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Leszek KNOPIK
Klaudiusz MIGAWA

MULTI-STATE MODEL OF MAINTENANCE POLICY

WIELOSTANOWY MODEL DECYZJI EKSPLOATACYJNYCH*

Preventive replacement is applied to improve the device availability or increase the profit per unit time of the maintenance system. In this paper, we study age-replacement model of technical object for n-state system model. The criteria function applied in this paper describe profit per unit time or coefficient of availability. The probability distribution of a unit's failure time is assumed to be known, and preventive replacement strategy will be used over very long period of time. We investigate the problem of maximization of profit per unit time and coefficient availability for increasing the failure rate function of the lifetime and for a wider class of lifetime. The purpose of this paper is to obtain conditions under which the profit per unit time approaches a maximum. In this paper we shows that the criteria function (profit per unit time or coefficient availability) can be expressed using the matrix calculation method. Finally, a numerical example to evaluate an optimal replacement age is presented.

Keywords: maintenance, preventive replacement, profit per unit time, availability, lifetime distribution, failure rate function, IFR class, MTFR class.

Wymiany prewencyjne stosuje się w celu podnoszenia gotowości systemów eksploatacji maszyn i wzrostu dochodu na jednostkę czasu systemu eksploatacji. W pracy analizuje się model wymian obiektów technicznych według wieku dla n-stanowego systemu. Funkcja kryterialna stosowana w pracy wyraża zysk przypadający na jednostkę czasu lub współczynnik gotowości. Zakłada się, że rozkład prawdopodobieństwa czasu do uszkodzenia obiektu technicznego jest znany i strategia wymian prewencyjnych będzie stosowana na długim przedziale czasowym. Bada się problem maksymalizacji zysku na jednostkę czasu i współczynnika gotowości dla rosnącej funkcji intensywności uszkodzeń lub funkcji intensywności z szerszej klasy. Celem tej pracy jest sformułowanie warunków, przy których zysk na jednostkę czasu osiąga maksimum. W pracy pokazano, że badaną funkcję kryterialną (zysk na jednostkę czasu lub współczynnik gotowości) można wyrazić za pomocą metod rachunku macierzowego. Na końcu pracy przedstawiono przykład numeryczny oceny optymalnego wieku wymiany dla rzeczywistego procesu eksploatacji.

***Słowa kluczowe**: utrzymanie, wymiana prewencyjna, zysk na jednostkę czasu, gotowość, rozkład czasu życia, funkcja intensywności uszkodzeń, klasa IFR, klasa MTFR.*

1. Introduction

Industrial system management requires implementation of various operational activities. The crucial tasks in which the role of economic optimization will increase include maintaining the operation system as well as the replacement of technical objects. Maintenance and replacement are not only technical issues but also an economic problem. Maintenance strategy is focused mostly on preventive maintenance mainly in the area of operational research and management studies. Age replacement strategy for technical objects implies preventive replacement of the object when it reaches age T or corrective (failure) replacement before it reaches age T. Preventive replacements are less expansive and cheaper than the corrective ones. It is known that the time of preventive replacement is usually shorter than the time of corrective replacement. Fundamental facts for age replacements are included in papers [2, 3]. A review of results connected with preventive replacements is to be found in papers [5, 21, 24]. Certain generalizations of the question of preventive replacements were arrived at in papers [12, 13]. Much later, methods of preventive replacements for multistate systems were analyzed. Testing multistate systems was introduced in [14-19, 25-26]. The article [7] uses fuzzy variables (fuzzy sets), while paper [22] examines multistate systems with components requiring minimal repair. Using simulation methods for testing preventive renewals was presented in paper [20]. This paper examines operation systems in which the technical object may at a given moment appear in one of the n states. For such systems optimal preventive replacements basing on the criteria function expressing the profit per time unit or availability coefficient. The structure of criteria function is based on the values of semi-Markov process [6, 9], as opposed to the classic approach based on the theory of renewal. The most quoted work in defining the criteria function based on the elements of renewal theory is paper [23]. The paper includes an analysis of sufficient conditions for existing of maximum profit per time unit and maximum asymptotic coefficient of availability of n-state operation system. Values of criteria function depend on the lifetime distribution, the mean value of preventive replacement time, mean repair time value, mean values of remaining at other states, profits per time unit, transition probability matrixes embedded in the semi-Markov process of Markov chain.

