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INFLUENCE OF AN L^P-PERTURBATION ON HARDY–SOBOLEV INEQUALITY WITH SINGULARITY A CURVE

Idowu Esther Ijaodoro and El Hadji Abdoulaye Thiam

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Abstract. We consider a bounded domain Ω of $\mathbb{R}^N$, $N \geq 3$, h and b continuous functions on Ω. Let Γ be a closed curve contained in Ω. We study existence of positive solutions $u \in H_0^1(\Omega)$ to the perturbed Hardy–Sobolev equation:

$$-\Delta u + hu + bu^{1+\delta} = \rho_\Gamma^{-\sigma} u^{2^*_\sigma - 1} \quad \text{in } \Omega,$$

where $2^*_\sigma := \frac{2(N-\sigma)}{N-2}$ is the critical Hardy–Sobolev exponent, $\sigma \in [0,2)$, $0 < \delta < \frac{4}{N-2}$ and ρ_Γ is the distance function to Γ. We show that the existence of minimizers does not depend on the local geometry of Γ nor on the potential h. For $N = 3$, the existence of ground-state solution may depends on the trace of the regular part of the Green function of $-\Delta + h$ and or on b. This is due to the perturbative term of order $1 + \delta$.

Keywords: Hardy–Sobolev inequality, positive minimizers, parametrized curve, mass, Green function.

Mathematics Subject Classification: 35J91, 35J20, 35J75.

1. INTRODUCTION

The Hardy–Sobolev inequality with a cylindrical weight states, for $N \geq 3$, $0 \leq k \leq N-1$ and $\sigma \in [0,2)$, that

$$\int_{\mathbb{R}^N} |\nabla v|^2 dx \geq C \left(\int_{\mathbb{R}^N} |z|^{-\sigma} |v|^{2^*_\sigma} dx \right)^{2/2^*_\sigma} \qquad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{1.1}$$

where $x = (t, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $C = C(N, \sigma, k) > 0$, $2^*_\sigma := \frac{2(N-\sigma)}{N-2}$ is the critical Hardy–Sobolev exponent and $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$v \longmapsto \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx \right)^{1/2}.$$

Inequality (1.1) can be obtained by interpolating between Hardy (which corresponds to the case $\sigma = 2$ and $k \neq N-2$) and Sobolev (which is the case $\sigma = 0$) inequalities. This inequality is invariant by scaling on $\mathbb{R}^N$ and by translations in the t-direction.

When $\sigma = 2$ and $k \neq N-2$, the best constant is $\left(\frac{N-k-2}{2}\right)^2$ but it is never achieved. For $\sigma \in [0, 2)$, the best constant C in (1.1) is given by

$$S_{N,\sigma} := \min \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{1}{2^*_\sigma} \int_{\mathbb{R}^N} |z|^{-\sigma} |v|^{2^*_\sigma} dx, \ v \in \mathcal{D}^{1,2}(\mathbb{R}^N) \right\}. \tag{1.2}$$

In the case $\sigma \in [0, 2)$ and $k = 0$, $S_{N,\sigma}$ is achieved by the standard bubble

$$c_{N,\sigma}(1 + |x|^{2-\sigma})^{\frac{2-N}{2-\sigma}},$$

see for instance Aubin [19], Talenti [1] and Lieb [16]. When $k = N-1$, the support of the minimizer is contained in a half space, see Musina [17].

For $1 \leq k \leq N-2$ and $\sigma \in (0, 2)$, Badiale and Tarentello [2] proved the existence of a minimizer w for (1.2). They were motivated by questions from astrophysics. Later Mancini, Fabbri and Sandeep used the moving plane method to prove that $w(t, z) = \theta(|t|, |z|)$, for some positive function θ. An interesting classification result was also derived in [7] when $\sigma = 1$, that every minimizer is of the form

$$c_{N,k}((1 + |z|)^2 + |t|^2)^{\frac{2-N}{2}},$$

up to scaling in $\mathbb{R}^N$ and translations in the t-direction.

Since in this paper we are interested with Hardy–Sobolev inequality with weight singular at a given curve, our asymptotic energy level is given by $S_{N,\sigma}$ with $k = 1$ and $\sigma \in [0, 2)$.

