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THE FASTEST MOTION OF A POINT ON THE PLANE

Abstract

This paper provides an analysis of time optimal control problem of motion of a material point on the plane, without friction. The point is controlled by a force whose absolute value is limited. In the analysis of this problem, the maximum principle is applied

INTRODUCTION

A material point (a car) of the mass equal to one moves on the plane without friction. The point is controlled by a force whose absolute value is limited by one. The initial position and the initial vector of velocity are also given. It is necessary to minimize time of the motion. We offer a complete solution to the problem.

Let the position of the point at time t be $x(t)=\left(x_1(t),x_2(t)\right)\in\mathbb{R}^2$ and its velocity $y(t)=\left(y_1(t),y_2(t)\right)\in\mathbb{R}^2$. Let the value of the force at time t be $u(t)=\left(u_1(t),u_2(t)\right)\in\mathbb{R}^2$. There is a control constraint $|u(t)|\le 1$, where $|u|=\sqrt{\langle u,u\rangle}$. The trajectory $\left(x(t),y(t)\right)$ must satisfy the endpoint constraints: at the initial time $t=0$, the initial position $x(0)$ is equal to $\hat{x}_0\in\mathbb{R}^2$ and the initial velocity $y(0)$ is equal to $\hat{y}_0\in\mathbb{R}^2$; at the final time $t=T$, the final position $x\left(T\right)$ is equal to $\hat{x}_T\in\mathbb{R}^2$ and the final velocity $y(T)$ is equal to $\hat{y}_T\in\mathbb{R}^2$. It necessary to minimize the time of the process T. Since $m=1$, by the Newton law we have $u(t)=\ddot{x}(t)=\dot{y}(t)$. Thus, the problem has the form:

$$T\to\min, \tag{1}$$

subject to the constraints

$$\begin{gathered}\dot{x}=y,\ \ \dot{y}=u,\ \ |u|\le 1,\\ x(0)=\hat{x}_0,\ \ x(T)=\hat{x}_T,\ \ y(0)=\hat{y}_0,\ \ y(T)=\hat{y}_T\end{gathered} \tag{2}$$

where $x\in\mathbb{R}^2$, $y\in\mathbb{R}^2$, $u\in\mathbb{R}^2$, $|u|=\sqrt{\langle u,u\rangle}$. One-dimensional version of this problem was studied, e.g., in [1],[2],[3]. We will study two-dimensional case.

1. MAXIMUM PRINCIPLE

Let $\left(x(t),y(t),u(t)\,|\,t\in\left[0,T\right]\right)$ be a solution to the problem. Let $\psi_x\in\left(\mathbb{R}^2\right)^*,\psi_y\in\left(\mathbb{R}^2\right)^*,\psi_t\in\mathbb{R}^1$, where $\left(\mathbb{R}^2\right)^*$ is the space of two-dimensional rowvectors. Set $H=\psi_x y+\psi_y u+\psi_t$. Condition of the maximum principle [3] have the form:

$$-\dot{\psi}_x = H_x = 0, \quad -\dot{\psi}_y = H_y = \psi_x, \quad -\dot{\psi}_t = H_t = 0,$$
$$-\psi_t(T) = \alpha_0 \geq 0, \quad H = \psi_x y + \psi_y u + \psi_t = 0, \tag{3}$$
$$\max_{|v|\leq 1} \psi_y v = \psi_y u, \quad (\psi_x, \psi_y) \neq (0,0),$$

where ψ_x and ψ_y are absolutely continous functions. It follows that the function ψ_t is equal to the non-positive constant $(-\alpha_0)$.

Obviously, conditions (3) of the maximum principle are equivalent to the following system of conditions:

$$\dot{\psi}_x = 0, \ -\dot{\psi}_y = \psi_x, \ \psi_x y + \psi_y u = \alpha_0 \geq 0, \ u = \frac{\psi_y}{|\psi_y|} \text{ if } \ \psi_y \neq 0 \ . \tag{4}$$

Let us add the following conditions to this system

$$\dot{x} = y, \quad \dot{y} = u \tag{5}$$

Thus obtained new system (4) and (5) we call *expended*. A pairs (x,u) such that there exists a pair (ψ_x, ψ_y), satisfying to the expanded system, we call *Pontryagin's extremal*. We will seek for Pontryagin's extremals without worrying about how they satisfy the boundary conditions.