In Chapter 2, the basic markings and assumptions used in the paper were presented. In Chapter 3 a model of technical object operation was prepared and the criteria function defining the profit per time unit at infinite time horizon was created. The crucial goal of this chapter is to introduce into the research criteria function in matrix form. In Chapter 4 sufficient conditions for occurrence of maximum profit per unit are analyzed, as well as asymptotic availability factor. Conditions for occurrence of extreme criteria function were formulated for the class of distributions with increasing failure rate function (IFR) as well as for a wider class introduced by one of the authors of the class (MTFR). Numerical example of the optimization of the maintenance system was analyzed in Chapter 5. In the example it was asserted that

(*) Tekst artykułu w polskiej wersji językowej dostępny w elektronicznym wydaniu kwartalnika na stronie ***www.ein.org.pl*** the lifetime of technological object has Weibull distribution. Data for this example were obtained from an existing municipal bus operation process.

2. Markings and assumptions included in the paper

The paper will examine semi-Markov model of preventive age replacement. An n-state semi-Markov process X(t) is considered with the space of states S = {1, 2, …, n}. If X(t) = i, then the considered technical object at moment t is at state i. It is asserted that 1 is the state of failure-free functioning, 2 is the state of repair, n is the state of preventive replacement, while remaining states i, where $2 \leq i \leq n-1$ are other states of system maintenance.

By z_i, i = 1, 2, …, n, profit per unit is marked per state i. The paper assumes that $z_1 > 0$, $z_i \leq 0$ for $2 \leq i \leq n$. If the technical object is in state 1, it brings profit, while if technical object is in state i, where $2 \leq i \leq n$, then technological object generates loss.

It is assumed that $\tau_0 < \tau_1 < \tau_2 < ... < \tau_k < ...$ are leap moments and $v_k < \tau_k - \tau_{k-1}$ for $k \geq 1$, $v_0 = 0$ are times of remaining at states of process X(t). The semi-Markov process is fully defined if the initial distribution is known:

$$P\{X(0) = i\} = p_i^{(0)}, \ i = 1,2,...,n$$

and its semi-Markov kernel defined by matrix:

$$Q(t) = [Q_{ij}(t)], \ i,j = 1,2,...,n,$$

where:

$$Q_{ij}(t) = P\{X(\tau_{k+1}) = j, \tau_{k+1} - \tau_k < t | X(\tau_k) = i\}, i,j = 1,2,...,n$$

Sequence $X(\tau_k)$, $k \in N$ of random variables is Markov chain with transition probability matrix:

$$P = [p_{ij}] = [Q_{ij}(\infty)] \ \ for \ \ i,j = 1,2,...,n$$

is called embedded Markov chain. Random variables T_i, i = 1, 2, …, n stand for times of remaining in the state and have distribution functions in the form of:

$$F_i(t) = P\{T_i < t\} = P\{\tau_{k+1} - \tau_k < t | X(\tau_k) = j\}$$

or otherwise:

$$F_i(t) = \sum_{j=1}^{n} Q_{ij}(t), \ i = 1,2,...,n. \quad (1)$$

Distribution function of remaining at state i, before moving to state j, is defined as follows:

$$F_{ij}(t) = P\left\{\tau_{k+1} - \tau_k < t \middle| X(\tau_{k+1}) = j, \ X(\tau_k) = i\right\}, \ for \ i,j = 1,2,...,n, k \in N. \quad (2)$$