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, h and b continuous function on Ω. Let $\Gamma \subset \Omega$ be a smooth closed curve. In this paper, we are concerned with the existence of minimizers for the infimum

$$\begin{aligned} \mu_\sigma(\Omega, \Gamma, h, b) := \inf_{u \in H^1_0(\Omega) \setminus \{0\}} & \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{2} \int_\Omega h u^2 dx \\ & + \frac{1}{2+\delta} \int_\Omega b u^{2+\delta} dx - \frac{1}{2^*_\sigma} \int_\Omega \rho_\Gamma^{-\sigma} |u|^{2^*_\sigma} dx, \end{aligned} \tag{1.3}$$

where $\sigma \in [0, 2)$, $2^*_\sigma := \frac{2(N-\sigma)}{N-2}$, $0 < \delta < \frac{4}{N-2}$ and $\rho_\Gamma(x) := \mathrm{dist}(x, \Gamma)$ is the distance function to Γ. Here and in the following, we assume that $-\Delta + h$ defines a coercive bilinear form on $H_0^1(\Omega)$ and that $b \leq 0$. We are interested with the effect of b and/or the location of the curve Γ on the existence of minimizer for $\mu_\sigma(\Omega, \Gamma, h, b)$.

When there is no perturbation, and $\sigma = 0$, problem (1.3) reduces to the famous Brezis–Nirenberg problem [3]. In this case, for $N \geq 4$ it is enough that $h(y_0) < 0$ to get a minimizer, whereas for $N = 3$, the existence of minimizers is guaranteed by the positiveness of a certain mass, see Druet [6].

Here, we deal with the case $\sigma \in [0, 2)$. Our first result deals with the case $N \geq 4$. Then we have

Theorem 1.1. *Let $N \geq 4$, $\sigma \in [0, 2)$ and Ω be a bounded domain of $\mathbb{R}^N$. Consider Γ a smooth closed curve contained in Ω. Let h and b be continuous function such that the linear operator $-\Delta + h$ is coercive and $b \leq 0$. We assume that*

$$b(y_0) < 0, \tag{1.4}$$

for some $y_0 \in \Gamma$. Then $\mu(\Omega, \Gamma, h, b)$ is achieved by a positive function $u \in H_0^1(\Omega)$.

In contrast, to the result of the second author and Fall [9], inequality (1.4) in Theorem 1.1 shows that there is no influence of the curvature of Γ nor the potential h. This is due to the influence of the added perturbation term in (1.3).

For $N = 3$, we let $G(x, y)$ be the Dirichlet Green function of the operator $-\Delta + h$, with zero Dirichlet data. It satisfies

$$\begin{cases} -\Delta_x G(x,y) + h(x) G(x,y) = 0 & \text{for every } x \in \Omega \setminus \{y\}, \\ G(x,y) = 0 & \text{for every } x \in \partial\Omega. \end{cases} \tag{1.5}$$

In addition, there exists a continuous function $\mathbf{m} : \Omega \to \mathbb{R}$ and a positive constant $c > 0$ such that

$$G(x, y) = \frac{c}{|x - y|} + c\,\mathbf{m}(y) + o(1) \qquad \text{as } x \to y. \tag{1.6}$$

This function $\mathbf{m} : \Omega \to \mathbb{R}$ is the *mass* of $-\Delta + h$ in Ω. Our second main result is the following.

Theorem 1.2. *Let $\sigma \in [0, 2)$ and Ω be a bounded domain of $\mathbb{R}^3$. Consider a smooth closed curve Γ contained in Ω. Let h and b be continuous functions such that the linear operator $-\Delta + h$ is coercive and $b \leq 0$. We assume that*

$$\begin{cases} b(y_0) < 0 & \textit{for} \quad 2 < \delta < 4, \\ m(y_0) > c b(y_0) & \textit{for} \quad \delta = 2, \\ m(y_0) > 0 & \textit{for} \quad 0 < \delta < 2, \end{cases} \tag{1.7}$$

for some positive constant c and $y_0 \in \Gamma$. Then $\mu_\sigma\left(\Omega, \Gamma, h, b\right)$ is achieved by a positive function $u \in H_0^1(\Omega)$.

The literature about Hardy–Sobolev inequalities on domains with various singularities is very hudge. The existence of minimizers depends on the curvatures at a point of the singularity. For more details, we refer to Ghoussoub–Kang [10], Ghoussoub–Robert [11, 12], Demyanov–Nazarov [5], Chern–Lin [4], Lin–Li [15], Fall–Thiam [9], Fall–Minlend–Thiam in [8] and the references therein. We refer also to Jaber [13,14] and Thiam [20–22] and references therein, for Hardy–Sobolev inequalities on Riemannian manifold. Here also the impact of the scalar curvature at the point singularity plays an important role for the existence of minimizers in higher dimensions $N \geq 4$. The paper [13] contains also existence result under positive mass condition for $N = 3$.

