i.e. situation when there are only two firms on the market. It is assumed in this model that both firms produce a standardized product (with the same parameters) and know the market demand curve. Based on this, each firm determines its output, taking into account that its competitor also make decisions about their own output similar product. Moreover, the final price of the product will depend on the total production (both firms together) that hits the market.

Thus, the essence of Cournot model is that each firm takes the output of its rival constant. Based on the data and information about the market demand for the product, the firm makes its own decision on the establishment of such volumes of its production, which would provide the maximum profit (based on compliance with the rules of equality of marginal revenue and marginal cost). Thus, the main problem of this model is to determine at which of output both firms reach equilibrium.

Cournot oligopoly model is the most actively studied, although initially the Cournot ideas have been criticized for their simplicity. Various modifications of the model have been made by many scientists. This enabled to improve it.

In particular, T. Puu in 1991 [29], while studying Cournot model, introduced another type of economic conditions, i.e. iso-elastic demand with different constant marginal costs, under which meaningful unimodal reaction function were developed. Since the model has been discussed in numerous amounts of publications [13, 32]. Several models were generalized by using adaptive rules and heterogeneous participants [5, 7, 8, 9, 10, 12, 37].

New properties of the Cournot-Puu model were proposed by T. Puu in one of his recent publications [36].

The research of Cournot model showed that it has an ample dynamic behavior. Some authors considered the quantitative oligopoly with homogeneous expectations and found a variety of complex dynamics, such as the appearance of strange attractors with fractal dimensions [2, 3]. The complex chaotic behavior in Cournot-Puu duopoly model has been studied in recent works [7, 14, 22].

Discrete dynamics of the triopoly game with homogeneous expectations is considered in the following works [1, 6]. The authors of these works have shown that the dynamics of Cournot oligopoly games may never reach the point of equilibrium and in the long run bounded periodic or chaotic behavior may be observed. Model with heterogeneous players were studied later, like in the works [18, 23].

B. Rosser also made its contribution to the theory of oligopoly. In the work [33] he made a detailed review of the theoretical development of oligopoly, namely, heterogeneous expectations, dynamics and stability of the market. Onozaki et al. investigated the stability, chaos and multiple attractors of heterogeneous two-dimensional cobweb model in the paper [27].

Recent studies of the duopoly and triopoly dynamics of Cournot model with heterogeneous players are presented in the works [4, 5, 19]. Problems of construction and study of models with N heterogeneous players are alternative in this direction.

Considering the numerous studies that show the chaotic dynamics in the Cournot-Puu oligopoly model (duopoly and triopoly model with homogeneous and heterogeneous players), there is the problem of control the chaos that occurs in these models. Some methods, such as DFC-method [15], OGY-method for controlling the chaos, pole placement method [24] were applied to the Cournot-Puu duopoly model. But studies of oligopolistic market only in case of duopoly is very limited, and therefore the question of building a generalized model arises naturally.

Some aspects of the nonlinear model of oligopoly in the case N firms were considered in recent works [25, 31]. Therefore, our alternative future research is to build a generalized model of Cournot-Puu and to investigate the stability of the equilibrium point and to apply of methods of control the chaos that occurs in this model.

In this work the generalized model of oligopoly Cournot-Puu is considered and the concept of Cournot equilibrium is introduced. A significant result is to establish conditions under which the equilibrium point is stable.

GENERALIZED COURNOT-PUU MODEL

To construct a model, we need to describe the behavior of market participants: motivation of their behavior, conditions in the market and the restrictions which they face.

Let n firms operate in an oligopolistic markets, $n \geq 2$. (If $n = 1$ we have a situation of monopoly.) Denote the oligopolist firms by $F_1, F_2, \ldots, F_n$, which produce quantities $q_1, q_2, \ldots, q_n$ respectively. Let's introduce assumption of Cournot and Puu to get the reaction functions.

Cournot assumption (generalized). Each firm i (i=1,2,…,n) expects its rival j (j=1,2,…,n, $j \neq i$) to offer the same quantity for sale in the current period as it did in the preceding period.