For distribution function $F_{1i}(x)$ defined by formula (2) it is assumed that $F_{1i}(x) = F_1(x)$ exists for i = 2, 3, 4, …, n. While constructing criteria function, the limit theorem is used for finite semi-Markov processes [6, 8]. It is assumed that mean values ET_i, i = 1, 2, …, n are finite, positive and Markov chain $X(\tau_k)$, $k = 0,1,2,...$, has one ergodic class. These assumptions allow one to formulate limit theorem of the form:

$$P_j = \lim_{t\to\infty} P\{X(t) = j\} = \lim_{t\to\infty} P\{X(t) = j \mid X(0) = i\} \ \ for \ \ i = 1,2,...,n,$$

where:

$$P_j = \frac{p_j^* ET_j}{\sum_{k=1}^{n} p_k^* ET_k}, \quad (3)$$

where p_j^*, $j = 0,1,2,...,n$ is a limit distribution of embedded Markov chain $X(\tau_k)$, where $k \in N$ with transition probability matrix $P = [p_{ij}]$, where $p_{ij} = Q_{ij}(\infty)$, $i,j = 0,1,2,...,n$. Limit probabilities p_j^*, $j = 0,1,2,...,n$ are a solution for a system of linear equations:

$$\sum_{i=1}^{n} p_i^* p_{ij} = p_j^* \text{ with condition } \sum_{i=1}^{n} p_i^* = 1, \text{ where i, j = 1, 2, …, n.}$$

3. Criteria function

Let X(t) be semi-Markov process with continuous kernel Q(t). The counting process is defined:

$$K_j(t) = \int_0^t I\{X(u) = j\} du,$$

Where I is an indicator determined as follows:

$$I\{X(u) = j\} = \begin{cases} 1 & for \ \ X(u) = j, \\ 0 & for \ \ X(u) \neq j. \end{cases}$$

It is total time of remaining of process X(t) at state i as well as in interval [0, t]. The value:

$$L(t) = \sum_{i=1}^{n} z_i EK_i(t)$$

is the expected profit per time unit in interval [0, t]. The limit:

$$L = \lim_{t\to\infty} \frac{L(t)}{t}$$

is the expected profit per time unit for infinite time interval. The limit is the basis for building criteria function. From the definition of the process $K_j(t)$, j = 1, 2, …, n, it is:

$$\lim_{t\to\infty}\frac{EK_j(t)}{t}=P_j,\ \text{j=1, 2, …, n,}$$

thus:

$$L=\sum_{i=1}^{n} z_i P_i.$$

According to (3) the following is true:

$$L=\frac{\sum_{i=1}^{n} z_i p_i^* ET_i}{\sum_{i=1}^{n} p_i^* ET_i}. \tag{4}$$

The unit is replacement at age T or when it failed, whichever comes first. $T_1(x)$ defines the time of replacement or failure. Variable $T_1(x)$ may be written as:

$$T_1(x)=\begin{cases}T_1, & if \quad T_1<x,\\ x, & if \quad T_1\geq x.\end{cases} \tag{5}$$

Using formula (5), a semi-Markov process is obtained with transition probability matrix P(x) of embedded Markov chain. Elements of the first verse of matrix P(x) depend on x. For $p_{1n}(x)$ is:

$$p_{1n}(x) = P\{X(\tau_{k+1}) = n \mid X(\tau_k) = 1\} =$$
$$P\{X(\tau_{k+1}) = n \mid X(\tau_k) = 1, T_1< x\}\ P\{T_1 < x \mid X(\tau_k) = 1\} +$$
$$P\{X(\tau_{k+1}) = n \mid X(\tau_k) = 1, T_1 \geq x\}\ P\{T_1 \geq x \mid X(\tau_k) = 1\}.$$

