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Positive output controllability of linear discrete–time invariant systems*

by

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Abstract: This paper studies the output controllability of discrete linear time invariant systems (LTI) with non-negative input constraints. Some geometrical arguments and positive invariance concepts are used to derive the necessary and/or sufficient conditions for the positive output controllability of discrete LTI systems. The paper also provides several academic examples, which support the theoretical results.

Keywords: discrete LTI systems, positive control, controllability, positive output controllability

1. Introduction

Controllability is one of the fundamental concepts in control theory. Studying controllability is meant to verify if a controller can be applied to generate a desired state space behavior. The system is said to be controllable if it is possible to transfer any initial state to any final state using an admissible control sequence. When the state and input control are not subject to any condition (i.e. $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$), Kalman, Ho and Narendra (1962) give an algebraic criterion to verify if a system is controllable or not.

In many situations, the systems may be subject to constraints, which change their controllability properties. So, controllability was extended to encompass many associated concepts, like positive controllability (Saperstone and Yorke, 1971), complete controllability (Shen, Shi and Sun, 2010), asymptotic controllability (Bacciotti and Mazzi, 2011), controllability of fractional systems (Klamka, 2019), controllability of positive systems (Kaczorek, 2002; Klamka, 1991)... etc.

The purpose of this work is to investigate controllability of discrete linear time invariant systems under non-negative input constraints. This type of problems is motivated by engineering systems, appearing in many investigations, like

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antivibration control of pendulum systems (Saperstone and Yorke, 1971), and non prehensile mechanisms (Lynch and Mason, 1999). In particular, the constrained controllability has attracted the attention of several authors. In Evans and Murthy (1977), necessary and sufficient conditions are proposed for the positive controllability – controllability of a system with a positive control of discrete linear time invariant systems. For continuous linear time invariant systems, the question is studied in Yoshida and Tanaka (2007), Brammer (1972) and Saperstone and Yorke (1971). For nonlinear systems, Brammer (1972) gives additional sufficient conditions of positive controllability (PC).

Studying controllability is not restricted as to the states, but also as to the outputs. In most of the engineering applications, tasks are defined for outputs. In fact, having control over the output of the system has a significant importance, understandably more so than over the states. For example, take the control of a multilink cable-driven manipulator, where the task is typically defined in terms of end effector pose rather than the joint positions and velocities, which can define the system's state (Lan, Oetomo and Halgamuge, 2013). Output controllability property of a linear invariant-time system is to verify the ability of an external input to move the output from any initial condition to any final condition in a finite time (Chen, 1970; Klamka, 2019). The necessary and sufficient conditions for output controllability of linear time-invariant systems are addressed in, for example, Garcia-Planas and Dominguez-Garcia (2013), Ogata (2010), Kaczorek (2006) and Klamka (2019). When the control is positive, Eden et al. (2016) proposed the necessary and sufficient conditions for the positive output controllability of continuous linear time invariant systems and also an additional sufficient condition to aid in the practical evaluation of positive output controllability.

In this paper, we study the positive output controllability (POC) of discrete LTI systems. The necessary and sufficient conditions of positive output controllability are given. Moreover, we state some practical sufficient condition to establish the positive output controllability.

The structure of the paper is as follows. In the next section, the mathematical notation regarding the positive matrix is presented. We recall also some definitions and propositions concerning cone theory and positive invariance. In Section 3, the problem presentation is provided and we give some definitions, related to controllability, reachability, positive controllability, output controllability... etc. Section 4 introduces the necessary and sufficient conditions for POC of discrete LTI systems. It also gives additional sufficient conditions, which are useful in practical evaluation, these conditions are then further extended to be necessary and sufficient for different cases of systems. Finally, in Section 5, we illustrate these results with some examples.

2. Elementary cone theory and positive invariance

First, we introduce some notations. For $n \in \mathbb{N}$, $\mathbb{R}^n_+$ denotes the non-negative orthant in $\mathbb{R}^n$, and the i^{th} standard basis vector of $\mathbb{R}^n$ will be denoted by e_i. The superscript T denotes matrix transposition. The vector $v = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T \in \mathbb{R}^n$ is said to be positive (non-negative) if for all $i = 1, 2, \cdots, n$, $v_i > 0$ ($v_i \geq 0$). For any vectors v, $u \in \mathbb{R}^n$, the inner product is denoted $\langle v, u \rangle = u^T v$.

Now, we recall some basic definitions concerning the characterization of the cone set (Luenberger, 1968; Klamka, 2019; Tarbouriech and Castelan, 1993; Kaczorek, 2011).

Definition 2.1 (Luenberger, 1968; Kaczorek, 2011) *A set $X \in \mathbb{R}^n$ is said to be a cone if for all $x \in X$ and $\alpha \geq 0$, $\alpha x \in X$. The set X is a convex cone if for all $x, y \in X$ and $\alpha, \beta \in \mathbb{R}_+$, $\alpha x + \beta y \in X$.*

Definition 2.2 *Let $G = \begin{bmatrix} g_1 & \cdots & g_m \end{bmatrix} \in \mathbb{R}^{n \times m}$, the image (span) of the matrix G is defined as the set*

$$Im(G) := \{x \in \mathbb{R}^n \quad | \quad x = \sum_{i=1}^{m} \alpha_i g_i, \quad \alpha_i \in \mathbb{R}\}.$$

The positive span of the matrix G is defined as the set

$$span_+(G) := \{x \in \mathbb{R}^n \quad | \quad x = \sum_{i=1}^{m} \alpha_i g_i, \quad \alpha_i \geq 0\}.$$