According to this assumption, the general reaction functions of each firm are as follows:

$$\begin{aligned} &q_1(t+1) = f_1\left(q_2(t), q_3(t), \ldots, q_n\right), \\ &q_2(t+1) = f_2\left(q_1(t), q_3(t), \ldots, q_n\right), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &q_n(t+1) = f_n\left(q_1(t), q2(t), \ldots, q_{n-1}\right). \end{aligned} \quad (1)$$

Reaction function is a curve that shows the output produced by one firm for each given output of another firm. The set of points on the reaction curve shows what the reaction will be of one of the firms (when choosing the amount of own manufacture) to the decision of other firms regarding their output. Thus, each of the functions $q_i(t+1)$ is a reaction curve of oligopolist i on output offered by other oligopolists.

Puu assumption 1 (generalized). The market demand is assumed to be iso-elastic, so that price p is reciprocal to the total demand q, i.e., $p = 1/q$.

Puu assumption 2 (generalized). Goods are perfect substitutes, so that demand equals supply, i.e., $q = q_1 + q_2 + \ldots + q_n$.

Puu assumption 3 (generalized). The competitors have constant but different marginal costs, denoted by c_i, $i = 1, \ldots, n$.

Based on these assumptions, the profit of firm F_i (i=1,2,…,n) becomes:

$$U_i(t+1) = \frac{q_i(t+1)}{q_i(t+1) + \sum\limits_{j=1, j\neq i}^{n} q_j(t)} - c_i q_i(t+1). \quad (2)$$

Each of e firms wants to reach such output that would maximize its income:

$$\frac{\partial U_i(t+1)}{\partial q_i(t+1)} = \frac{q_i(t+1) + \sum\limits_{j=1, j\neq i}^{n} q_j(t) - \left(1 + \sum\limits_{j=1, j\neq i}^{n} \frac{\partial q_j(t)}{\partial q_i(t+1)}\right) q_i(t+1)}{\left(q_i(t+1) + \sum\limits_{j=1, j\neq i}^{n} q_j(t)\right)^2} - c_i = 0. \quad (3)$$

Hence, given the Cournot assumption that:

$$\frac{\partial q_j(t)}{\partial q_i(t+1)} = 0, \quad i \neq j, \quad (4)$$

obtain the equation:

$$\sum_{j=1, j\neq i}^{n} q_j(t) - c_i \left[q_i(t+1) + \sum_{j=1, j\neq i}^{n} q_j(t) \right]^2 = 0, \quad i = \overline{1,n}. \quad (5)$$

The solutions of equations (5) are the reaction function for firms $F_1, F_2, \ldots, F_n$. Then we will have a system of equations:

$$q_i(t+1) = \sqrt{\sum_{j=1, j\neq i}^{n} \frac{q_j(t)}{c_i}} - \sum_{j=1, j\neq i}^{n} q_j(t), \quad i = \overline{1,n}. \quad (6)$$

We need to solve the system (6) to find the equilibrium points. We obtain two equilibrium points: a trivial $\underbrace{(0,0,\ldots,0)}_{n}$ and non-trivial $\left(q_1^*, q_2^*, \ldots, q_n^*\right)$. All the future research will deal with the only nontrivial point, called the *Cournot equilibrium* or *Nash equilibrium*.

The value of the equilibrium point we can write as:

$$q_i^* = (n-1) \frac{-(n-2)c_i + \sum\limits_{j=1, j\neq i}^{n} c_j}{\left(\sum\limits_{j=1}^{n} c_j\right)^2}, \quad i = \overline{1,n}. \quad (7)$$

METHODOLOGY OF THE EQUILIBRIUM POINT STABILITY ASSESSMENT

Let us investigate the stability of the equilibrium point $\left(q_1^*, q_2^*, \ldots, q_n^*\right)$. We linearize the system (6) at the equilibrium point. Denote:

$$\delta q_i(t) = q_i(t) - q_i^*, \quad i = 1, 2, \ldots, n, \quad (8)$$

and proceed to deviations:

$$\begin{aligned} \delta q_i(t+1) + q_i^* &= \sqrt{\sum_{j=1, j\neq i}^{n} \frac{q_j^*}{c_i}} - \sum_{j=1, j\neq i}^{n} q_j^* + \\ &+ \sum_{j=1}^{n} \left[\frac{\partial q_i(t+1)}{\partial q_j(t)} \right]_{\left(q_1^*, q_2^*, \ldots, q_n^*\right)} \cdot \delta q_j(t), \quad i = \overline{1,n}. \end{aligned} \quad (9)$$