For (5), the following is true:

$$P\ \{X(\tau_{k+1}) = n \mid X(\tau_k) = 1, T_1 \geq x\} = 1.$$

Using qualities of conditional probability, the following is obtained:

$$P\{X(\tau_{k+1}) = n \mid X(\tau_k) = 1, T_1< x\}$$

$$P\{X(\tau_{k+1}) = n, T_1< x \mid X(\tau_k) = 1\}/ P\{T_1 < x \mid X(\tau_k) = 1\} = Q_{1n}(x) / F_1(x),$$

thus:

$$p_{1n}(x) = Q_{1n}(x) + R_1(x)\ \text{, where } R_1(x) = 1 - F_1(x).$$

For (2), the following is true:

$$p_{1n}(x) = p_{1n}F_{1n}(x) + R_1(x).$$

Similarly, for probability $p_{1i}(x)$ where $2 \leq i \leq n - 1$, the following is true:

$$p_{1i}(x) = P\{X(\tau_{k+1}) = i \mid X(\tau_k) = 1\} =$$
$$P\{X(\tau_{k+1}) = i \mid X(\tau_k) = 1, T_1< x\}\ P\{T_1 < x \mid X(\tau_k) = 1\} +$$
$$P\{X(\tau_{k+1}) = i \mid X(\tau_k) = 1, T_1 \geq x\}\ P\{T_1 \geq x \mid X(\tau_k) = 1\}.$$

Definition (5) concludes in:

$$P\{X(\tau_{k+1}) = i \mid X(\tau_k) = 1, T_1 \geq x\} = 0,$$

which results in:

$$P\{X(\tau_{k+1}) = i \mid X(\tau_k) = 1, T_1< x\} = Q_{1i}(x) / F_1(x)$$

as well as:

$$p_{1i}(x) = Q_{1i}(x) = p_{1i}F_{1i}(x) \text{ for i = 2, 3, …, n –1.}$$

Now, the matrix P(x) of transition probabilities is as follows:

$$P(x)=\begin{bmatrix}0 & p_{12}F_{12}(x) & p_{13}F_{13}(x) & \cdots & p_{1n}F_{1n}(x)+R_1(x)\\ p_{21} & 0 & p_{23} & \cdots & p_{2n}\\ p_{31} & p_{32} & 0 & \cdots & p_{3n}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ p_{n1} & p_{n2} & p_{n3} & \cdots & 0\end{bmatrix}.$$

On the basis of (4), criteria function may be written in the form:

$$g(x)=\frac{ET_1(x)z_1p_1^*(x)+\sum_{i=2}^{n} ET_iz_ip_i^*(x)}{ET_1(x)p_1^*(x)+\sum_{i=2}^{n} ET_ip_i^*(x)}, \tag{6}$$

where $p_i^*(x)$, i = 1, 2, …, n are limit probabilities of Markov chain with transition matrix P(x), while $ET_1(x)$ is the value of mean random variable $T_1(x)$. Mean value $ET_1(x)$ is calculated from the formula:

$$ET_1(x)=\int_0^x dF_1(t)+xP\{T_1\geq x\}.$$

Integration by parts results in:

$$ET_1(x)=\int_0^x R_1(t)dt\ . \tag{7}$$

Limit probabilities $p_i^*(x)$, i = 1, 2, …, n meet the following system of linear equations:

$$\begin{bmatrix}-1 & p_{21} & p_{31} & \cdots & p_{n1}\\ p_{12}F_{12}(x) & -1 & p_{32} & \cdots & p_{n2}\\ p_{13}F_{13}(x) & p_{23} & -1 & \cdots & p_{n3}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ p_{1n}F_{1n}(x)+R(x) & p_{2n} & p_{3n} & \cdots & -1\end{bmatrix}\begin{bmatrix}p_1^*(x)\\ p_2^*(x)\\ p_3^*(x)\\ \cdots\\ p_n^*(x)\end{bmatrix}=\begin{bmatrix}0\\0\\0\\ \cdots\\0\end{bmatrix}. \tag{8}$$