Let us note that the last two conditions of system (4) imply

$$\psi_x y + |\psi_y| = \alpha_0 \geq 0. \tag{6}$$

This conditio holds for all $t \in [0,T]$, but, as is known, it suffices to check this conditio only at one point of the interval. This remark will be used in the sequel.

2. EQUATIONS OF MOTION

Let us study the expended system on an interval $\Delta \subset [0,T]$:

$$\dot{\psi}_x = 0, \quad -\dot{\psi}_y = \psi_x, \quad (\psi_x, \psi_y) \neq (0,0), \quad \psi_x y + |\psi_y| = \alpha_0 \geq 0,$$
$$u = \frac{\psi_y}{|\psi_y|} \ \text{ if } \ \psi_y \neq 0, \qquad \dot{x} = y, \ \dot{y} = u. \tag{7}$$

The first two conditions imply that ψ_x is a constant vector, while ψ_y is a linear function, i.e.,

$$\psi_y = kt + b, \ \psi_x = -k, \ k^2 + b^2 > 0, \tag{8}$$

where $k, b \in \mathbb{R}^2$. The following two cases are possible.

1) The function ψ_y does not vanish on Δ. In this case

$$u(t) = \frac{kt+b}{|kt+b|} \tag{9}$$

is a continous function, and the motion is uniquely defined by the conditions:

$$\dot{x}(t) = y(t), \ \dot{y}(t) = u(t) = \frac{kt+b}{|kt+b|}, \ x(t_0) = x_0, \ y(t_0) = y_0,$$
$$-ky_0 + |kt_0 + b| \geq 0, \ k^2 + b^2 > 0, \ |kt+b| > 0, \ t_0 \in \Delta. \tag{10}$$

In oreder to find one of such extremals, one has to choose arbitrary $k, b, x_0, y_0 \in \mathbb{R}^2$ and t_0 so that the following conditions hold

$$-ky_0 + |kt_0 + b| \geq 0, \ \ k^2 + b^2 > 0, \ \ |kt_0 + b| > 0, \tag{11}$$

and then to solve a system

$$\dot{x}(t) = y(t), \ \dot{y}(t) = u(t) = \frac{kt+b}{|kt+b|}, \ x(t_0) = x_0, \ y(t_0) = y_0 \tag{12}$$

To the left and to the right from the point t_0 until the following conditio hold

$$|kt+b|>0. \tag{13}$$

The interval Δ can be chosen as the maximal interval such that the conditio (13) is fulfiled.

2) The function ψ_y vanishes at a point $\tau\in\Delta$, and hence ψ_y has the form $\psi_y = k(t-\tau)$, where $k\in\mathbb{R}^2, k\neq 0$. In this case

$$u(t)=k^0\,\mathrm{sign}(t-\tau),\ \ k^0=\frac{k}{|k|}, \tag{14}$$

i.e., that function $u(t)$ is piecewise costant, taking only two values k^0 and $-k^0$ and having one switching at the point τ (the so-called bang-bang control). Essentially, this is a one-dimensional case, very similar to the time optimal control problem of the form

$$\begin{aligned} &T\to\min,\quad \dot{x}=y,\quad \dot{y}=u,\quad |u|\le 1,\\ &x(0)=\hat{x}_0,\quad x(T)=\hat{x}_T,\ \ y(0)=\hat{y}_0,\ \ y(T)=\hat{y}_T \end{aligned} \tag{15}$$

where $x,y,u\in\mathbb{R}^1$.

But let us return to our problem. In the considered case, the condition $\alpha_0\ge 0$ is equivalent to the following one: $-ky_\tau\ge 0,$ where $y_\tau=y(\tau)$. In order to obtain an extremal of this type, one has to choose arbitrary $k^0,x_\tau,y_\tau\in\mathbb{R}^2$ and τ so that the consitions

$$|k^0|=1,\ \ k^0y_\tau\le 0,\ \ x_\tau^2>r^2, \tag{16}$$

hold, and then to solve a system

$$\dot{x}(t)=y(t),\quad \dot{y}(t)=u(t)=k^0\,\mathrm{sign}(t-\tau),\quad x(\tau)=x_\tau,\quad y(\tau)=y_\tau \tag{17}$$

To the left and to the right of the point t_0.