Definition 2.3 (Luenberger, 1968) *Let $X \subset \mathbb{R}^n$. X^- be defined as the negative polar cone of the set X, is the set of all $y \in \mathbb{R}^n$ such that $\langle y, x \rangle \leq 0$, $\forall x \in X$.*

Remark 2.1
1. *The negative polar cone and the positive span of a matrix always form convex cones by Definition 2.1.*
2. *According to Tarbouriech and Castelan (1993), a convex cone X of $\mathbb{R}^n$ can be characterized by a matrix $D \in \mathbb{R}^{q \times n}$*

$$X = cone(D) = \{x \in \mathbb{R}^n \quad | \quad Dx \leq 0\}.$$

□

If the LTI system admits some domains in its state space, from which any state vector trajectory cannot escape, these domains are called positively invariant sets of the system. The existence and characterization of positively invariant sets of dynamical systems is therefore a basic issue for many constrained regulation problems. To analyze the desired properties of a closed-loop LTI system under a linear state feedback, it suffices to study the discrete LTI system given by:

$$x(k+1) = Ax(k), \quad x(0) = x_0, \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$. The solution of the system (1) is given by $x(k) = A^k x_0$.

Definition 2.4 *A nonempty set $X \in \mathbb{R}^n$ is a positively invariant set with respect to system (1) if and only if for any initial state $x(0) \in X$, the trajectory of the state vector $x(k)$ remains in X (i.e. $x(k) \in X$, $\forall k \in \mathbb{N}$).*

The set X can be a polyhedron, a vector space or a cone. In the last cases, the positive invariance is equivalent to the well-known property of A-invariance of subspaces (Wonham, 1985).

Proposition 2.1 (Tarbouriech and Castelan, 1993) *A cone X is positively invariant with respect to system (1) if and only if $AX \subset X$.*

We can deduce that the cone X is positively invariant with respect to system (1) if and only if $A^k X \subset X$, $\forall k \in \mathbb{N}^*$. From Remark 2.1 and Proposition 2.1, a characterization of positive invariant cone with respect to system (1) is given as follows (Tarbouriech and Castelan, 1993; Castelan and Hennet, 1993):

Proposition 2.2 *$X = cone(D)$ is positively invariant with respect to system (1), if and only if the following is verified:*

$$Dx(k) \leq 0 \Rightarrow Dx(k+1) \leq 0, \quad \forall x(k) \in X, \forall k \geq 0. \tag{2}$$

A very interesting and useful characterization of the positive invariance of a cone $X = cone(D)$ is the following result from Castelan and Hennet (1993) and Tarbouriech and Castelan (1993).

Proposition 2.3 *Let $D \in \mathbb{R}^{q \times n}$, the $cone(D)$ is positively invariant with respect to system (1) if and only if there exists a non-negative matrix $H \in \mathbb{R}^{q \times q}$ such that $DA = HD$.*

3. Problem formulation and auxiliary results

This paper is concerned with the following system

$$x(k+1) = Ax(k) + Bu(k) \tag{3}$$

$$y(k) = Cx(k) \tag{4}$$

where the state $x \in \mathbb{R}^n$, the m dimensional input $u \in \mathbb{R}^m$ and the p dimensional output $y \in \mathbb{R}^p$. The system dynamics is given by A, B and C, which are matrices with appropriate dimensions. In particular, the paper studies the positive output controllability of the system (3)-(4). The concept of positive output controllability will be defined in the sequel.

3.1. Controllability

The controllability property means that the input may be chosen in order to drive the state from any initial state x_0 to any final state x_f. In addition, the properties of reachability and null controllability are defined in a similar manner with $x_0 = 0$ and $x_f = 0$ respectively. Formally, controllability, reachability and null controllability are defined as in Callier and Desoer (1991), Castelan and Hennet (1993).

DEFINITION 3.1 *A discrete LTI system (3) is controllable if for all* $x_0, x_f \in \mathbb{R}^n$, *there exists a finite integer* $N > 0$ *and an input sequence* $\big(u(0)\, u(1) \cdots u(N-1)\big)$ *such that* $x(0) = x_0$ *and* $x(N) = x_f$.

DEFINITION 3.2 *A discrete LTI system (3) is reachable if for all* $x_f \in \mathbb{R}^n$, *there exists a finite integer* $N > 0$ *and an input sequence* $\big(u(0) \quad u(1) \cdots u(N-1)\big)$ *such that* $x(0) = 0$ *and* $x(N) = x_f$.

DEFINITION 3.3 *A discrete LTI system (3) is null controllable if for all* $x_0 \in \mathbb{R}^n$, *there exists a finite integer* $N > 0$ *and an input sequence* $\big(u(0)\, u(1) \cdots u(N-1)\big)$ *such that* $x(0) = x_0$ *and* $x(N) = 0$.

REMARK 3.1 *For discrete LTI systems, the controllability is equivalent to the reachability and implies the null controllability (see, pp. 288 in Callier and Desoer, 1991).* □

The concept of controllability has been extended to positive controllability, i.e. controllability subject to the non-negative input constraint $u(.) \in \cup_N^+$, where $\cup_N^+ = \{u(.) \in \mathbb{R}^m \mid u(k) \to \mathbb{R}_+^m,\ k = 0, 1, \ldots, N-1\}$, where $N \geq 1$.

DEFINITION 3.4 *A discrete LTI system (3) is positively controllable* (PC) *if for all* $x_0, x_f \in \mathbb{R}^n$, *there exists a finite integer* $N > 0$ *and a non-negative input sequence* $u(.) \in \cup_N^+$ *such that* $x(0) = x_0$ *and* $x(N) = x_f$.