The proof of Theorem 1.1 and Theorem 1.2 rely on test function methods. Namely to build appropriate test functions allowing to compare $\mu_\sigma(\Omega, \Gamma, h, b)$ and $S_{N,\sigma}$. We find a continuous family of test functions $(u_\varepsilon)_{\varepsilon>0}$ concentrating at a point $y_0 \in \Gamma$ which yields $\mu(\Omega, \Gamma, h, b) < S_{N,\sigma}$, as $\varepsilon \to 0$, provided (1.7) holds. In Section 4, we consider the case $N = 3$. Due to the fact that the ground-state w for $S_{3,\sigma}$, $\sigma \in (0, 2)$ is not known explicitly, it is not radially symmetric, it is not smooth and $S_{3,\sigma}$ is only invariant under translations in the $t-$direction; we could only construct a discrete family of test functions $(\Psi_{\varepsilon_n})_{n\in\mathbb{N}}$ that leads to the inequality $\mu_\sigma(\Omega, \Gamma, h, b) < S_{3,\sigma}$. These are similar to the test functions $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ in dimension $N \geq 4$ near the concentration point y_0, but away from it is substituted with the regular part of the Green function $G(x, y_0)$, which makes appear the mass $\mathbf{m}(y_0)$ and/or $b(y_0)$ in its first order Taylor expansion, see (1.6).

The paper is organized as follows: In Section 2, we recall some geometric and analytic preliminaries results relating to the local geometry of the curve Γ and the decay estimates of the ground state w of $S_{N,\sigma}$. In Sections 3 and 4, we construct a test function for $\mu_\sigma(\Omega, \Gamma, h, b)$ in order to prove Theorem 1.1 and Theorem 1.2. Their proof is completed in Section 5.

2. PRELIMINARIES RESULTS

Let $\Gamma \subset \mathbb{R}^N$ be a smooth closed curve. Let $(E_1; \dots; E_N)$ be an orthonormal basis of $\mathbb{R}^N$. For $y_0 \in \Gamma$ and $r > 0$ small, we consider the curve $\gamma : (-r, r) \to \Gamma$, parameterized by arc length such that $\gamma(0) = y_0$. Up to a translation and a rotation, we may assume that $\gamma'(0) = E_1$. We choose a smooth orthonormal frame field $(E_2(t); \dots; E_N(t))$ on the normal bundle of Γ such that $(\gamma'(t); E_2(t); \dots; E_N(t))$ is an oriented basis of $\mathbb{R}^N$ for every $t \in (-r, r)$, with $E_i(0) = E_i$.

We fix the following notation, that will be used a lot in the paper,

$$Q_r := (-r, r) \times B_{\mathbb{R}^{N-1}}(0, r),$$

where $B_{\mathbb{R}^{N-1}}(0, r)$ denotes the ball in $\mathbb{R}^{N-1}$ with radius r centered at the origin. Provided $r > 0$ small, the map $F_{y_0} : Q_r \to \Omega$, given by

$$(t, z) \mapsto F_{y_0}(t, z) := \gamma(t) + \sum_{i=2}^{N} z_i E_i(t),$$

is smooth and parameterizes a neighborhood of $y_0 = F_{y_0}(0,0)$. We consider $\rho_\Gamma : \Gamma \to \mathbb{R}$ the distance function to the curve given by

$$\rho_\Gamma(y) = \min_{\overline{y} \in \mathbb{R}^N} |y - \overline{y}|.$$

In the above coordinates, we have

$$\rho_\Gamma\left(F_{y_0}(x)\right) = |z| \quad \text{for every } x = (t,z) \in Q_r. \tag{2.1}$$

Clearly, for every $t \in (-r,r)$ and $i = 2, \dots N$, there are real numbers $\kappa_i(t)$ and $\tau_i^j(t)$ such that

$$E_i'(t) = \kappa_i(t)\gamma'(t) + \sum_{j=2}^{N} \tau_i^j(t) E_j(t). \tag{2.2}$$