We linearize the system (9), substitute the value of the equilibrium point (7) and write the resulting system in matrix form:

$$\delta q(t+1) = J \cdot \delta q(t), \quad (10)$$

where:

$$\begin{aligned} &\delta q(t+1) = \left(\delta q_1(t+1), \delta q_2(t+1), \ldots, \delta q_n(t+1)\right)^T, \\ &\delta q(t) = \left(\delta q_1(t), \delta q_2(t), \ldots, \delta q_n(t)\right)^T. \end{aligned} \quad (11)$$

J is Jacobi matrix of the linearized system:

$$J=\begin{bmatrix} 0 & p_1 & \dots & \dots & \dots & \dots & \dots & p_1 \\ p_2 & 0 & p_2 & \dots & \dots & \dots & \dots & p_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ p_i & \dots & \dots & p_i & \underbrace{0}_{i-\text{th position}} & p_i & \dots & p_i \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ p_n & \dots & \dots & \dots & \dots & \dots & p_n & 0 \end{bmatrix}, \qquad (12)$$

elements p_i of this matrix are given by:

$$p_i=\frac{c_1+c_2+\ldots+(3-2n)c_i+\ldots+c_n}{2(n-1)c_i}, \quad i=\overline{1,n}. \qquad (13)$$

The stability of system (10) is governed by its characteristic equation:

$$\det(J-\lambda I)=0, \qquad (14)$$

or

$$\lambda^n+a_1\lambda^{n-1}+\ldots+a_{n-1}\lambda+a_n=0. \qquad (15)$$

As it is known [34], the construction of the analytical form of the coefficients of the characteristic polynomial (15) can be carried out using the principal minors of Jacobi matrix J.

The coefficient at λ^{n-1} is equal to the trace of the matrix, taken as negative. As in our case all diagonal elements are equal to zero, then:

$$a_1=-trJ=0. \qquad (16)$$

Free member a_n of the characteristic polynomial (15) of Jacobi matrix J is equal to the determinant of this matrix multiplied by $(-1)^n$ where n is the order of matrix. So:

$$a_n=(-1)^n\begin{vmatrix} 0 & p_1 & \dots & p_1 \\ p_2 & 0 & \dots & p_2 \\ \dots & \dots & \dots & \dots \\ p_n & p_n & \dots & 0 \end{vmatrix}. \qquad (17)$$

We construct coefficients a_i, $i=\overline{2,n-1}$ at λ^m, $m=\overline{n-2,1}$ by the formula:

$$a_i=(-1)^{n-m}\sum_{j=1}^{k}\Delta_j, \quad i=\overline{2,n-1},$$
$$m=n-i, \quad k=\binom{n}{i}=\frac{n!}{i!(n-i)!}, \qquad (18)$$

where: Δ_j, $j=\overline{1,k}$ are the principal minors of Jacobi matrix J of order $n-m$, formed by deletion of m rows with numbers $i_1, i_2, \ldots, i_m$ and m columns with the same numbers.

By the well-known theorem of von Neumann, the equilibrium point $\left(q_1^*, q_2^*, \ldots, q_n^*\right)$ is asymptotically stable if for all its eigenvalues λ of Jacobi matrix J the following condition holds:

$$|\lambda|<1. \qquad (19)$$

Consider the space A of all coefficients of the characteristic polynomials of the order n. Condition (19) defines in this space the geometrical domain of asymptotical stability. The analytical description of this stability domain can be constructed with the help of the classical Routh-Hurwitz procedure in the form of non-linear inequalities. This procedure can be described as follows [34].