The system of equations is homogenous and has infinite number of solutions. Replacing the last equation of system (8) with normalization condition of the form:

$$\sum_{i=1}^{n} p_i^*(x)=1, \tag{9}$$

the system of linear equations has the form:

$$\begin{bmatrix} -1 & p_{21} & p_{31} & \cdots & p_{n1} \\ p_{12}F_{12}(x) & -1 & p_{32} & \cdots & p_{n2} \\ p_{13}F_{13}(x) & p_{31} & -1 & \cdots & p_{n3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} p_1^*(x) \\ p_2^*(x) \\ p_3^*(x) \\ \cdots \\ p_n^*(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 1 \end{bmatrix}. \tag{10}$$

The $u_i(x)$ marks algebraic completion of i-th one in n-th verse of the matrix of the equation system (10). Taking into consideration the right side of the system (10), the solution of the system has the form $p_i^*(x) = u_i(x) / W(x)$ where W(x) is the determinant of the system (10). Embedding $u_i(x)$ in formula (6) instead of $p_i^*(x)$, the criteria function (6) does not change its value. From determinant value, the numerator of criteria function (6) may be written as:

$$L(x; z_1, z_2, \ldots, z_n) = \begin{vmatrix} -1 & p_{21} & p_{31} & \cdots & p_{n1} \\ p_{12}(x) & -1 & p_{32} & \cdots & p_{n2} \\ p_{13}(x) & p_{23} & -1 & \cdots & p_{n3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ z_1 ET_1(x) & z_2 ET_2 & z_3 ET_3 & \cdots & z_n ET_n \end{vmatrix}. \tag{11}$$

Denominator M(x) of criteria function (6) is expressed as: M(x) = L(x; 1, 1, …, 1). For the formula (11), the following is true that the criteria function (6) may now be written as coefficient of determinants

$$g(x) = L(x; z_1, z_2, \ldots, z_n) / L(x; 1, 1, \ldots, 1). \tag{12}$$

The latter equation is an important result of the work. Representation of criteria function g(x) in the form of (12) makes it possible to calculate the value of function without determining limit probabilities $p_i^*(x)$. In paper [9] the criteria function g(x) is expressed in the form (6). The A_1 means algebraic completion of element $z_1ET_1(x)$. It is known that A_1 does not depend on x, $z_2, z_3, \ldots, z_n$. $D_i(x)$ denotes the algebraic completion of i-th one in matrix of equation system (10). Limit probability $p_i^*(x)$ is expressed by the formula:

$$p_i^*(x) = D_i(x) / W(x), \quad i = 1, 2, \ldots, n, \tag{13}$$

where W(x) is the indicator of the matrix of equation system (10). The matrix obtained from the matrix of the system (10) by crossing out the n-th verse and the n-th column has the dominant diagonal. In this case it means that the sum of the elements of each column, bypassing the element on the diagonal, is lower than 1. On the basis of the qualities of matrix [4] the sing of completion of element z_nET_n may be written in the form of $(-1)^{n-1}$. On the basis of (13) it is concluded that the sign of the determinant W(x) is also equal to $(-1)^{n-1}$. The determinant of completion $D_1(x)$ is the determinant from independent matrix from x, $z_1, z_2, \ldots, z_n$ and the sign of this determinant equals $(-1)^{n-1}$. The numerator and the denominator of the criteria function g(x) given by the formula (13) is multiplied by W(x) $(-1)^{n-1}$. Asserting that the above markings of the criteria function g(x) given in the formula (6) may be written in the form:

$$g(x) = \frac{Az_1ET_1(x) + \sum_{i=2}^{n} z_i ET_i E_i(x)}{AET_1(x) + \sum_{i=2}^{n} ET_i E_i(x)}, \tag{14}$$

where $E_i(x) = D_i(x)\,(-1)^{n-1}$, $A = A_1(-1)^{n-1}$. It is now known that for each $x \geq 0$ is $E_i(x) \geq 0$ and $A \geq 0$. From the form of the matrix of equation system (10) it is concluded that $E_i(x)$ is a linear function of the distribution function $F_1(x)$. There exist a continuous G_i i H_i so that $E_i(x) = G_iF_1(x) + H_i$. Embedding x = 0, it is concluded that $H_i \geq 0$.