The quantity $\kappa_i(t)$ is the curvature in the $E_i(t)$-direction while $\tau_i^j(t)$ is the torsion from the osculating plane spanned by $\{\gamma'(t); E_j(t)\}$ in the direction E_i. We note that provided $r > 0$ small, κ_i and τ_i^j are smooth functions on $(-r,r)$. Moreover, it is easy to see that

$$\tau_i^j(t) = -\tau_j^i(t) \quad \text{for} \quad i,j = 2, \dots, N. \tag{2.3}$$

The curvature vector is $\kappa : \Gamma \to \mathbb{R}^N$ is defined as

$$\kappa(\gamma(t)) := \sum_{i=2}^{N} \kappa_i(t) E_i(t)$$

and its norm is given by

$$|\kappa\gamma(t)| := \sqrt{\sum_{i=2}^{N} \kappa_i^2(t)}.$$

Next, we derive the expansion of the metric induced by the parameterization F_{y_0} defined above. For $x = (t,z) \in Q_r$, we define

$$\begin{aligned} g_{11}(x) &= \partial_t F_{y_0}(x) \cdot \partial_t F_{y_0}(x), \\ g_{1i}(x) &= \partial_t F_{y_0}(x) \cdot \partial_{z_i} F_{y_0}(x), \\ g_{ij}(x) &= \partial_{z_j} F_{y_0}(x) \cdot \partial_{z_i} F_{y_0}(x). \end{aligned}$$

We have the following result.

Lemma 2.1. *There exits $r > 0$, only depending on Γ and N, such that for ever $x = (t, z) \in Q_r$*

$$\begin{cases} g_{11}(x) = 1 + 2\sum_{i=2}^{N} z_i\kappa_i(0) + 2t\sum_{i=2}^{N} z_i\kappa_i'(0) + \sum_{ij=2}^{N} z_iz_j\kappa_i(0)\kappa_j(0) \\ \qquad\qquad + \sum_{ij=2}^{N} z_iz_j\beta_{ij}(0) + O(|x|^3), \\ g_{1i}(x) = \sum_{j=2}^{N} z_j\tau_j^i(0) + t\sum_{j=2}^{N} z_j\left(\tau_j^i\right)'(0) + O(|x|^3), \\ g_{ij}(x) = \delta_{ij}, \end{cases} \tag{2.4}$$

where $\beta_{ij}(t) := \sum_{l=2}^{N} \tau_i^l(t)\tau_j^l(t)$.

Proof. To alleviate the notations, we will write $F = F_{y_0}$. We have

$$\partial_t F(x) = \gamma'(t) + \sum_{j=2}^{N} z_jE_j'(t) \quad \text{and} \quad \partial_{z_i}F(x) = E_i(t). \tag{2.5}$$

Therefore

$$g_{ij}(x) = E_i(t)\cdot E_j(t) = \delta_{ij}. \tag{2.6}$$

By (2.2) and (2.5), we have

$$g_{1i}(x) = \sum_{l=2}^{N} z_lE_l'(t)\cdot E_i(t) = \sum_{j=2}^{N} z_j\tau_j^i(t) \tag{2.7}$$

and

$$\begin{aligned} g_{11}(x) &= \partial_t F(x)\cdot\partial_t F(x) \\ &= 1 + 2\sum_{i=2}^{N} z_i\kappa_i(t) + \sum_{ij=2}^{N} z_iz_j\kappa_i(t)\kappa_j(t) + \sum_{ij=2}^{N} z_iz_j\left(\sum_{l=2}^{N}\tau_i^l(t)\tau_j^l(t)\right). \end{aligned} \tag{2.8}$$

By Taylor expansions, we get

$$\kappa_i(t) = \kappa_i(0) + t\kappa_i'(0) + O\left(t^2\right)$$

and

$$\tau_i^k(t) = \tau_i^k(0) + t\left(\tau_i^k\right)'(0) + O\left(t^2\right).$$

Using these identities in (2.8) and (2.7), we get (2.4), thanks to (2.6). This ends the proof. $\square$

As a consequence we have the following result.