At first we construct the parameters:

$$b_0=\sum_{i=0}^{n}a_i, \text{ where } a_0=1, \quad b_1=\sum_{i=0}^{n}a_i(n-2i),$$

$$b_r=\sum_{i=0}^{n}a_i\sum_{k=0}^{n}(-1)^k\binom{n-i}{r-k}\binom{i}{k},$$

$$\text{where: } \binom{i}{k}=\begin{cases} \dfrac{i!}{k!(i-k)!}, & i\ge k,\ k\ge 0, \\ 0, & i<k, \\ 0, & k<0, \end{cases} \qquad (20)$$

$$b_n=1-a_1+a_2-\ldots+(-1)^{n-1}a_{n-1}+(-1)^n a_n.$$

Then we construct the matrix:

$$\begin{pmatrix} b_1 & b_3 & b_5 & \dots & \dots & \dots \\ b_0 & b_2 & b_4 & \dots & \dots & \dots \\ 0 & b_1 & b_3 & \dots & \dots & \dots \\ 0 & b_0 & b_2 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \qquad (21)$$

and its principal (diagonal) minors Δ_r, $r=\overline{1,n}$ of order r, that are built from the first r column and the first r row of the upper left corner of the matrix.

The conditions of asymptotical stability are:

$$b_0>0, \quad \Delta_r>0, \quad r=1,2,\ldots,n, \qquad (22)$$

and the boundaries of the stability domain in the space A determined with the help of the above-described Routh-Hurwitz procedure by the non-linear equalities:

$$b_0=0, \quad \Delta_r=0, \quad r=1,2,\ldots,n. \qquad (23)$$

On the boundaries (23) the absolute values of some eigenvalues of the Jacobi matrix are equal 1 and the plethora of different bifurcation phenomena exist [34].

Detailed description of the Routh-Hurwitz procedure for two- and three-dimensional case, and geometric construction of the stability domain is considered in studies of M. Sonis [34, 35].

STABILITY OF THE EQUILIBRIUM POINT OF THE COURNOT-PUU DUOPOLY MODEL

As an example, in this section we will explore in detail the stability of the Cournot equilibrium of the duopoly model. In the case of duopoly there are only two firms F_1 and F_2 on the market in the same industry, with output q_1 and q_2, respectively.

According to the generalized model (6), Cournot-Puu duopoly model is as follows (see also [7]):

$$q_1(t+1)=\sqrt{\frac{q_2(t)}{c_1}}-q_2(t),$$
$$q_2(t+1)=\sqrt{\frac{q_1(t)}{c_2}}-q_1(t). \quad (24)$$

Functions $q_1(t+1)$ and $q_2(t+1)$ with parameter values $c_1 = 1$, $c_2 = 6{,}25$ and initial conditions $q_1(0) = q_2(0) = 0{,}01$ have the form as shown in Fig. 1.

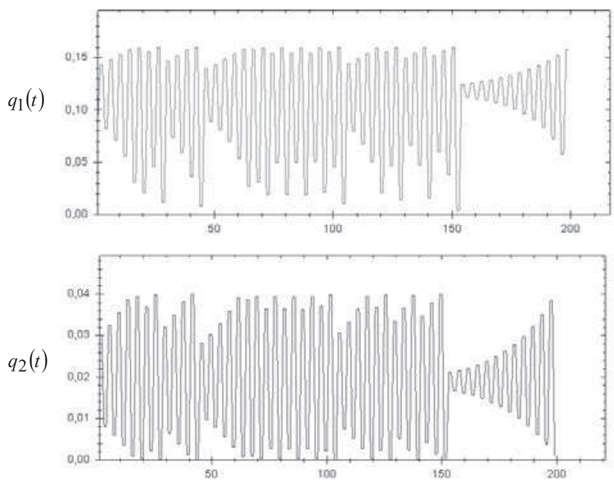

Fig. 1. The reaction functions of firms F_1 and F_2

Nontrivial equilibrium point of the system (24) – Cournot equilibrium (Nash equilibrium) – is a point of intersection of the reaction curves and according to the expression (7), has the value:

$$q_1^* = \frac{c_2}{(c_1+c_2)^2}, \quad q_2^* = \frac{c_1}{(c_1+c_2)^2}. \quad (25)$$

