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Necessary and sufficient conditions for constant prices in a Sraffa's model*

by

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Abstract: We give a simple and direct proof of necessary and sufficient conditions to have constant equilibrium prices in a simple Sraffa's production model. We make some remarks on the linearity of the relation between profit rate and wage rate.

Keywords: Sraffa's price system, uniform organic composition of capital, standard system, labour value theory

1. Introduction

Since the publication in 1960 of "*Production of Commodities by Means of Commodities*" by Piero Sraffa (Sraffa, 1960) a lot of reviews, papers and books on Sraffa's work appeared, which demonstrates the deep impact of this book on classical and modern economic theory. Several authors have been concerned with the relationship between profit rate and wage rate in a simple Sraffa's production model and, in particular, on the construction and properties of the so-called "standard system" and "standard commodity" (sometimes with mathematical inaccuracies). The related literature is very wide; here we quote only Abraham-Frois and Berrebi (1979), Giorgi (1987), Giorgi and Zuccotti (2012), Kurz and Salvadori (1995), Pasinetti (1979), Woods (1978, 1990). All these works contain further references. Other useful items are given at the end of the present paper in the list of references. The willing reader shall have the opportunity to deepen the various economic questions related to the various Sraffa's models.

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Curiously, as far as we are aware, there is no direct, complete and "autonomous" proof concerning the necessary and sufficient conditions, which assure that in a simple Sraffa's production model the equilibrium prices are constant. The proof of Burmeister (1968) is incomplete, as it is concerned only with sufficient conditions. The proofs of Pasinetti (1979) and Woods (1985) are unsatisfactory from the mathematical point of view. The proof of Schefold (1976) is a corollary of a more general result and it is not clear if the related conditions are necessary and sufficient to have constant prices. The proof of Kurz and Salvadori (1995) is inserted in a long chain of equivalences and is not therefore a direct proof.

The aim of the present note is to give a simple, complete and direct proof of the said property (constant prices) for a simple Sraffa's production model. In Section 2 we give a short presentation of the said model and obtain the related equilibrium solutions. In Section 3 we give the main results of the present note and make some remarks on the same results. We note that the case of constant prices in a simple Sraffa's production model is closely related to the so-called "uniform organic composition of capital", where the labour theory of value of Karl Marx is valid, see, e.g., Burmeister (2008), Kurz and Salvadori (1995). The final Section 4 is concerned with some conclusions.

Throughout the paper the following conventions will be used for matrices and vectors. Let A be a real matrix of order (m, n); we denote by $[0]$ the zero matrix of order (m, n).

- $A \geqq [0]$ denotes a *nonnegative matrix,* i.e. a matrix, whose entries a_{ij}, $i = 1, ..., m$; $j = 1, ..., n$, are nonnegative: $a_{ij} \geqq 0$, $\forall i, j$.
- $A \geq [0]$ denotes a *semipositive matrix,* i.e. it holds that $A \geqq [0]$, $A \neq [0]$.
- $A > [0]$ denotes a *positive matrix,* i. e. a matrix whose entries a_{ij}, $i = 1, ..., m$; $j = 1, ..., n$, are positive: $a_{ij} > 0$, $\forall i, j$.

The same convention is used for vectors of $\mathbb{R}^n$; $[0]$ denotes the zero vector of $\mathbb{R}^n$. A_i, $i = 1, ..., m$, denotes the i-th row of A; A^j, $j = 1, ..., n$, denotes the j-th column of A.

2. The simple Sraffa's production model

Let us consider:

- a semipositive square matrix A, of order n, $A = [a_{ij}]$, $i, j = 1, ..., n$, which is the matrix of input-output coefficients. For economic reasons it is convenient to assume that every row of A is semipositive:

$$A_i \geq [0]\,,\ i = 1, ..., m;$$

and every column of A is semipositive:

$$A^j \geq [0]\,,\ j = 1, ..., n.$$

• $p^\top = [p_1, p_2, ..., p_n] > [0]$ is the vector of commodity prices ("equilibrium prices").

• $\ell^\top = [\ell_1, \ell_2, ..., \ell_n] > [0]$ is the vector of direct input coefficients.

• $r \geqq 0$ is the (uniform) rate of profit.

• $w \geqq 0$ is the (uniform) wage rate, paid at the end of production.