DEFINITION 3.5 *A discrete LTI system (3) is positively reachable* (PR) *if for all* $x_f \in \mathbb{R}^n$, *there exists a finite integer* $N > 0$ *and a non-negative input sequence* $u(.) \in \cup_N^+$ *such that* $x_0 = 0$ *and* $x(N) = x_f$.

The following property helps us to determine the necessary and sufficient conditions for positive controllability of the system (3):

PROPOSITION 3.1 *A discrete LTI system (3) is PC if and only if it is positively reachable.*

PROOF As the necessary condition is obvious, we omit its proof. To show sufficiency, let x_0, x_f be two vectors of $\mathbb{R}^n$. By the positive reachability, then, for N sufficiently large there is a non-negative input sequence $\big(u(0)\, u(1) \cdots u(N-1)\big)$ such that

$$x_f = \sum_{i=0}^{N-1} A^{(N-1-i)} B u(i).$$

And for the final state $(-A^N x_0)$ there is a non-negative input sequence $\big(v(0)\, v(1) \cdots v(N-1)\big)$ such that

$$-A^N x_0 = \sum_{i=0}^{N-1} A^{(N-1-i)} B v(i).$$

By the superposition property of linear systems it can therefore be seen that the input $(u+v)(.) \in \cup_N^+$ and its corresponding trajectory is

$$\begin{aligned} x(k) &= A^k x_0 + \sum_{i=0}^{k-1} A^{(k-1-i)} B(u+v)(i) \\ &= A^k x_0 + \sum_{i=0}^{k-1} A^{(k-1-i)} B u(i) + \sum_{i=0}^{k-1} A^{(k-1-i)} B v(i). \end{aligned}$$

And, we have

$$\begin{aligned} x(N) &= A^N x_0 + \sum_{i=0}^{N-1} A^{(N-1-i)} B u(i) + \sum_{i=0}^{N-1} A^{(N-1-i)} B v(i) \\ &= \sum_{i=0}^{N-1} A^{(N-1-i)} B u(i) = x_f, \\ x(0) &= x_0. \end{aligned}$$

This completes the proof. ∎

The following proposition gives the necessary and sufficient conditions for PC of the discrete LTI system (3), and it can be proven by using only well known and elementary geometric properties of sets of controllability.

Proposition 3.2 *The discrete LTI system* (3) *is PC if and only if there is no nonzero vector* $v \in \mathbb{R}^n$ *such that*

$$\big\langle v, A^k B u \big\rangle \leq 0, \quad \forall u \geq 0,\ \forall k \geq 0. \tag{5}$$

Proof Let us recall the positive reachable cone of (3) and its negative polar cone:

$$\begin{aligned} R_s &= \left\{ \sum_{i=0}^{N-1} A^{(N-1-i)} B u(i) : \forall N \geq 1 \text{ and } \forall u(.) \in \cup_N^+ \right\}, \\ R_s^- &= \left\{ v \in \mathbb{R}^n : \left\langle v, \sum_{i=0}^{N-1} A^{(N-1-i)} B u(i) \right\rangle \leq 0,\ \forall N \geq 1 \text{ and } \forall u(.) \in \cup_N^+ \right\}. \end{aligned}$$

Let us also define the following set

$$X = \left\{ v \in \mathbb{R}^n : \big\langle v, A^k B u \big\rangle \leq 0,\ \forall u \geq 0 \text{ and } \forall k \geq 0 \right\}.$$

Using Proposition 3.1, the discrete LTI system (3) is PC if and only it is PR, or, equivalently, that $R_s = \mathbb{R}^n$. Therefore, it will be sufficient to prove that $R_s^- = X$. It is clear that $X \subset R_s^-$. For the inverse inclusion, let $v \in R_s^-$, then

$$\sum_{i=1}^{N} \left\langle v, A^{(N-i)} Bu(i-1) \right\rangle \leq 0, \quad \forall u(.) \in \cup_N^+.$$

So, by taking a special choice of $u(.) \in \cup_N^+$, like

$$u(0) = u, \qquad for\ i = 0,$$
$$u(i) = 0, \qquad for\ i \neq 0,$$

we then have

$$\left\langle v, A^{N-1} Bu \right\rangle = \left\langle v, A^k Bu \right\rangle \leq 0, \quad \forall k = N-1 \geq 0 \text{ and } \forall u \geq 0.$$

■

3.2. Output controllability

The output controllability property of system (3) with output (4) is defined as follows:

Definition 3.6 *A discrete LTI system (3)-(4) is output controllable if for all* $y_0, y_f \in \mathbb{R}^p$*, there exists a finite integer* $N > 0$ *and an input trajectory* $u(.) \in \mathbb{R}^m$ *such that* $y(0) = y_0$ *and* $y(N) = y_f$*.*

Consistently with Definitions 3.4 - 3.5, positive output controllability and output reachability are defined as follows

Definition 3.7 *A discrete LTI system (3)-(4) is positive output controllable* (POC) *if for all* $y_0, y_f \in \mathbb{R}^p$*, there exists a finite integer* $N > 0$ *and a positive input trajectory* $u(.) \in \mathbb{R}^m$ *such that* $y(0) = y_0$ *and* $y(N) = y_f$*.*

Definition 3.8 *A discrete LTI system (3)-(4) is positive output reachable* (POR) *if for all* $y_f \in \mathbb{R}^p$*, there exists a finite integer* $N > 0$ *and a positive input trajectory* $u(.) \in \mathbb{R}^m$ *such that* $y(0) = 0$ *and* $y(N) = y_f$*.*

Remark 3.2 *Output reachability and output null controllability properties are analogous to Definitions 3.6 - 3.7, with* $y_0 = 0$ *and* $y_f = 0$*, respectively.* □