Profit of the duopolists at the Cournot equilibrium is, respectively:

$$U_1^* = \frac{c_2^2}{(c_1+c_2)^2}, \quad U_2^* = \frac{c_1^2}{(c_1+c_2)^2}. \quad (26)$$

Let us investigate the stability of the equilibrium point (25). We linearize the system (24) near the equilibrium point (25), as it was done for a generalized model in the preceding paragraph, and we obtain the Jacobi matrix:

$$J=\begin{pmatrix} 0 & \dfrac{c_2-c_1}{2c_1} \\ \dfrac{c_1-c_2}{2c_2} & 0 \end{pmatrix}. \quad (27)$$

The eigenvalues of the matrix Jacobi J of the linearized system are the solutions of the characteristic polynomial:

$$\lambda^2 + a_1\lambda + a_2 = \lambda^2 - trJ\,\lambda + \det J = 0, \quad (28)$$

where:
$a_1 = -trJ = 0,$

$$a_2 = \det J = \begin{vmatrix} 0 & \dfrac{c_2-c_1}{2c_1} \\ \dfrac{c_1-c_2}{2c_2} & 0 \end{vmatrix} =$$
$$= -\frac{(c_2-c_1)(c_1-c_2)}{4c_1c_2} = \frac{(c_2-c_1)^2}{4c_1c_2}. \quad (29)$$

Equilibrium point (q_1^*, q_2^*) is asymptotically stable if for all the eigenvalues λ of the Jacobi matrix J condition (19) holds. Routh-Hurwitz procedure for $n = 2$ is as follows:

We construct the parameters (20):

$$b_0 = 1 + a_1 + a_2,$$
$$b_1 = 2 - 2a_2,$$
$$b_2 = 1 - a_1 + a_2. \quad (30)$$

Then we construct a matrix:

$$\begin{pmatrix} b_1 & 0 \\ b_0 & b_2 \end{pmatrix}, \quad (31)$$

and its principal minors:

$$\Delta_1 = b_1,$$
$$\Delta_2 = \begin{vmatrix} b_1 & 0 \\ b_0 & b_2 \end{vmatrix}. \quad (32)$$

Classical conditions of asymptotic stability are:

$$b_0 > 0, \quad \Delta_1 > 0, \quad \Delta_2 > 0. \quad (33)$$

It means that:

$$b_0 > 0, \quad b_1 > 0, \quad \Delta_2 = b_1 b_2 > 0, \quad (34)$$

namely:

$$b_0 > 0, \quad b_1 > 0, \quad b_2 > 0. \tag{35}$$

Conditions (35) according to the values of the parameters b_i, $i = 0,1,2$ (30) and the coefficients a_i, $i = 1,2$ (29) can be written as:

$$\begin{aligned} &1 - trJ + \det J > 0, \\ &2 - 2\det J > 0, \\ &1 - trJ + \det J > 0. \end{aligned} \tag{36}$$

Or:

$$\begin{aligned} &\det J > trJ - 1, \\ &\det J < 1, \\ &\det J > -trJ - 1. \end{aligned} \tag{37}$$

Fig. 2. shows the domain of attraction (stability), which is the triangle ABC in the space of eigenvalues $\{a_1,a_2\}$ with vertices:

$$A(-2,1),\ B(2,1),\ C(0,-1). \tag{38}$$

The sides of the triangle of stability are defined by the following straight lines, *the divergence boundary:*