We make the assumption that the vector x of total outputs is unitary, i.e. $x_i = 1,\ i = 1, ..., n$, or, equivalently, $x = e$, where $e \in \mathbb{R}^n$ is the *sum vector* of $\mathbb{R}^n$:

$$e^\top = [1, 1, ..., 1] .$$

We make the assumption that the total net output is a semipositive vector, i.e.

$$(e - Ae) \geq [0] ,$$

or, with $y = e - Ae$, $y \geq [0]$. We assume, moreover, that prices are "normalized", i.e. we impose that the value of the total net output is equal to one:

$$p^\top y = 1.$$

Finally, we assume that also the total direct labour requirement is "normalized", i.e. we put

$$\ell^\top e = 1,$$

so that the wage rate measures the total of the wages paid: $w\ell^\top e = w$.

Then, the price system of the present Sraffa's model is described by

$$\begin{cases} p^\top = p^\top A + rp^\top A + w\ell^\top , \\ p > [0] , \\ r \geqq 0,\ w \geqq 0,\ p^\top y = 1, \end{cases}$$

i.e.

$$\begin{cases} (1 + r)p^\top A + w\ell^\top = p^\top , \\ p > [0] , \\ r \geqq 0,\ w \geqq 0,\ p^\top y = 1. \end{cases}$$

We are interested in obtaining the solutions of the said system, under the assumption that $w > 0$ (if $w = 0$, in order to obtain solutions $p > [0]$, we have to make additional assumptions on the matrix A, for example that A is *indecomposable;* see the next section). We have

$$p^\top (I - (1 + r)A) = w\ell^\top$$

from which

$$p^\top (1+r)\left[\frac{1}{1+r}I - A\right] = w\ell^\top .$$

Now we have

$$[I-(1+r)A]^{-1} \geq [0] \iff \left[\frac{1}{1+r}I - A\right]^{-1} \geq [0]$$

and this is true (see, e. g., Debreu and Herstein, 1953; Takayama, 1985; Woods, 1978) if and only if

$$\frac{1}{1+r} > \lambda^*(A)$$

where $\lambda^*(A)$ is the Frobenius eigenvalue of A, which is positive, thanks to the assumptions made on the rows and columns of A. The last inequality can be rewritten in the form

$$0 \leqq r < \frac{1}{\lambda^*(A)} - 1,$$

from which we have $\lambda^*(A) < 1$.

If we put

$$r^* = \frac{1}{\lambda^*(A)} - 1$$

(Sraffa uses the notation R, instead of r^* and calls R the "maximum profit rate"), we have

$$p^\top = w\ell^\top [I-(1+r)A]^{-1}, \quad \forall r \in [0, r^*) .$$

Hence, we may rewrite the normalization condition $p^\top y = 1$ in the form

$$1 = p^\top y = w\ell^\top [I-(1+r)A]^{-1} y.$$

Since $\ell > [0]$ and every column of $[I-(1+r)A]^{-1}$ is semipositive, we have

$$\ell^\top [I-(1+r)A]^{-1} > [0], \quad \forall r \in [0, r^*) .$$

But as $y \geq [0]$, we have also

$$\ell^\top [I-(1+r)A]^{-1} y > 0, \quad \forall r \in [0, r^*) .$$

Hence, we have

$$w = \frac{1}{\ell^\top [I-(1+r)A]^{-1} y} > 0, \quad \forall r \in [0, r^*), \tag{1}$$

and also

$$p^\top = \frac{1}{\ell^\top \left[I-(1+r)A\right]^{-1} y}\ell^\top \left[I-(1+r)A\right]^{-1}, \quad \forall r \in [0, r^*), \tag{2}$$

with $p > [0]$, $\forall r \in [0, r^*)$.

In the last relation it appears evident that, with w given by (1), the equilibrium prices depend on the profit rate r. In the following section we shall find necessary and sufficient conditions to have

$$p = \bar{p} = \text{ a constant vector, for } r \in [0, r^*).$$

3. The main results

In the present section we assume that A is *indecomposable* (or *irreducible*), i.e. that there is no permutation matrix P such that

$$PAP^{-1} = PAP^\top = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with A_{11} being square submatrix of A (and hence also A_{22} square submatrix of A) and at least one of A_{12}, A_{21} being a zero matrix. See, e.g., Abraham-Frois and Berrebi (1979), Debreu and Herstein (1953), Pasinetti (1979), Takayama (1985), Woods (1978). The assumption of A being indecomposable means that all commodities produced by the system are (following the terminology of Sraffa) *basic commodities,* i.e. every commodity enters, directly or indirectly, in the production of all commodities (included itself).