4. The main results

4.1. Necessary and sufficient conditions for POC of discrete LTI systems

To determine the necessary and sufficient conditions for the positive output controllability of the system (3) with output (4), the following proposition is first stated:

Proposition 4.1 *A discrete LTI system (3)-(4) is POC if and only if it is POR.*

Proof The proof is similar to the one of Proposition 3.1, where x_0, x_f and $x(k)$ are replaced by y_0, y_f and $y(k) = Cx(k)$, respectively. ∎

The following theorem gives the necessary and sufficient conditions for *POC* of discrete time invariant systems, based on the separating hyperplane theorem (Luenberger, 1968):

Theorem 4.1 *A discrete LTI system is POC if and only if there is no nonzero vector $v \in \mathbb{R}^p$ such that*

$$\left\langle v, CA^k Bu \right\rangle \leq 0, \quad \forall u \geq 0 \text{ and } \quad \forall k \geq 0. \tag{6}$$

Proof The proof is similar to the one of Proposition 3.2, where R_s, R_s^- and X are replaced by

$$R_o = \left\{ \sum_{i=0}^{N-1} CA^{(N-1-i)} Bu(i) : \forall N \geq 1 \text{ and } \forall u(.) \in \cup_N^+ \right\},$$

$$R_o^- = \left\{ v \in \mathbb{R}^p : \left\langle v, \sum_{i=0}^{N-1} CA^{(N-1-i)} Bu(i) \right\rangle \leq 0, \quad \forall N \geq 1 \text{ and } \forall u(.) \in \cup_N^+ \right\}$$

and

$$Y = \left\{ v \in \mathbb{R}^p : \left\langle v, CA^k Bu \right\rangle \leq 0, \quad \forall u \geq 0 \text{ and } \forall k \geq 0 \right\},$$

respectively. Therefore, we omit the details. ∎

Remark 4.1 $\mathbb{B} = span_+(B)$. *Rearranging (6) to the form*

$$\left\langle (A^T)^k C^T v, Bu \right\rangle \leq 0.$$

It can be seen that Theorem 4.1 states that the discrete LTI system (3)-(4) is POC if and only if there is no vector $z_0 \in Im(C^T)$ with dynamic systems described by

$$z(k+1) = A^T z(k), \quad z(0) = z_0 = C^T v, \tag{7}$$

such that $z(k) \in \mathbb{B}^-$, $\forall k \geq 0$. □

Theorem 4.2 *A discrete LTI system is POC if and only if there is no matrix $D \in \mathbb{R}^{q \times n}$ such that*

(i) $DA^T = HD$ *where H is a non-negative matrix.*
(ii) $\exists v \in \mathbb{R}^q$ *such that* $DC^T v \leq 0$.
(iii) $cone(D) \subset \mathbb{B}^-$.

Proof Assume that there exists a matrix $D \in \mathbb{R}^{q\times n}$ that satisfies the given conditions. Then, by Proposition 2.3, the first condition implies that the $cone(D)$ is positive invariant for the dynamics (7). The second condition shows that there exists $z_0 \in Im(C^T)$ such that $DC^T v = Dz_0 \le 0$. Hence, $z(k) \in cone(D)$ (see (2)). The third condition implies that $z(k) \in \mathbb{B}^-$, i.e. $\langle z(k), Bu\rangle \le 0$, $\forall k \ge 0$. As a result, the discrete LTI system is not POC by Theorem 4.1.

Conversely, assume that the system (3)-(4) is not POC. Then, by the interpretation of Remark 2.1, there is an initial state $z_0 \in Im(C^T)$ such that $z(k)$, given by (7), lies in the negative polar *cone* $\mathbb{B}^-$, $\forall k \ge 0$. Let $E = \{z(k) \mid z(k) \in \mathbb{B}^-,\ \forall k \ge 0\}$. It is clear that E is a cone in $\mathbb{B}^-$. Then, for every $z_0 \in E$, we have

$$\langle z_0, Bu\rangle \le 0 \implies \left\langle (A^T)^k z_0, Bu\right\rangle = \langle z(k), Bu\rangle \le 0,$$

where the last inequality is established as $z(k) \in \mathbb{B}^-$, $\forall k \ge 0$. Therefore, the set E is a positively invariant cone in $\mathbb{B}^-$. By Proposition 2.3, this means that there is a matrix $D \in \mathbb{R}^{q\times n}$ and a non-negative matrix $H \in \mathbb{R}^{q\times q}$ such that $DA^T = HD$. The matrix D is exactly the matrix $B^T(A^T)^k$, i.e. there is $z_0 = C^T v \in Im(C^T)$ such that $z(k) \in \mathbb{B}^-$. Consequently, $\langle z(k), Bu\rangle = \left\langle (A^T)^k z_0, Bu\right\rangle = \left\langle B^T(A^T)^k C^T v, u\right\rangle \le 0$. Finally, since $u \ge 0$, then $B^T(A^T)^k C^T v \le 0$ and there is $v \in \mathbb{R}^q$ such that $DC^T v \le 0$. ■

4.2. Sufficient conditions

4.2.1. Sufficient conditions for PC and POC of discrete LTI systems

This section gives sufficient conditions for PC and POC of the discrete LTI system (3)-(4). The following theorems show the cases where (5) and (6) are satisfied for every $v \in \mathbb{R}^n$ and $v \in \mathbb{R}^p$, respectively.