$$1 + a_1 + a_2 = 0 \quad \text{or} \quad \det J = trJ - 1, \tag{39}$$

the flip boundary,

$$1 - a_1 + a_2 = 0 \quad \text{or} \quad \det J = -trJ - 1, \tag{40}$$

and *the flutter boundary*,

$$a_2 = 1 \quad \text{or} \quad \det J = 1. \tag{41}$$

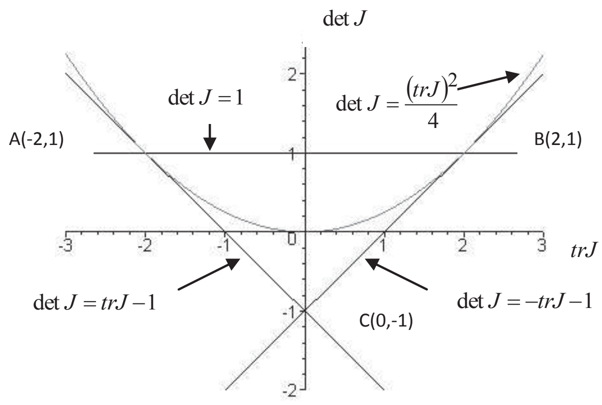

Fig. 2. Stable region

Obviously, in our case, the conditions $\det J > trJ - 1$ and $\det J > -trJ - 1$ are satisfied ($trJ = 0$), loss of stability occurs when the absolute value of eigenvalues becomes equal to unity, i.e., when $\det J = 1$ either $\frac{(c_2 - c_1)^2}{4c_1c_2} = 1$.

Denote the ratio of marginal costs, $\frac{c_2}{c_1} = c_r$, then $\det J = \frac{(c_2 - c_1)^2}{4c_1c_2} = \frac{(c_r - 1)^2}{4c_r}$.

Stability domain of Cournot equilibrium will be:

$$\frac{(c_r - 1)^2}{4c_r} < 1, \tag{42}$$

or

$$c_r^2 - 6c_r + 1 < 0. \tag{43}$$

Namely:

$$c_{r_1} < c_r < c_{r_2}, \tag{44}$$

where: $c_{r_1}, c_{r_2} > 0$ are the roots:

$$c_{r_{1,2}} = 3 \pm \sqrt{8}, \tag{45}$$

of the quadratic equation:

$$c_r^2 - 6c_r + 1 = 0. \tag{46}$$

Thus, the dynamic process is stable, if the value c_r falls inside the interval bounded by the obtained solution, i.e.:

$$3 - \sqrt{8} < c_r < 3 + \sqrt{8}. \tag{47}$$

Without loss of generality we will assume that $c_2 \ge c_1$ (i.e., $c_r \ge 1$), then we will obtain a narrowing of this interval:

$$1 \le c_r < 3 + \sqrt{8}. \tag{49}$$

From the condition of inalienability output for both firms and properties of their reaction functions, we determined the entire range of values related marginal costs c_r [11]:

$$\frac{4}{25} \le c_r \le \frac{25}{4}. \tag{50}$$

Taking into account the assumption that $c_r \ge 1$, we will have a range of values c_r

$$1 \le c_r \le \frac{25}{4}. \tag{51}$$

Thus, we have found that the equilibrium point is stable in the interval (see equation (49)):

$$1 \le c_r < 3 + \sqrt{8}.$$

So, the equilibrium point is unstable in the second part of the interval:

$$3 + \sqrt{8} \le c_r \le 25/4. \tag{52}$$

Limit cycles and chaos exist in the system at these values c_r. Bifurcation diagram for firms F_2 with output q_2 with respect to the ratio c_r of marginal costs is presented in Fig. 3.

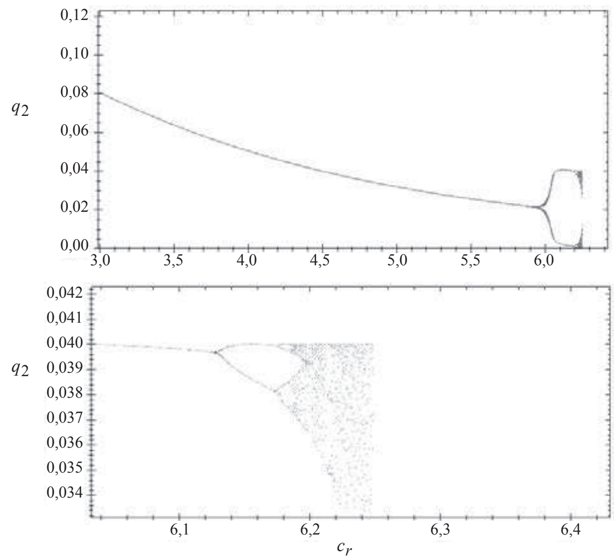

Fig. 3. Bifurcation diagram of the firm F_2 with the production q_2

CONCLUSIONS

In this paper, we generalize Cournot-Puu duopoly model when there are N firms on the oligopolistic market. It is considered that each firm-oligopolist produces the same standard products, which it has to sell for the same price (established based on the size of the total production in the industry). In such conditions, each company in this market (through decision on its own output) can influence the total output, and thus its market price. In addition, each firm is characterized by a function of optimal reaction. This function describes the optimal output (one that maximizes profits) of one firm according to the decision on the output of other firms.