Consider a system of more general fprm, with switching of the contro lat a point τ and with initial data given at a point t_0 (which may be different from the point τ):

$$\dot{x}(t)=y(t),\quad \dot{y}(t)=u(t)=k^0\,\mathrm{sign}(t-\tau),\quad x(t_0)=x_0,\quad y(t_0)=y_0. \tag{18}$$

System (17) is easily integrable. Consider an interval $\Delta=(t',t'')$ such that $\tau\notin\Delta$. In this case $\mathrm{sign}(t-\tau)$ is a constant on Δ equal to σ, where $\sigma=\pm1$. Assume that $t_0\in\Delta$. Conditions $\dot{y}=u=k^0\sigma,\ y(t_0)=y_0$ imply

$$x=\frac{\sigma k_0}{2}(t-t_0)^2+y_0(t-t_0)+x_0. \tag{19}$$

We have studied the case, where $b=-k\tau$, i.e., k and b are linearly dependent. In the case, where k and b are linearly independent, the integration of equations for x and y is more difficult, although again $x(t)$ and $y(t)$ can be expressed in terms of elementary functions. Let us find formulas for $x(t)$ and $y(t)$ in the latter case.

3. INTEGRATION OF EQUATIONS OF MOTION IN THE CASE OF LINEARLY INDEPENDENT *k* AND *b*

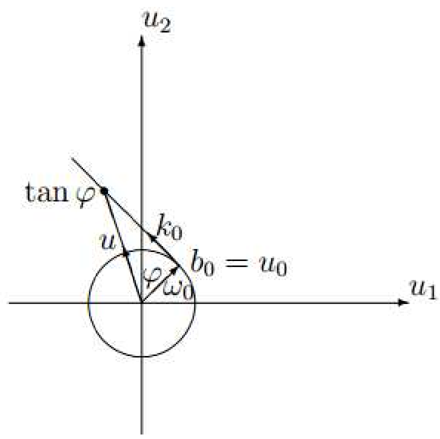
This motion is defined by equations (12):

$$\dot{x}(t)=y(t),\ \dot{y}(t)=u(t)=\frac{kt+b}{|kt+b|},\ x(t_0)=x_0,\ y(t_0)=y_0,\ \ t\in(0,T),$$

where $t_0\in(0,T)$. We assume that k and b are linearly independent and then (taking into account the possibility of multiplication by a positive constant) we can represent the function $\psi_y=kt+b$ in the form $\psi_y=\lambda(t-t_0)k_0+b_0$ where $\lambda\in\mathbb{R},\ \lambda\neq 0,\ b_0\in\mathbb{R}^2,\ |b_0|=1,\ k_0=Ab_0$. Here A is the rotation matrix by the angle $\frac{\pi}{2}$ counterclockwise:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that $\psi_y(t_0) = b_0 = u(t_0)$. Denote $u_0 := u(t_0)$. It is convenient to use the complex plane so that the real axis is u_1, while the imaginary axis is u_2. Then $u = u_1 + iu_2$. Let $\omega(t) = \varphi(t) + \omega_0$, $\varphi(t_0) = 0$, $u_0 = u(t_0) = e^{i\omega_0} = b_0$, $u(t) = e^{i\omega(t)} = e^{i(\varphi(t)+\omega_0)} = u_0 e^{i\varphi(t)}$.



Pic. 1. Plane of control

It is clear that (see the picture 1.) $\tan\varphi(t) = \lambda(t - t_0)$, and hence $\varphi(t) = \operatorname{arc}\tan\lambda(t - t_0)$. Since $\dot{y} = \frac{dy}{d\varphi}\frac{d\varphi}{dt} = u = u_0 e^{i\varphi}$, we have

$$\frac{dy}{d\varphi} = \frac{dt}{d\varphi} u_0 e^{i\varphi}.$$

Moreover, the condition $\tan\varphi(t) = \lambda(t - t_0)$ implies that

$$\frac{dy}{d\varphi} = \frac{1}{\lambda\cos^2\varphi}.$$

Thus

$$\frac{dy}{d\varphi} = \frac{u_0}{\lambda\cos^2\varphi}(\cos\varphi + i\sin\varphi).$$

Consequently,

$$dy = \frac{u_0}{\lambda}\left(\frac{1}{\cos\varphi} + i\frac{\sin\varphi}{\cos^2\varphi}\right)d\varphi,$$

Whence

$$y = \frac{u_0}{\lambda}\left(\int\frac{1}{\cos\varphi}d\varphi + i\int\frac{\sin\varphi}{\cos^2\varphi}d\varphi\right).$$