Assuming that:

$$B_1 = \sum_{i=2}^{n} z_i ET_i G_i,\ C_1 = \sum_{i=2}^{n} z_i ET_i H_i,\ B = \sum_{i=2}^{n} ET_i G_i,\ C = \sum_{i=2}^{n} ET_i H_i \tag{15}$$

the criteria function (14) may be written as:

$$g(x) = \frac{Az_1ET_1(x) + B_1F_1(x) + C_1}{AET_1(x) + BF_1(x) + C}. \tag{16}$$

It is easy to notice that $g(0) = C_1 / C \leq 0$ as well as $g(\infty) = (A\, z_1ET_1 + B_1 + C_1) / (A\, ET_1 + B + C)$.

4. Sufficient conditions for existing of maximum criteria function

In this subsection of the paper the sufficient conditions are formulated for the occurrence of maximum criteria function assuming that the time to failure T_1 is a random variable of the function of increasing failure rate $\lambda_1(t)$. This fact is written as follows: $T_1 \in$ IFR (Increasing Failure Rate). The second discussed distribution class is the class with unimodal failure rate included in MTFR (Mean Time to Failure or Repair) class. The qualities of MTFR class were examined in detail in papers [1, 10, 11]. Derivative g'(x) of criteria function is expressed by the formula:

$g'(x) = \{A[B_1 - Bz_1][ET_1(x)f_1(x) - F_1(x)R_1(x)] + R_1(x)A(Cz_1 - C_1) + f_1(x)(B_1C - BC_1)\}/M^2(x)$,

where M(x) is the denominator in formula (16). In conclusion, the derivative g'(x) may be written as:

$$g'(x) = \frac{1}{M^2(x)}\left(A\alpha r_1(x) + \beta R_1(x) + f_1(x)\gamma\right), \tag{17}$$

where:

$$\begin{aligned} \alpha &= A(B_1 - Bz_1), \\ \beta &= A(Cz_1 - C_1), \\ \gamma &= B_1C - BC_1, \\ r_1(x) &= ET_1(x) f_1(x) - F_1(x) R_1(x). \end{aligned} \tag{18}$$

At the beginning of Chapter 2 it was asserted that $z_1 > 0$, $z_i \leq 0$ for $2 \leq i \leq n$. This assertion as well as formulas (15), (16), (17) and (18) allow one to formulate conclusions 1 and 2.

Conclusion 1. If $z_1 > 0$, $z_i \leq 0$, $F_{1i}(x) = F_1(x)$ for i = 2, 3, 4, …, n, then inequality $\beta > 0$ is true.

Conclusion 2. If $z_i = z_j$ for $2 \leq i \leq n$, $2 \leq j \leq n$, then $\gamma = 0$.

The theses of conclusions 1 and 2 are very useful while formulating criteria for occurrence of maximum availability rate (conclusions

7 and 8). Below it is assumed that the function of failure rate $\lambda_1(x)$ is continuous for $t \geq 0$.

Conclusion 3. If $T_1 \in$ IFR, $\beta + \gamma f(0^+) > 0$, $\alpha < 0$, $\gamma < 0$, $A\alpha(ET_1 \lambda_1(\infty) - 1) + \beta + \gamma\lambda_1(\infty) < 0$, then the criteria function g(x) reaches exactly one maximum.