We recall that a permutation matrix P is given by permuting the rows (or columns) of the identity matrix I.

One of the more important consequences of this assumption is that there exists a positive right-sided eigenvector of A and a positive left-sided eigenvector of A *if and only if* the related eigenvalue is the Frobenius eigenvalue $\lambda^*(A)$. See on that also in Blakley and Gossling (1967) and Gantmacher (1959). In this case, the solution of the model with $w = 0$ is given by

$$p^\top A = \lambda^*(A) p^\top, \quad p > [0], \; p^\top y = 1,$$

i.e.

$$p^\top A = \frac{1}{1+r^*} p^\top, \quad p > [0], \; p^\top y = 1,$$

where, as before, $r^* = \frac{1}{\lambda^*(A)} - 1$. This entails the asumption that A is not only indecomposable, but also "productive", i.e. $\lambda^*(A) < 1$, or, equivalently, there exists a vector $x > [0]$ such that $(I - A)x > [0]$. (The indecomposability assumption allows for rewriting the previous inequality as $(I-A)x \geq [0]$). These conditions are, in turn, equivalent to many other conditions, among which the celebrated "Hawkins-Simon conditions":

$$(1-a_{11})>0,\ \begin{vmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix}>0,...,|I-A|>0.$$

Regarding the above, see, e.g., Berman and Plemmons (2014), Nikaido (1968, 1970), Takayama (1985), or Woods (1978).

Theorem 1 *Let A be indecomposable; the price solution p of Sraffa's model, described by (2), is constant, with $r\in[0,r^*)$, if and only if the pair (A,ℓ) verifies the relation*

$$\ell^\top A=\lambda^*(A)\ell^\top.$$

Proof (*i*) Let p be constant with r varying in the interval $[0,r^*)$: hence if p is an equilibrium solution for $r=0$, it will be an equilibrium solution also for the other values of r in the said interval. But for $r=0$ we have $w(0)=1$. Indeed, from

$$p^\top-p^\top A=rp^\top A+w\ell^\top,\quad r\in[0,r^*),$$

we have

$$p^\top e-p^\top Ae=rp^\top Ae+w\ell^\top e,\quad r\in[0,r^*),$$

i.e. as we have, $\ell^\top e=1$ and $p^\top y=1$, i.e. $p^\top e-p^\top Ae=1$,

$$1=rp^\top Ae+w,\quad r\in[0,r^*),$$

that is,

$$w=1-rp^\top Ae,$$

which becomes, with $r=0$, $w=1$.

Then we have

$$p^\top=w\ell^\top[I-A]^{-1}=\ell^\top[I-A]^{-1}\equiv v^\top.$$

Hence, the constant equilibrium price vector "passe partout" is

$$p^\top=v^\top,\quad r\in[0,r^*).$$

Therefore, the relation

$$w=1-rp^\top Ae,\quad r\in[0,r^*),$$

becomes a *linear relation*:

$$w=1-rv^\top Ae,\quad r\in[0,r^*).$$

Obviously, there exists only one straight line passing through the distinct points $(r, w) = (0, 1)$ and $(r, w) = (r^*, 0)$:

$$w = 1 - \frac{r}{r^*}, \quad r \in [0, r^*]\,.$$

Now we insert the expressions for p and w just found, in the fundamental equilibrium equation

$$p^\top (I - A) = rp^\top A + w\ell^\top, \quad r \in [0, r^*)\,.$$

We get

$$v^\top (I - A) = r^* v^\top A + \left(1 - \frac{r}{r^*}\right) \ell^\top, r \in [0, r^*)\,, \quad r \in [0, r^*)\,,$$

i.e., having $v^\top = \ell^\top (I - A)^{-1}$,

$$\ell^\top = rv^\top A + \ell^\top - \frac{r}{r^*}\ell^\top, \quad r \in [0, r^*)\,,$$

meaning that

$$r\left[v^\top A - \frac{1}{r^*}\ell^\top\right] = [0]\,, \quad r \in [0, r^*)\,.$$

For $0 < r < r^*$ the previous relation holds if and only if

$$\left[v^\top A - \frac{1}{r^*}\ell^\top\right] = [0]\,,$$

i.e.