Lemma 2.2. *There exists $r > 0$ only depending on Γ and N, such that for every $x \in Q_r$, we have*

$$\sqrt{|g|}(x) = 1 + \sum_{i=2}^{N} z_i\kappa_i(0) + t\sum_{i=2}^{N} z_i\kappa_i'(0) + \frac{1}{2}\sum_{ij=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) + O(|x|^3), \tag{2.9}$$

where $|g|$ stands for the determinant of g. Moreover $g^{-1}(x)$, the matrix inverse of $g(x)$, has components given by

$$\begin{cases} g^{11}(x) = 1 - 2\displaystyle\sum_{i=2}^{N} z_i\kappa_i(0) - 2t\sum_{i=2}^{N} z_i\kappa_i'(0) + 3\sum_{ij=2}^{N} z_i z_j\kappa_i(0)\kappa_j(0) + O(|x|^3), \\ g^{i1}(x) = -\displaystyle\sum_{j=2}^{N} z_j\tau_j^i(0) - t\sum_{j=2}^{N} z_j\left(\tau_j^i\right)'(0) + 2\sum_{j=2}^{N} z_l z_j\kappa_l(0)\tau_j^i(0) + O(|x|^3), \\ g^{ij}(x) = \delta_{ij} + \displaystyle\sum_{lm=2}^{N} z_l z_m \tau_l^j(0)\tau_m^i(0) + O(|x|^3). \end{cases} \tag{2.10}$$

Proof. We write

$$g(x) = id + H(x),$$

where id denotes the identity matrix on $\mathbb{R}^N$ and H is a symmetric matrix with components $H_{\alpha\beta}$, for $\alpha, \beta = 1, \dots, N$, given by

$$\begin{cases} H_{11}(x) = 2\displaystyle\sum_{i=2}^{N} z_i\kappa_i(0) + 2t\sum_{i=2}^{N} z_i\kappa_i'(0) + \sum_{ij=2}^{N} z_i z_j\kappa_i(0)\kappa_j(0) \\ \qquad\qquad + \displaystyle\sum_{ij=2}^{N} z_i z_j\beta_{ij}(0) + O(|x|^3), \\ H_{1i}(x) = \displaystyle\sum_{j=2}^{N} z_i\tau_j^i(0) + O\left(|x|^2\right), \\ H_{ij}(x) = 0. \end{cases} \tag{2.11}$$

We recall that as $|H| \to 0$,

$$\sqrt{|g|} = \sqrt{\det(I + H)} = 1 + \frac{\operatorname{tr} H}{2} + \frac{(\operatorname{tr} H)^2}{4} - \frac{\operatorname{tr}(H^2)}{4} + O\left(|H|^3\right). \tag{2.12}$$

Now by (2.11), as $|x| \to 0$, we have

$$\begin{aligned} \frac{\operatorname{tr} H}{2} = \sum_{i=2}^{N} z_i\kappa_i(0) + t\sum_{i=2}^{N} z_i\kappa_i'(0) + \frac{1}{2}\sum_{ij=2}^{N} z_i z_j\kappa_i(0)\kappa_j(0) \\ + \frac{1}{2}\sum_{ij=2}^{N} z_i z_j\beta_{ij}(0) + O(|x|^3), \end{aligned} \tag{2.13}$$

so that

$$\frac{(\text{ tr } H)^2}{4} = \sum_{ij=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) + O(|x|^3). \tag{2.14}$$

Moreover, from (2.11), we deduce that

$$\begin{aligned}\text{tr } (H^2)(x) &= \sum_{\alpha=1}^{N} \left(H^2(x)\right)_{\alpha\alpha} = \sum_{\alpha\beta=1}^{N} H_{\alpha\beta}(x) H_{\beta\alpha}(x) \\ &= \sum_{\alpha\beta=1}^{N} H^2_{\alpha\beta}(x) = H^2_{11}(x) + 2\sum_{i=2}^{N} H^2_{i1}(x),\end{aligned}$$

so that

$$-\frac{\text{tr } (H^2)}{4} = -\sum_{ij=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) - \frac{1}{2}\sum_{ijl=2}^{N} z_i z_j \tau_i^l(0)\tau_j^l(0) + O(|x|^3). \tag{2.15}$$

Therefore plugging the expression from (2.13), (2.14) and (2.15) in (2.12), we get

$$\sqrt{|g|}(x) = 1 + \sum_{i=2}^{N} z_i \kappa_i(0) + t\sum_{i=2}^{N} z_i \kappa_i'(0) + \frac{1}{2}\sum_{ij=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) + O(|x|^3).$$

The proof of (2.9) is thus finished.