Let us calculate the integrals in the real and imaginary parts of this expression. We have

$$\int\frac{\sin\varphi}{\cos^2\varphi}d\varphi = \frac{1}{\cos\varphi} + C, \qquad \int\frac{d\varphi}{\cos\varphi} = \ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right| + C.$$

Consequently

$$y=\frac{u_0}{\lambda}\left(\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|+\frac{i}{\cos\varphi}\right)+C\ ,$$

where $C=C_1+iC_2$. Since $\varphi(t_0)=0$, we have $y_0:=y(t_0)=\frac{iu_0}{\lambda}+C$, whence $C=y_0-\frac{iu_0}{\lambda}$. Thus. We get

$$y=\frac{u_0}{\lambda}\left(\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|+\frac{i}{\cos\varphi}\right)+y_0-\frac{iu_0}{\lambda}\ , \tag{20}$$

or

$$y=\frac{u_0}{\lambda}\left(\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|+i\tan\frac{\varphi}{2}\tan\varphi\right)+y_0\,,$$

Taking into account that $y=y_1+iy_2$, $u_0=\cos\omega_0+i\sin\omega_0$, $y_0=y_{10}+iy_{20}$, we obtain the following expressions for y_1 and y_2 in Cartesian coordinates:

$$y_1=\frac{1}{\lambda}\left(\cos\omega_0\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|-\sin\omega_0\tan\frac{\varphi}{2}\tan\varphi\right)+y_{10}\,,$$

$$y_2=\frac{1}{\lambda}\left(\sin\omega_0\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|+\cos\omega_0\tan\frac{\varphi}{2}\tan\varphi\right)+y_{20}\,,$$

$$\varphi=\arctan\lambda(t-t_0)$$

Furthermore, since $\dot{x}=y$, we have $\frac{dx}{d\varphi}=\frac{dt}{d\varphi}y=\frac{1}{\lambda\cos^2\varphi}y$. This combined with (20) implies

$$dx=\frac{u_0}{\lambda^2\cos^2\lambda}\left(\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|+\frac{i}{\cos\varphi}\right)d\varphi+\frac{\lambda y_0-iu_0}{\lambda^2\cos^2\varphi}d\varphi\ .$$

Consequently,

$$x=\frac{u_0}{\lambda^2}\left(\int\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|\frac{d\varphi}{\cos^2\varphi}+i\int\frac{d\varphi}{\cos^3\varphi}\right)+\frac{\lambda y_0-iu_0}{\lambda^2}\int\frac{d\varphi}{\cos^2\varphi}\,. \tag{21}$$

Let us calculate the integrals in this expression. We have

$$\int\frac{d\varphi}{\cos^2\varphi}=\tan\varphi+C\,, \tag{22}$$

$$\int\frac{d\varphi}{\cos^3\varphi}=\int\frac{\cos\varphi}{\cos^4\varphi}=\int\frac{d\sin\varphi}{\left(1-\sin^2\varphi\right)^2}\ . \tag{23}$$

In order to calculate the latter integral, let us find the integral $\int \frac{dz}{\left(z^2-1\right)^2}$. Using the method of undetermined coefficients, we obtain

$$\frac{1}{\left(z^2-1\right)^2}=\frac{1}{4}\left(-\frac{1}{z-1}+\frac{1}{(z-1)^2}+\frac{1}{z+1}+\frac{1}{(z+1)^2}\right).$$

Integrating this function we get

$$\int \frac{dz}{\left(z^2-1\right)^2}=\frac{1}{4}\ln\left|\frac{z+1}{z-1}\right|+\frac{1}{2}\cdot\frac{z}{1-z^2}+C\,.$$

Consequently,

$$\int \frac{d\sin\varphi}{\left(1-\sin^2\varphi\right)^2}=\frac{1}{4}\ln\left|\frac{1+\sin\varphi}{1-\sin\varphi}\right|+\frac{1}{2}\cdot\frac{\sin\varphi}{\cos^2\varphi}+C\,.$$

This combined with (23) implies

$$\int \frac{d\varphi}{\cos^3\varphi}=\frac{1}{4}\ln\left|\frac{1+\sin\varphi}{1-\sin\varphi}\right|+\frac{1}{2}\cdot\frac{\sin\varphi}{\cos^2\varphi}+C\,.$$

We deduce from this that

$$\int \frac{d\varphi}{\cos^3\varphi}=\frac{1}{2}\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|+\frac{1}{2}\cdot\frac{\sin\varphi}{\cos^2\varphi}+C\,. \tag{24}$$