Theorem 4.3 *The system (3) is PC if for all $v \in \mathbb{R}^n$, there is $k \ge 0$ and $u \ge 0$ such that $\left\langle v, A^k Bu\right\rangle > 0$.*

Proof Proposition 3.2 shows that the inner product (5) must not be negative or null. So, if there exists a control input $u \ge 0$ such that the inner product (5) remains non-negative, then the system is PC. ■

Theorem 4.4 *The system (3)-(4) is POC if for all $v \in \mathbb{R}^p$ there is $k \ge 0$ and $u \ge 0$ such that $\left\langle v, CA^k Bu\right\rangle > 0$.*

Proof Theorem 4.1 shows that the inner product (6) must not be negative or null. So, if there exists a control input $u \ge 0$ such that the inner product (6) remains non-negative, then the system is POC. ■

Using Remark 4.1 and Theorem 4.1, the following theorem provides a sufficient condition for the positive output controllability by looking at the behavior of $z(k)$ as a solution of (7).

Theorem 4.5 *The linear system (3)-(4) is POC if there is no* $z_0 \in Im(C^T)$ *such that*

$$\left\langle z_0, A^k Bu \right\rangle \leq 0, \quad \forall k \geq 0, \forall u \geq 0. \tag{8}$$

Proof It is clear from Remark 4.1 by taking $z_0 = C^T v$ in the inner product (6). ■

4.2.2. Necessary and sufficient conditions for PC and POC of discrete LTI systems: case discussion

In this section, we give the necessary and sufficient conditions for the PC and POC of discrete LTI system (3)-(4), first if the system is single input (i.e. $u \in \mathbb{R}$), secondly, when it is single output (i.e. the vector $v \in \mathbb{R}$), and finally when both of them are not single. These cases are given as follows:

- Case 1: $m = 1$ and $p \geq 1$.
- Case 2: $m > 1$ and $p = 1$.
- Case 3: $m > 1$ and $p > 1$.

First, we begin with the situation where we have a single input. In this paragraph the matrix B will be replaced by a vector $b \in \mathbb{R}^n$.

Proposition 4.2 *A discrete* LTI *system (3) with a single input control* $u \in \mathbb{R}^+$ *is not* PC.

Proof Consider a discrete LTI system (3) such that $A^k b = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}^T$, then $\forall v \in \mathbb{R}^n$ we have

$$\left\langle v, A^k bu \right\rangle = \left\langle \begin{bmatrix} v_1 \cdots v_n \end{bmatrix}, \begin{bmatrix} f_1 \cdots f_n \end{bmatrix}^T u \right\rangle = u \sum_{i=1}^{n} v_i f_i, \ \forall k \geq 0, \forall u(.) \in \mathbb{R}^+.$$

If we take $v_i = -f_i$, $\forall i = 1 \ldots n$, then

$$u \sum_{i=1}^{n} v_i f_i = -u \sum_{i=1}^{n} (f_i)^2 \leq 0.$$

So, by Proposition 3.2, the system (3) is not PC. ■

Proposition 4.3 *A discrete* LTI *system (3)-(4) with a single input control* $u \in \mathbb{R}^+$ *is not* POC.

Proof Consider a discrete LTI system (3)-(4) such that $CA^k b = \begin{bmatrix} l_1 & \cdots & l_p \end{bmatrix}^T$, then $\forall v \in \mathbb{R}^p$ we have

$$\left\langle v, CA^k bu \right\rangle = \left\langle \begin{bmatrix} v_1 \cdots v_p \end{bmatrix}, \begin{bmatrix} l_1 \cdots l_p \end{bmatrix}^T u \right\rangle = u \sum_{i=1}^{p} v_i l_i, \quad \forall k \geq 0, \forall u(.) \in \mathbb{R}^+.$$

If we take $v_i = -l_i$, $\forall i = 1 \dots p$, then

$$u\sum_{i=1}^{p} v_i l_i = -u\sum_{i=1}^{p}(l_i)^2 \leq 0.$$

So, by Theorem 4.1, the system (3)-(4) is not POC.

Remark 4.2 *Proposition 4.2 and Proposition 4.3 show that if $m = 1$, the LTI system is not PC neither POC. These results are illustrated in Example 3.* □

In the case where we have a single output, the matrix CA^kB and the inner product (6) are as follows

$$CA^kB = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix},$$

$$\langle v, CA^kBu\rangle = v\sum_{i=1}^{m} f_i u_i.$$

Proposition 4.4 *A discrete LTI system (3)-(4) with $m > 1$ and $p = 1$ is POC if and only if for every $v \in \mathbb{R}$, there exist $i_1, i_2 \in \{1, 2, ..., m\}$ such that $f_{i_1} \times f_{i_2} < 0$.*

Proof Assume that (3)-(4) with $m > 1$ and $p = 1$ is POC, then by Theorem 4.1, there is no nonzero $v \in \mathbb{R}$ such that $\langle v, CA^kBu\rangle = v\sum_{i=1}^{m} f_i u_i \leq 0$, for every $u \in \mathbb{R}_+^m$. If the components f_i, for all $i \in \{1, 2, ..., m\}$ are all with the same sign or null, then there is $v \in \mathbb{R}$ with different sign from f_i and for all $u \in \mathbb{R}_+^m$ the inner product $\langle v, CA^kBu\rangle \leq 0$.