$$v^\top A = \frac{1}{r^*}\ell^\top,$$

i.e., owing to the definitions of v and r^* :

$$\ell^\top (I - A)^{-1} A = \frac{\lambda^*(A)}{1 - \lambda^*(A)}\ell^\top. \tag{3}$$

But, since $0 < \lambda^*(A) < 1$, we have

$$(1 - \lambda^*(A))\ell^\top (I - A)^{-1} A = \lambda^*(A)\ell^\top.$$

We remark that A and $(I - A)^{-1}$ *commute*: indeed, from the identity

$$A - AA = A - AA$$

we have

$$A(I - A) = (I - A)A,$$

from which

$$A = (I - A)A(I - A)^{-1}$$

and also

$$(I - A)^{-1}A = A(I - A)^{-1}.$$

Hence, we can rewrite (3) in the form

$$(1 - \lambda^*(A))\ell^\top A(I - A)^{-1} = \lambda^*(A)\ell^\top,$$

from which we obtain

$$(1 - \lambda^*(A))\ell^\top A = \lambda^*(A)\ell^\top(I - A),$$

i.e.

$$\ell^\top A - \lambda^*(A)\ell^\top A = \lambda^*(A)\ell^\top - \lambda^*(A)\ell^\top A,$$

that is,

$$\ell^\top A = \lambda^*(A)\ell^\top.$$

ii) Conversely, let us suppose that

$$\ell^\top A = \lambda^*(A)\ell^\top.$$

We prove that the triplet (p, r, w), with $p^\top = v^\top \equiv \ell^\top(I-A)^{-1}$, $r \in [0, r^*)$, $w = 1 - \frac{r}{r^*}$, is a solution (with a constant price vector!) of the fundamental Sraffa's system. We get at once $p > [0]$, $p^\top y = 1$, and so

$$p^\top y = v^\top y = \ell^\top(I - A)^{-1}(I - A)e = \ell^\top e = 1.$$

It remains to prove that there holds

$$p^\top(I - (1 + r)A) = w\ell^\top.$$

It is sufficient to perform the previous proof, *i*), backwards, taking into account that $(I - A)^{-1}A$ and $A(I - A)^{-1}$ commute. □

Obviously, the vector of labour requirements, which assures constant prices in the model prevoiusly described, coincides with the equilibrium price solution p^* with $w = 0$: both are left-hand sided positive eigenvectors of A, associated to the Frobenius eigenvalue $\lambda^*(A)$:

$$(p^*)^\top A = \lambda^*(A)(p^*)^\top$$

or $(p^*)^\top A^j \big/ p_j^* = k$, constant for all $j = 1, ..., n$. This relation has an economic interpretation (see Woods, 1985): as $(p^*)^\top A^j$ is the value of capital employed in the j-th industry, $(p^*)^\top A^j \big/ p_j^*$ is that industry's value-capital/output ratio. Therefore, we can say that the equilibrium prices of Sraffa's model are constant if and only if there is a uniform value-capital/output ratio in all industries at any feasible rate of profit. But, p^* being proportional to ℓ, with $\ell^\top A = \lambda^*(A)\ell^\top$, as A being indecomposable, we have also

$$(p^*)^\top A = \alpha \ell^\top, \quad \alpha > 0,$$

or $(p^*)^\top A^j \big/ \ell_j = \alpha$, $\alpha > 0$. So, equilibrium prices are invariant if and only if there is a uniform value-capital/labour ratio in all industries at any feasible rate of profit.

Another equivalent way to express the necessary and sufficient conditions to have constant equilibrium prices in the previous model is:

$$v^\top \equiv \ell^\top (I - A)^{-1} = \frac{1}{1 - \lambda^*(A)} \ell^\top. \tag{4}$$

Indeed, by multiplying both members of the last equality by $(I - A)$, we get

$$\ell^\top = \frac{1}{1 - \lambda^*(A)} \ell^\top (I - A),$$

i.e.

$$(1 - \lambda^*(A))\ell^\top = \ell^\top - \ell^\top A,$$

that is,

$$\lambda^*(A)\ell^\top = \ell^\top A. \tag{5}$$

In economic terms, relation (4) is perhaps more meaningful than relation (5), as it says that the ratio between the "total labour requirement" v_j and the direct labour requirement ℓ_j is constant, for every industry $j = 1, ..., n$, the constant being given by $\frac{1}{1-\lambda^*(A)}$. This just leads to the labour theory of value: prices are proportional to the total labour requirements, not only for $r = 0$, i.e. with $p^\top = \ell^\top (I - A)^{-1} \equiv v^\top$, but also for the other feasible values of r (case of "uniform organic composition of capital"). Pasinetti (1973, 1979) calls the vector v "vector of vertically integrated labour coefficients". Following Marx's terminology, v is the vector of the quantities of labour "embodied" in the different commodities.