The model is a system of nonlinear equations that has both trivial and non-trivial equilibrium points. Nontrivial point of equilibrium is Cournot (Nash) equilibrium. In this type of equilibrium each firm makes a decision, which enables to maximize its profit, anticipating the same behavior of competitor. In oligopoly equilibrium occurs at a lower price, more products and less overall profit compared to pure monopoly. Given the first two parameters (lower price and more products), oligopoly can be considered the best option for a market economy than monopoly.

The process of investigating the stability of the Cournot equilibrium point in the case of oligopoly is a time-consuming task. It can be carried out using the Routh-Hurwitz procedure. The article presents the study of the stability of equilibrium point for the duopoly. The value of the system parameter c_r, at which the equilibrium point is stable, is established.

REFERENCES

1. **Agiza H.N., Bischi G.I. and Kopel M. 1999.** *Multistability in a Dynamic Cournot Game with Three Oligopolists.* Math. Comput. Simulation, Vol. 51, 63–90.
2. **Agiza H.N, Hegazi A.S. and Elsadany A.A. 2001.** *The dynamics of Bowley's model with bounded rationality.* Chaos, Solitons and Fractals, Vol. 9, 1705–1717.
3. **Agiza H.N, Hegazi A.S. and Elsadany A.A. 2002.** *Complex dynamics and synchronization of duopoly game with bounded rationality.* Mathematics and Computers in Simulation, Vol. 58, 133–146.
4. **Agiza H.N. and Elsadany A.A. 2003.** *Nonlinear dynamics in the Cournot duopoly game with heterogeneous players.* Physica A, Vol. 320, 512–524.
5. **Agiza H.N. and Elsadany A.A. 2004.** *Chaotic dynamics in nonlinear duopoly game with heterogeneous players.* Applied Math, and computation, Vol. 149, 843–860.
6. **Agliari A., Gardini L. and Puu T. 2000.** *The dynamics of a triopoly game.* Chaos, Solitons and Fractals, Vol. 11, 2531–2560.
7. **Agliari A., Gardini L. and Puu T. 2006a.** *Global bifurcation in duopoly when the Cournot point is destabilized via a subcritical Neimark bifurcation.* International Game Theory Review, Vol. 8, No. 1, 1–20.
8. **Agliari A., Chiarella C. and Gardini L. 2006.** *A Reevaluation of the Adaptive Expectations in Light of Global Nonlinear Dynamic Analysis.* Journal of Economic Behavior and Organization, Vol. 60, 526–552.
9. **Agliari A. 2006.** *Homoclinic connections and subcritical Neimark bifurcations in a duopoly model with adaptively adjusted productions.* Chaos, Solitons and Fractals, Vol. 29, 739–755.
10. **Ahmed E., Agiza, H.N. and Hassan, S.Z. 2000.** *On modifications of Puu's dynamical duopoly.* Chaos, Solitons and Fractals, Vol. 11, 1025-1028.
11. **Alekseyev I.V., Khoma I.B., Kavalets I.I., Kostrobii P.P. and Hnativ B.V. 2012.** *Matematychni modeli rehuliuvannia khaosu v umovakh olihopolistychnoho rynku.* Rastr-7, Lviv.
12. **Angelini N., Dieci R. and Nardini F. 2009.** *Bifurcation analysis of a dynamic duopoly model with heterogeneous costs and behavioural rules.* Mathematics and Computers in Simulation, Vol. 79, 3179–3196.
13. **Bischi G.I. and others. 2009.** *Nonlinear Oligopolies: Stability and Bifurcations.* Springer-Verlag, New York.
14. **Bischi G. I., Lamantia F. and Sushko I. 2012.** *Border collision bifurcations in a simple oligopoly model with constraints.* International Journal of Applied Mathematics and Statistics, Vol. 26, Issue No. 2.
15. **Chen L. and Chen G. 2007.** *Controlling chaos in an economic model.* Physica A, No. 374, 349–358.
16. **Chukhray N. 2012.** *Competition as a strategy of enterprise functioning in the ecosystem of innovations.* Econtechmod, Vol. 1, No. 3, 9–16.
17. **Cournot A.A. 1838.** *Recherches sur les principes mathematiques de la theorie des richesses.* Hachette, Paris.
18. **Den-Haan W.J. 2001.** *The importance of the number of different agents in a heterogeneous asset-pricing model.* Journal of Economic Dynamics and Control, Vol. 25, 721–746.