This is not possible, since it is contradictory to what we assumed (i.e. that (3)-(4) is POC). So, the components f_i, for all $i \in \{1, 2, ..., m\}$, are not all with the same sign neither null, or at least two components of CA^kB have opposed sign and are not null. In this case, there are many choices of $u \in \mathbb{R}_+^m$ for which $\langle v, CA^kBu\rangle > 0$. Conversely, if there is $i_1, i_2 \in \{1, 2, ..., m\}$ such that $f_{i_1} \times f_{i_2} < 0$, then for every $v \in \mathbb{R}$ there is $u \in \mathbb{R}_+^m$ with $u_i = 0$, for all $i \neq i_1$ and $i \neq i_2$ such that $\langle v, CA^kBu\rangle = v(f_{i_1}u_{i_1} + f_{i_2}u_{i_2}) > 0$. Then, by Theorem 4.4, the system (3)-(4) is POC. ■

Remark 4.3 *Proposition 4.4 shows that, the components of the matrix CA^kB must not all have the same sign nor null. If it is not the case, then there is $v \in \mathbb{R}$ for all $u \in \mathbb{R}_+^m$ such that the inner product (6) is negative or null, and then, by Theorem 4.1, the system is not POC. We illustrate this result in Example 1.*□

Finally, in this subsection, sufficient and necessary conditions for PC ($with\ m > 1$) and POC ($with\ m > 1\ and\ p > 1$) are proposed. First, we need to define the matrix A^kB and the inner product (5) as follows

$$A^kB = \begin{pmatrix} g_{1,1} & \cdots & g_{1,m} \\ \vdots & \ddots & \vdots \\ g_{n,1} & \cdots & g_{n,m} \end{pmatrix},$$

$$\langle v, A^k Bu \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} (v_j g_{j,i}) u_i.$$

PROPOSITION 4.5 *A discrete LTI system (3) with $m > 1$ is PC if and only if for every $v \in \mathbb{R}^n$ there exists $i_1 \in \{1, 2, ..., m\}$ such that $\sum_{j=1}^{n} v_j g_{j,i_1} > 0$.*

PROOF Assume that (3) with $m > 1$ is PC, then, by Proposition 3.2, there is no nonzero vector $v \in \mathbb{R}^n$ for every $u \in \mathbb{R}_+^m$ such that

$$\langle v, A^k Bu \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} (v_j g_{j,i}) u_i \leq 0.$$

This is not possible if for all $i \in \{1, 2, ..., m\}$, $\sum_{j=1}^{n} v_j g_{j,i} \leq 0$ (for example $v_j = -g_{j,i}$, $\forall i \in \{1, 2, ..., m\}$). So, at least there is $i_1 \in \{1, 2, ..., m\}$ such that $\sum_{j=1}^{n} v_j g_{j,i_1} > 0$. In this case, there are many choices of u, for which $\langle v, A^k Bu \rangle > 0$. Conversely, if for every $v \in \mathbb{R}^n$ there is $i_1 \in \{1, 2, ..., m\}$ such that $\sum_{j=1}^{n} v_j g_{j,i_1} > 0$, then there is $u \in \mathbb{R}_+^m$ with $u_i = 0$, $\forall i \neq i_1$ such that $\langle v, A^k Bu \rangle = \sum_{j=1}^{n} v_j g_{j,i_1} u_{i_1} > 0$, and then by Theorem 4.3 the system (3) is PC. ∎

We move now to the POC of the discrete LTI system (3)-(4) with $m > 1$ and $p > 1$. First, we need to define the matrix CA^kB and the inner product (6) as follows

$$CA^kB = \begin{pmatrix} f_{1,1} & \cdots & f_{1,m} \\ \vdots & \ddots & \vdots \\ f_{p,1} & \cdots & f_{p,m} \end{pmatrix},$$

$$\langle v, CA^k Bu \rangle = \sum_{i=1}^{m} \sum_{j=1}^{p} (v_j f_{j,i}) u_i.$$

PROPOSITION 4.6 *A discrete LTI system (3)-(4) with $p > 1$ and $m > 1$ is POC if and only if for every $v \in \mathbb{R}^p$ there exists $i_1 \in \{1, 2, ..., m\}$ such that $\sum_{j=1}^{p} v_j f_{j,i_1} > 0$.*

PROOF The proof is similar to the proof of Proposition 4.5. ∎

An additional sufficient condition for PC and POC is the following one:

PROPOSITION 4.7 *A discrete LTI system (3) with m even and non-null is PC if the following two assertions hold, for all $k \in \mathbb{N}$*

i) The matrix A^kB has only two opposed non-null elements in $\frac{m}{2}$ rows.

ii) The matrix A^kB has only one non-null element in each column.

PROOF Assume that the matrix A^kB has only two opposed non-null elements in each row and only one non-null element in each column. So, the matrix A^kB is exactly like

$$A^kB = \begin{pmatrix}\alpha_{i_1}e_{i_1} & \beta_{i_1}e_{i_1} & \cdots & \alpha_{i_{\frac{m}{2}}}e_{i_{\frac{m}{2}}} & \beta_{i_{\frac{m}{2}}}e_{i_{\frac{m}{2}}}\end{pmatrix},$$

for all $i_1, ..., i_{\frac{m}{2}} \in \{1, 2, ..., n\}$ and $\alpha_{i_j} \times \beta_{i_j} < 0$ for all $j \in \{1, 2, ... \frac{m}{2}\}$, and then the inner product (5) becomes

$$\langle v, CA^kBu\rangle = \sum_{j=1}^{\frac{m}{2}} v_{i_j}(\alpha_{i_j}u_{(j_1)} + \beta_{i_j}u_{(j_2)}).$$

Since $\alpha_{i_j} \times \beta_{i_j} < 0$ for all $j \in \{1, 2, ... \frac{m}{2}\}$ then whatever the sign of v_{i_j} there is $u_{(j_1)}, u_{(j_2)} \in \{u_1, u_2, ..., u_m\}$ for all $j \in \{1, 2, ... \frac{m}{2}\}$ such that the inner product (5) turns positive. Then, by Theorem 4.3, the system (3) is PC. ■

PROPOSITION 4.8 *A discrete LTI system (3)-(4) with m even and non-null and $p > 1$ is POC if the following two assertions hold, for all $k \in \mathbb{N}$*

i) The matrix CA^kB has only two opposed non-null elements in $\frac{m}{2}$ rows.
ii) The matrix CA^kB has only one non-null element in each column.