Proof. It is assumed that $\lambda_1(\infty)$ is the border of the function $\lambda_1(t)$ with $t \to \infty$ or upper limit of function $\lambda_1(t)$ with $t \to \infty$. Let $s(x) = \alpha r(x) + \beta + \gamma\lambda_1(x)$, where $r(x) = \lambda_1(t)\, ET_1(x) - F(x)$. Formula (17) results in the mark of derivative g'(x) is the same as the mark of function s(x). The assumptions $\alpha < 0$, i $\gamma < 0$ result in the function s(x) is continuous and decreasing from $s(0) = \beta + \gamma f_1(0^+)$ to $s(\infty) = A\alpha(ET_1 \lambda_1(\infty) - 1) + \beta + \gamma\lambda_1(\infty)$. If $s(0) > 0$ and $s(\infty) < 0$, there exists exactly one x_0 so that $s(x_0) = 0$ and $g'(x_0) = 0$. Thus g(x) reaches one maximum.

Conclusion 4. If $T_1 \in$ IFR, $\alpha < 0$, $\beta + \gamma f_1(0^+) < 0$, $\lambda_1(\infty) = \infty$, then the criteria function g(x) reaches exactly one maximum.

Proof. If $\lambda_1(\infty) = \infty$, then $s(\infty) < 0$. Conclusion 3 results in the fact that criteria function g(x) reaches exactly one maximum.

Conclusion 4 results in the following conclusion:

Conclusion 5. If $T_1 \in$ IFR, $\alpha < 0$, $f_1(0^+) = 0$, $\lambda_1(\infty) = \infty$, then the criteria function g(x) reaches exactly one maximum.

Conclusions 3, 4 and 5 include sufficient conditions for occurrence of maximum function g(x) for distributions of times to failure from IFR class. Below, conditions for existing of maximum for MTFR class are formulated.

Definition. Random variable $T_1 \in$ MTFR, if $r_1(x) \geq 0$ for every $x \geq 0$.

MTFR class includes certain random variables with unimodal failure rate functions.

Conclusion 6. Let the time to failure T_1 have a distribution with unimodal failure rate function $\lambda_1(t)$. The equality $T_1 \in$ MTFR is true only when $ET_1\, \lambda_1(\infty) \geq 1$.

Replacing $z_1 = 1$, $z_2 = 0$, $z_3 = 0$,…, $z_n = 0$, the criteria function g(x) expresses asymptotic availability coefficient. For the availability coefficient $g(0) = 0$ as well as $g(\infty) = A\, ET_1 / (A\, ET_1 + B + C)$.

Conclusion 7. If $T_1 \in$ IFR, $\alpha < 0$, $\lambda_1(\infty) > (1 - \alpha/\beta) / ET_1$, then the availability coefficient reaches maximum value.

Proof. It is known on the basis of conclusion 2 that for availability coefficient $\gamma = 0$. It results from the fact that function $s(x) = \alpha\, r(x) + \beta$ decreases from $s(0) = \beta > 0$ to $s(\infty) = \alpha\, (ET_1\, \lambda_1(\infty) - 1) + \beta < 0$. Function s(x) changes the mark from „+" to „–" exactly once.

Conclusion 8. If $T_1 \in$ MTFR, $\alpha < 0$, $\lambda(t)$ is unimodal, $\lambda_1(\infty) > (1 - \alpha/\beta) / ET_1$, then availability coefficient g(x) reaches maximum value.

Proof. Derivative of function s(t) equals $s'(t) = \lambda'_1(x)\, [\alpha\, ET_1(x) + \gamma]$. If it results from the fact that function $\lambda_1(t)$ is unimodal, then the function s(t) is also unimodal. Paper [9] shows that unimodal nature of function s(x) may be proven without asserting differentiability of function $\lambda_1(t)$. In order for function s(t) to have precisely single zero position it is enough for the following inequalities to be true $s(0) = \beta > 0$ and $s(\infty) < 0$. Condition $s(\infty) < 0$ is equivalent to condition $\lambda_1(\infty) > (1 - \alpha/\beta) / ET_1$.