We stress that in the proof of Theorem 1 we have also obtained that the functional relation between w and r is linear, under the assumptions of the theorem. That is, if prices are constant, we have

$$w = 1 - \frac{r}{r^*}.$$

It is well-known that the case of constant prices is not the unique way to obtain the linearity of the said functional relation (the so-called "factor price frontier"). The construction of the "standard commodity" of Sraffa is another way to obtain linearity between wages and profit rates. See on that , e. g., Giorgi (1987), Giorgi and Zuccotti (2012), Miyao (1977). In particular, Miyao (1977) presents an interesting analysis, oriented at obtaining a "general standard commodity". The model, considered by Miyao, is quite similar to the model considered in the present paper. The same author assumes:

- $A \geq [0]$, indecomposable and productive (i. e. $\lambda^*(A) < 1$), of order n.
- $\ell > [0]$.
- $y = (I - A)x$, $x \geq [0]$ (x not fixed).

If we put, for brevity,

$$Z_r \equiv I - (1 + r)A,$$

Miyao considers the usual Sraffa's system

$$\begin{cases} p^\top Z_r = w\ell^\top \\ p > [0]\,, \ r \geqq 0, \ w \geqq 0 \end{cases}$$

and finds

$$\begin{cases} r \in [0, r^*] = \left[0, \frac{1}{\lambda^*(A)} - 1\right] \\ r < r^* : \begin{cases} p^\top = w\ell^\top Z_r^{-1} \\ w = \frac{1}{\ell^\top x} p^\top Z_r x \end{cases} \\ r = r^* : \begin{cases} p^\top = (p^*)^\top > [0] \\ (p^*)^\top A = \lambda^*(A)(p^*)^\top \\ w = w^* = 0. \end{cases} \end{cases}$$

Then, Miyao defines "general standard" commodity as a vector $x \geq [0]$, which makes linear the functional relation between wage rate and profit rate (the "factor price frontier"). This author finds that the necessary and suffcient conditions for the said linearity are

$$\left\{ \frac{p^\top A x}{p^\top y} = \frac{1}{r^*} \right\} \iff \left\{ \begin{array}{l} p_i'(r) = 0, \ \forall i = 1, ..., n, \ \forall r \in (r_0, r^0), \\ \text{ with } 0 \leqq r_0 < r^0 \leqq r^* \end{array} \right\} \iff \tag{6}$$

$$\iff \left\{ \frac{w}{p^\top y} \text{ is linear in } r \right\},$$

where $y = (I - A)x$, $x \geq [0]$ and r^* is, as usual, $\frac{1}{\lambda^*(A)} - 1$.

Let us now define the set X of the general standard commodities:

$$X = \left\{ x : x \geq [0]\,, \ x \text{ makes linear } \frac{w}{p^\top y} \right\}.$$

We note that:

a)

$$\left\{\exists \ell > [0] : \ell^\top A = \lambda^*(A)\ell^\top\right\} \Longrightarrow \{x \in X,\ \forall x \geq [0]\};$$

b)

$$\{\exists x^* > [0] : Ax^* = \lambda^*(A)x^*\} \Longrightarrow \{x^* \in X\}.$$

Miyao (1977) then solves the following (non trivial) question: which is the "dimension", i.e. the number of linearly independent elements of X? This author proves the following result.

Theorem 2 (Miyao, 1977) *Denote*

$$K = \begin{bmatrix} \ell^\top \\ \ell^\top A \\ \ell^\top A^{(2)} \\ \vdots \\ \ell^\top A^{(n-1)} \end{bmatrix},$$

where $A^{(k)}$ is the k-th power of A. It results that

$$rank(K) = n - rank(X) + 1,$$

i. e.

$$rank(X) = n + 1 - rank(K).$$

In other words, there will be m independent standard commodities if and only if

$$rank(K) = n - m + 1.$$

It turns out that there will be only one standard commodity if and only if

$$rank(K) = n,$$

i.e. the vectors $\ell^\top$, $\ell^\top A, ..., \ell^\top A^{(n-1)}$, are linearly independent. This fact has been proven also by Bidard (1978). Those Sraffa's systems, in which the said vectors are linearly independent are called by Schefold (1976) "regular systems". Schefold obtains interesting results, under "regularity" assumptions, also for Sraffa's joint production systems.