19. **Elabbasy E.M. and others, 2007.** *The dynamics of triopoly game with heterogeneous players*. International Journal of Nonlinear Science, Vol. 3, No. 2, 83–90.
20. *Encyclopedia of Business and Finance* by Kaliski B.S. 2001. MacMillan Reference Books Hardcover.
21. **Feshchur R., Samulyak V., Shyshkovskyi S. and Yavorska N. 2012.** *Analytical instruments of management development of industrial enterprises*. Econtechmod, Vol. 1, No. 3, 17–22.
22. **Jakimowicz A. 2012.** *Stability of the Cournot–Nash Equilibrium in Standard Oligopoly.* Acta Physica Polonica A, Vol. 121, B-50–B-53.
23. **Kirman A. and Zimmermann J.B. 2001.** *Economics with heterogeneous Interacting Agents.* Lecture Notes in Economics and Mathematical Systems. Springer, Berlin.
24. **Matsumoto A. 2006.** *Controlling the Cournot–Nash chaos*. Journal of Optimization Theory and Applications, No. 128, 379–392.
25. **Matsumoto A. and Szidarovszky F. 2011.** *Stability, Bifurcation, and Chaos in N-Firm Nonlinear Cournot Games*. Discrete Dynamics in Nature and Society.
26. **Moroz O., Karachyna N. and Filatova L. 2012.** *Economic behavior of machine-building enterprises: analytic and managerial aspects*. Econtechmod, Vol. 1, No. 4, 35–42.
27. **Onazaki T., Sieg G. and Yokoo M. 2003.** *Stability, chaos and multiple attractors: A single agent makes a difference*. Journal of Economic Dynamics and Control, Vol. 27, 1917–1938.
28. **Petrovich J.P. and Nowakiwskii I.I. 2012.** *Modern concept of a model design of an organizational system of enterprise management*. Econtechmod, Vol. 1, No. 4, 43–50.
29. **Puu T. 1991.** *Chaos in duopoly pricing*. Chaos, Solitons and Fractals, Vol. 6, No. 1, 573–581.
30. **Puu T. 2000.** *Attractors, Bifurcations, and Chaos: Nonlinear Phenomena in Economics*. Springer, New York.
31. **Puu T. 2007.** *On the Stability of Cournot Equilibrium when the Number of Competitors Increases*. Journal of Economic Behavior and Organization.
32. **Puu T. and Sushko I. 2002. (Ed. s).** *Oligopoly and Complex Dynamics: Models and Tools*. Springer, New York.
33. **Rosser B.** *The development of complex oligopoly dynamic theory*. (Available: http://www.belairsky.com/coolbit/econophys/complexoligopy.pdf).
34. **Sonis M. 1997.** *Linear Bifurcation Analysis with Applications to Relative Socio-Spatial* Dynamics. Discrete Dynamics in Nature and Society, Vol. 1, 45-56.
35. **Sonis M. 2000.** *Critical Bifurcation Surfaces of 3D Discrete Dynamics*. Discrete Dynamics in Nature and Society, Vol. 4, 333-343.
36. **Tramontana F., Gardini L. and Puu T. 2010.** *New properties of the Cournot duopoly with isoelastic demand and constant unit costs*. Working Papers Series in Economics, Mathematics and Statistics. WP-EMS, No. 1006.
37. **Tramontana F. 2010.** *Heterogeneous duopoly with isoelastic demand function*. Economic Modelling, Vol. 27, 350–357.