PROOF The proof is similar to the proof of Proposition 4.7. ■

5. Numerical examples

5.1. Example 1

Consider the discrete LTI system (A, B, C) defined by the following matrices:

$$A = \begin{bmatrix}1 & 2\\0 & 3\end{bmatrix},\ B = \begin{bmatrix}-1 & -3\\0 & -1\end{bmatrix} \text{and} \quad C = \begin{bmatrix}c_1 & c_2\end{bmatrix},$$

where $c_1, c_2 \in \mathbb{R}$. This system is controllable since $rank\begin{bmatrix}B & AB\end{bmatrix} = 2$. However, it is not PC. Indeed, let us consider the eigenvectors $v_1 = \begin{bmatrix}1 & -1\end{bmatrix}^T$ and $v_2 = \begin{bmatrix}0 & 1\end{bmatrix}^T$ of A^T. For all $u \geq 0$, we have

$$\begin{aligned}\langle v_2, A^kBu\rangle &= \left\langle \begin{pmatrix}0 & 1\end{pmatrix}, \begin{bmatrix}1 & 2^k\\0 & 3^k\end{bmatrix}\begin{bmatrix}-1 & -3\\0 & -1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}\right\rangle\\ &= \left\langle \begin{pmatrix}0 & 1\end{pmatrix}, \begin{bmatrix}-u_1 + (-3-2^k)u_2\\ -3^ku_2\end{bmatrix}\right\rangle\\ &= -3^ku_2 \leq 0, \quad \forall k \geq 1, \forall u \geq 0.\end{aligned}$$

So, by Proposition 3.2, the system (A, B) is not PC.

Using Theorem 4.1, this system is POC if

$$CA^kB = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} 1 & 2^k \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} -c_1 & (-3-2^k)c_1 - 3^k c_2 \end{bmatrix},$$

is positive spanning (i.e. there is no non-zero $v \in \mathbb{R}$ for all $u \geq 0$ such that $\langle v, CA^kBu \rangle \leq 0$). Since $CA^kB \in \mathbb{R}^{1\times 2}$, $\forall v \in \mathbb{R}$ and

$$\langle v, CA^kBu \rangle = v(-c_1u_1 + ((-3-2^k)c_1 - 3^kc_2)u_2),$$

then this is the case if the elements of CA^kB contain one positive term and one negative term. This means that, for all $v \in \mathbb{R}$, if $-c_1((-3-2^k)c_1 - 3^kc_2) < 0$, there is $u_1, u_2 \in \mathbb{R}_+$ such that $\langle v, CA^kBu \rangle > 0$. So, by Theorem 4.4, the system (A, B, C) is POC.

5.2. Example 2

The studies of the positive output controllability become an alternative issue if the system is not controllable neither PC. This example illustrates such a situation.

Consider the discrete LTI system (A, B, C) with multiple input ($m = 2$) as follows

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \; B = \begin{bmatrix} -1 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \text{ and } \quad C = \begin{bmatrix} -1 & -3 & 2 \end{bmatrix}.$$

This system is not controllable, since $rank[B \; AB \; A^2B] < 3$. The matrices A^k, A^kB and CA^kB are given by

$$A^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3^k & \beta \\ 0 & 0 & 2^k \end{bmatrix}, \; A^kB = \begin{bmatrix} -1 & 0 \\ 0 & 3^{k+1} \\ 0 & 0 \end{bmatrix} \text{ and } \quad CA^kB = \begin{bmatrix} 1 & -3^{k+2} \end{bmatrix},$$

where $\beta \in \mathbb{R}$. Taking the eigenvectors of A^T as $v_1 = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T$, $v_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ and $v_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, the inner product (5) is not positive for all $u \geq 0$, because

$$\langle v_1, A^kBu \rangle = -3^{k+1}u_1 \leq 0, \quad \forall k \geq 0, \forall u \geq 0.$$

Then, by Proposition 3.2, the system (A, B) is not PC. The inner product (6) is given as follows

$$\langle v, CA^kBu \rangle = v(u_1 - 3^{k+2}u_2), \quad \forall k \geq 0, \forall u \geq 0.$$

For all $v \in \mathbb{R}$ there is $u_1, u_2 \geq 0$ such that $\langle v, CA^kBu \rangle > 0$. Then, by Theorem 4.4, the system (A, B, C) is POC.

5.3. Example 3

Consider the discrete LTI system (A, B, C) defined by the following matrices:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n,n} \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_n \end{bmatrix} \text{ and } \ C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix},$$

where $c_i, a_{i,j} \in \mathbb{R}$, $\forall i, j \in \{1, 2, \ ... \ , n\}$. The matrices A^k, A^kB and CA^kB are given by

$$A^k = \begin{bmatrix} d_{1,1} & d_{1,2} & \cdots & d_{1,n} \\ 0 & d_{2,2} & \cdots & d_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_{n,n} \end{bmatrix}, \ A^kB = \begin{bmatrix} d_{1,n}b_n \\ d_{2,n}b_n \\ \vdots \\ d_{n,n}b_n \end{bmatrix} \text{ and } \ CA^kB = b_n \sum_{i=1}^{n} c_i d_{i,n},$$

where $d_{i,i} = a_{i,i}^k$ and $d_{i,j} \in \mathbb{R}$, $\forall i, j \in \{1, 2, \ ... \ , n\}$. Then, the inner product (5) becomes $\forall v = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^n$

$$\langle v, A^kBu \rangle = b_n u \sum_{i=1}^{n} v_i d_{i,n}, \quad \forall k \geq 0, \forall u \geq 0.$$

Since $u \geq 0$, by Proposition 3.2, the system (A, B, C) is PC if $\forall v = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^n$, and we have $\sum_{i=1}^{n} v_i d_{i,n} < 0$. This is not the case, so the system is not PC.