Let us consider again the first relation of (6). We have, with $r \in [0, r^*)$:

$$\left\{\frac{p^\top Ax}{p^\top y} = \frac{1}{r^*}\right\} \iff \left\{\frac{w\ell^\top Z_r^{-1}Ax}{w\ell^\top Z_r^{-1}y} = \frac{1}{r^*}\right\},$$

from which

$$\left\{ \frac{\ell^\top Z_r^{-1} A x}{\ell^\top Z_r^{-1} y} = \frac{1}{r^*} \right\} \iff \left\{ r^* \frac{\ell^\top Z_r^{-1} A x}{\ell^\top Z_r^{-1} y} = 1 \right\}.$$

Hence,

$$\left\{ r^* \ell Z_r^{-1} A x = \ell^\top Z_r^{-1} y \right\} \iff \left\{ 0 = r^* \ell Z_r^{-1} A x - \ell^\top Z_r^{-1} y \right\},$$

i.e.

$$0 = r^* \ell^\top Z_r^{-1} A x - \ell^\top Z_r^{-1} (I - A) x,$$

that is,

$$\left\{ 0 = \ell^\top Z_r^{-1} \left[r^* A - (I - A) \right] x \right\} \iff \left\{ 0 = \ell^\top Z_r^{-1} \left[-r^* A + I - A \right] x \right\},$$

meaning that

$$0 = \ell^\top Z_r^{-1} Z_{r^*} x, \tag{7}$$

or, as Z_r^{-1} and Z_{r^*} commute,

$$0 = \ell^\top Z_{r^*} Z_r^{-1} x. \tag{8}$$

Therefore, (7) or, equivalently, (8), are necessary and sufficient conditions to have $\frac{w}{p^\top y}$ linear in r, i.e. to have linearity of the distribution function between the parameters r and w. From these relations, which appear, with slightly different assumptions, also in Giorgi (1987) and in Giorgi and Zuccotti (2012), we can obtain immediately the above conditions a) and b).

On the grounds of what has been said, we note that a) and b) are not therefore "exclusive" conditions in order to obtain the said linearity. Consider, e.g., the following numerical example.

EXAMPLE 3 Let us consider

$$A = \begin{bmatrix} 0 & 0 & 0.25 \\ 0 & 0 & 0.25 \\ 0.4 & 0.24 & 0 \end{bmatrix},$$

$$\ell^\top = \left[\frac{2}{13}, \frac{6}{13}, \frac{5}{13} \right].$$

A is indecomposable and productive, as $\lambda^*(A) = 0.4 < 1$; hence $r^* = \frac{3}{2} = 1.5$. Moreover, $\ell^\top A \neq \lambda^*(A) \ell^\top$, and the effective productions are not standard, i.e. $Ae \neq \lambda^*(A) e$. The equilibrium price vector is

$$p^\top = \left[\frac{16 + 6r}{39}, \frac{24 - 6r}{39}, \frac{25}{39} \right].$$

with $r \in [0,\ 1.5)$ and

$$w = 1 - \frac{r}{1.5}.$$

Hence, w is a linear function of r.

4. Concluding remarks

The aim of the present paper is to give a simple and direct proof of the necessary and sufficient conditions, which assure that the equilibrium price vector of a simple production Sraffa's model is composed of constants. Indeed, the existing proofs are either incomplete or not totally satisfactory from a pure mathematical point of view, and in the last ten or twenty years, as far as I am aware, no new proof has appeared. The literature on the behaviour of prices and on the construction of the so-called "standard commodity" for Sraffa's models is too vaste to be fully taken into consideration in a single paper. Hence, we have preferred, for the reader's convenience, to give several additional bibliographical references on the said topics, instead of opening a discussion on the same, that would be too long and not proportionate to the main purpose of this paper, which is focused only on a mathematical feature of a Sraffa's model, without pretentions of developing an analysis on the various related economic questions.

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