Other integral in (21) can be found by integrating by parts

$$\int \ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\frac{d\varphi}{\cos^2\varphi}=\int \ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)d\tan\varphi=$$

$$=\ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\tan\varphi-\int \tan\varphi d\left\{\ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\right\}=$$

$$=\ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\tan\varphi-\int \tan\varphi\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\frac{1}{\cos^2\frac{\varphi}{2}\left(1-\tan\frac{\varphi}{2}\right)^2}d\varphi=$$

$$=\ln\left(\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right)\tan\varphi-\frac{2}{1-\tan^2\frac{\varphi}{2}}+C\,.$$

We have considered the case $1-\tan^2\frac{\varphi}{2}>0$. Similarly one can analyze the case $1-\tan^2\frac{\varphi}{2}<0$. As result we get

$$\int \ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|\frac{d\varphi}{\cos^2\varphi}=\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|\tan\varphi-\frac{2}{1-\tan^2\frac{\varphi}{2}}+C\,. \tag{25}$$

Relations (21), (22), (24), and (25) imply

$$x=\frac{u_0}{\lambda^2}\left\{\tan\varphi\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|+2-\frac{2}{1-\tan^2\frac{\varphi}{2}}+\frac{i}{2}\left(\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|+\frac{\sin\varphi}{\cos^2\varphi}\right)\right\}+\frac{\lambda y_0-iu_0}{\lambda^2}\tan\varphi+x_0 .$$

Moreover,

$$2-\frac{2}{1-\tan^2\frac{\varphi}{2}}=-\frac{2\tan^2\frac{\varphi}{2}}{1-\tan^2\frac{\varphi}{2}}=-\tan\varphi\tan\frac{\varphi}{2} .$$

Consequently,

$$x=\frac{u_0}{\lambda^2}\left\{\tan\varphi\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|-\tan\varphi\tan\frac{\varphi}{2}+\frac{i}{2}\left(\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|+\frac{\sin\varphi}{\cos^2\varphi}\right)\right\}+\frac{\lambda y_0-iu_0}{\lambda^2}\tan\varphi+x_0 .$$

It follows that

$$x=\frac{u_0}{\lambda^2}\left\{\tan\varphi\left(\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|-\tan\frac{\varphi}{2}\right)+\frac{i}{2}\left(\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|+\frac{\tan\varphi}{\cos\varphi}-2\tan\varphi\right)\right\}+$$
$$+\frac{\lambda y_0-iu_0}{\lambda^2}\tan\varphi+x_0,\qquad \varphi=\arctan\lambda\left(t-t_0\right) \tag{26}$$

Taking into account that $x=x_1+ix_2,\ u_0=\cos\omega_0+i\sin\omega_0,\ x_0=x_{10}+ix_{20},\ y_0=y_{10}+iy_{20}$, we obtain for Cartesian coordinates x_1 and x_2 :

$$x_1=\frac{1}{\lambda^2}\cos\omega_0\tan\varphi\left(\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|-\tan\frac{\varphi}{2}\right)-\frac{\sin\varphi_0}{2\lambda^2}\left(\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|+\frac{\tan\varphi}{\cos\varphi}-2\tan\frac{\varphi}{2}\right)+$$
$$+\frac{y_{10}}{\lambda}\tan\varphi+x_{10},$$

$$x_2=\frac{1}{\lambda^2}\sin\omega_0\tan\varphi\left(\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|-\tan\frac{\varphi}{2}\right)+\frac{\cos\varphi_0}{2\lambda^2}\left(\ln\left|\frac{1+\tan\frac{\varphi}{2}}{1-\tan\frac{\varphi}{2}}\right|+\frac{\tan\varphi}{\cos\varphi}-2\tan\frac{\varphi}{2}\right)+$$
$$+\frac{y_{20}}{\lambda}\tan\varphi+x_{20},$$

where $\varphi=\arctan\lambda\left(t-t_0\right)$.

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NAJSZYBSZY RUCH PUNKTU NA PŁASZCZYŹNIE

Streszczenie

W artykule zrobiono analizę zadania sterowania optymalnego dotyczącego ruchu punktu materialnego na płaszczyźnie, który odbywa się bez tarcia. Punkt jest sterowany za pomocą siły ograniczonej. W analizie tego problemu wykorzystano zasadę maksimum Pontryagina

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