5. Numeric example

In this chapter, numeric example illustrating results obtained in chapters 3 and 4 are analyzed. An 8-state process of municipal buses operation process is discussed. The following states have been singled out:

S_1 – completion of transport task,
S_2 – repair after failure,
S_3 – preventive replacement (repair),
S_4 – condition control after repair,
S_5 – fuel delivery,
S_6 – service on working day,
S_7 – periodic technical service,
S_8 – stoppage at depot parking space.

It is asserted that time to failure has Weibull distribution with reliability function: $R_1(t) = \exp(-(t/a)^c)$ for $a, c > 0$, $t \geq 0$. Rate function for this distribution has the form: $\lambda_1(t) = (c/a)\,(x/a)^{c-1}$, $t \geq 0$. Based on the statistic analysis of the data from operation for mean values of remaining at states, the following ratings were obtained: $ET_1 = 8.852$, $ET_2 = 3.619$, $ET_3 = 1.501$, $ET_4 = 0.164$, $ET_5 = 0.096$, $ET_6 = 0.122$, $ET_7 = 3.885$, $ET_8 = 5.659$. Unit profits resulting from working at states of the system were rated as: $z_1 = 4$, $z_2 = -2$, $z_3 = -0.2$, $z_4 = -0.2$, $z_5 = -0.2$, $z_6 = -0.2$, $z_7 = -0.2$, $z_8 = -1$. The rating of transition probability matrix for Markov chain embedded in the process is the following:

$$P = \begin{bmatrix} 0 & 0.239 & 0.104 & 0 & 0.657 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0.277 & 0 & 0 & 0 & 0.723 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.982 & 0.178 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0.234 & 0 & 0 & 0 & 0 & 0 & 0 & 0.766 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Values of factors A, B, B_1, C and C_1 were calculated using the unmarked factor method. For the values of parameters given above it was calculated: $\alpha = -5.778$, $\beta = 2.400$, $\gamma = -4.133$. For every $c \in \{2, 2.5, 3, 3.5\}$ the value of parameter b was calculated so that $ET_1 = 8.852$. Figure 1 presents charts of profit function per time unit depending on the moment of preventive replacement (repair). Each of the four completions of criteria function reach maximum value.

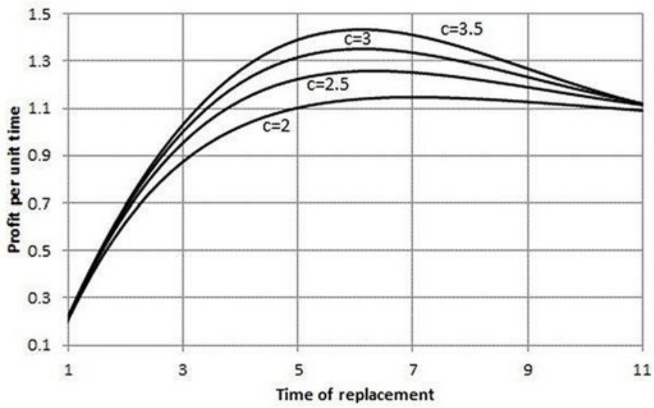

Fig. 1. Graphs of the criteria function g(x) for c {2, 2.5, 3, 3.5}

6. Conclusions

The paper discusses the issue of age replacement of technical objects for multi-state operation systems. The criteria function examined in this paper is the profit per time unit and availability coefficient. The first of the objectives of the article was the matrix representation of the criteria function (formula (12)), while the second was to show

how, with certain general assumptions, it was possible to formulate sufficient conditions for occurrence of the maximum of the criteria function.

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Leszek KNOPIK
Faculty of Management
UTP University of Science and Technology
ul. Fordońska 430, 85-890 Bydgoszcz, Poland

Klaudiusz MIGAWA
Faculty of Mechanical Engineering
UTP University of Science and Technology
al. prof. S. Kaliskiego 7, 85-796 Bydgoszcz, Poland

E-mails: knopikl@utp.edu.pl, klaudiusz.migawa@utp.edu.pl