The inner product (6) is given as follows

$$\langle v, CA^kBu \rangle = v b_n u \sum_{i=1}^{n} c_i d_{i,n}.$$

Since $u \geq 0$, by Theorem 4.1, the system (A, B, C) is POC if $\forall v \in \mathbb{R}$, so that we have $v \sum_{i=1}^{n} c_i d_{i,n} < 0$. But this is not the case, then the system is not POC.

For example, we take the system (A, B, C) as follows:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 2 \\ 0 & 0 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix} \text{ and } \ C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix},$$

where $c_1, c_2, c_3 \in \mathbb{R}$. This system is controllable since $rank \begin{bmatrix} B & AB & A^2B \end{bmatrix} = 3$.

The matrices A^k, A^kB and CA^kB are given by

$$A^k = \begin{bmatrix} 2^k & \alpha & \gamma \\ 0 & 5^k & \beta \\ 0 & 0 & 4^k \end{bmatrix}, \ A^kB = \begin{bmatrix} \gamma b_3 \\ \beta b_3 \\ 4^k b_3 \end{bmatrix} \text{ and } \ CA^kB = (c_1\gamma + c_2\beta + c_3 4^k) b_3,$$

where $\alpha, \gamma, \beta \in \mathbb{R}$. Taking the eigenvectors of A^T as $v_1 = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^T$, $v_2 = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}^T$ and $v_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, the inner product (5) is not positive for all $u \geq 0$, because there exists $v = -b_3 v_3 \in \mathbb{R}^3$ such that

$$\langle v, A^k Bu \rangle = -4^k b_3^2 u \leq 0, \quad \forall k \geq 0, \forall u \geq 0.$$

Then, by Proposition 3.2, the system is not PC. The inner product (6) is given as follows

$$\langle v, CA^k Bu \rangle = v(c_1\gamma + c_2\beta + c_3 4^k) b_3 u, \quad \forall k \geq 0, \forall u \geq 0.$$

Since $u \geq 0$, by Theorem 4.4, the system (A, B, C) is POC if $\forall v \in \mathbb{R}$, we have $v(c_1\gamma + c_2\beta + c_3 4^k)b_3 > 0$. If we take $v = -(c_1\gamma + c_2\beta + c_3 4^k)b_3$, then $v(c_1\gamma + c_2\beta + c_3 4^k)b_3 = -(c_1\gamma + c_2\beta + c_3 4^k)^2 b_3^2 < 0$. So, by Theorem 4.4, the system (A, B, C) is not POC.

5.4. Example 4

In this section we are going to illustrate the results of Propositions 4.7 and 4.8. We take the system (A, B, C) as follows:

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{and } C = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

The matrices A^k and $A^k B$ are given by

$$A^k = \begin{bmatrix} 2^k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1^k & 0 & 0 & 0 & 0 \\ 0 & 0 & (-2)^k & 0 & 0 & 0 \\ 0 & 0 & 0 & (-3)^k & 0 & 0 \\ 0 & 0 & 0 & 0 & 1^k & \beta \\ 0 & 0 & 0 & 0 & 0 & 4^k \end{bmatrix}$$

$$\text{and} \quad A^kB = \begin{bmatrix} 2^{k+1} & -2^k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2^k & -2^{k+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\beta \in \mathbb{R}$. The inner product (5) becomes

$$\langle v, A^kBu \rangle = v_1(2^{k+1}u_1 - 2^ku_2) + v_3(2^ku_3 - 2^{k+1}u_4) + v_5(-3u_5 + u_6),$$

then for all $v^T \in \mathbb{R}^5$ and $k \geq 0$ there is $u \in \mathbb{R}^6_+$ such that $\langle v, A^kBu \rangle > 0$. So, by Theorem 4.3, the system (A, B) is PC. As we note that the matrix A^kB has only two opposed non-null elements in $\frac{m}{2} = 3$ rows and only one non-null element in each column, then, by Proposition 4.7, the system $(A,\ B)$ is PC.

Now, let the matrix $CA^kB \in \mathbb{R}^{5\times 6}$ be as follows

$$CA^kB = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2^{k+1} & -2^k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & 3 \\ 0 & 0 & -2^k & 2^{k+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The inner product (6) becomes

$$\langle v, CA^kBu \rangle = v_2(2^{k+1}u_1 - 2^ku_2) + v_3(-6u_5 + 3u_6) + v_4(-2^ku_3 + 2^{k+1}u_4),$$

then for all $v^T \in \mathbb{R}^5$ and $k \geq 0$, there is $u \in \mathbb{R}^6_+$ such that $\langle v, CA^kBu \rangle > 0$. So, by Theorem 4.4, the system (A, B, C) is POC. As we note that the matrix CA^kB has only two opposed non-null elements in $\frac{m}{2} = 3$ rows and only one element in each column, then, by Proposition 4.8 the system $(A,\ B,\ C)$ is POC.

6. Conclusion

Based on the evaluation of the geometric properties of the system, necessary and sufficient conditions for the positive output controllability of discrete LTI systems have been established. These conditions were then applied to numerical examples to illustrate their usefulness. The investigation of positive output controllability can be considered as an interesting alternative when the system is not controllable neither positive controllable, as seen in Example 2. The subject of the further research will be to develop this work, and to extend it for discrete-time non-linear systems, and then for internally and externally positive discrete linear systems. POC of switched discrete LTI systems is also an open problem and definitely worth of attention.

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