By Lemma 2.1, we can write

$$g(x) = id + A(x) + B(x) + O(|x|^3),$$

where A and B are symmetric matrix with components $(A_{\alpha\beta})$ and $(A_{\alpha\beta})$, $\alpha, \beta = 1, \dots, N$, given respectively by

$$A_{11}(x) = 2\sum_{i=2}^{N} z_i \kappa_i(0), \quad A_{i1}(x) = \sum_{j=2}^{N} z_j \tau_j^i(0) \quad \text{and} \quad A_{ij}(x) = 0 \tag{2.16}$$

and

$$\begin{cases} B_{11}(x) = 2t\displaystyle\sum_{i=2}^{N} z_i \kappa'(0) + \sum_{i=2}^{N} z_i z_j \kappa_i(0)\kappa_j(0) + \sum_{ij=2}^{N} z_i z_j \beta_{ij}(0), \\ B_{i1}(x) = t\displaystyle\sum_{j=2} z_j \left(\tau_j^i\right)'(0) \quad \text{and} \quad B_{ij}(x) = 0. \end{cases} \tag{2.17}$$

We observe that, as $|x| \to 0$, we have

$$g^{-1}(x) = id - A(x) - B(x) + A^2(x) + O(|x|^3).$$

We then deduce from (2.16) and (2.17) that

$$
\begin{aligned}
g^{11}(x) &= 1 - A_{11}(x) - B_{11}(x) + A_{11}^2(x) + \sum_{i=1}^{N} A_{1i}^2(x) + O(|x|^3) \\
&= 1 - 2\sum_{i=2}^{N} z_i\kappa_i(0) - 2t\sum_{i=2}^{N} z_i\kappa'(0) + 3\sum_{i=2}^{N} z_iz_j\kappa_i(0)\kappa_j(0) \\
&\quad + 3\sum_{ij=2}^{N} z_iz_j\beta_{ij}(0) + O(|x|^3), \\
g^{i1}(x) &= -A_{1i}(x) - B_{1i}(x) + \sum_{\alpha=1}^{N} A_{i\alpha}A_{1\alpha} + O(|x|^3) \\
&= -A_{1i}(x) - B_{1i}(x) + A_{i1}(x)A_{11}(x) + \sum_{j=2}^{N} A_{ij}(x)A_{1j}(x) + O(|x|^3) \\
&= -\sum_{j=2}^{N} z_j\tau_j^i(0) - t\sum_{j=2} z_j\left(\tau_j^i\right)'(0) + 2\sum_{jl=2}^{N} z_lz_j\kappa_l(0)\tau_j^i(0)
\end{aligned}
$$

and

$$
\begin{aligned}
g^{ij}(x) &= \delta_{ij} - A_{ij}(x) - B_{ij}(x) + \left(A^2\right)_{ij}(x) + O(|x|^3) \\
&= \delta_{ij} - A_{ij}(x) - B_{ij}(x) + A_{1i}A_{1j} + \sum_{l=2}^{N} A_{il}(x)A_{jl}(x) + O(|x|^3) \\
&= \delta_{ij} + \sum_{lm=2}^{N} z_lz_m\tau_m^i(0)\tau_l^j(0) + O(|x|^3).
\end{aligned}
$$

This ends the proof. □

We recall that the best constant for the cylindrical Hardy–Sobolev inequality is given by

$$
S_{N,\sigma} = \min\left\{\frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 dx - \frac{1}{2^*_\sigma}\int_{\mathbb{R}^N} |z|^{-\sigma}|w|^{2^*_\sigma} dx \;:\; w \in \mathcal{D}^{1,2}(\mathbb{R}^N)\right\}.
$$

Further it is attained by a positive function $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, that satisfies the Euler–Lagrange equation

$$
-\Delta w = |z|^{-\sigma} w^{2^*_\sigma - 1} \quad \text{in } \mathbb{R}^N, \tag{2.18}
$$

see e.g. [2]. By [7], we have the last result of this section.

Lemma 2.3. *For $N \geq 3$, we have*

$$w(x) = w(t,z) = \theta(|t|,|z|) \quad \text{for a function} \quad \theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+. \tag{2.19}$$

Moreover, there exists two constants $0 < C_1 < C_2$, such that

$$\frac{C_1}{1+|x|^{N-2}} \leq w(x) \leq \frac{C_2}{1+|x|^{N-2}} \quad \text{in } \mathbb{R}^N. \tag{2.20}$$

3. EXISTENCE OF MINIMZERS FOR $\mu(\Omega, \Gamma, h, b)$ IN DIMENSION $N \geq 4$

We consider Ω a bounded domain of $\mathbb{R}^N$, $N \geq 3$ and $\Gamma \subset \Omega$ be a smooth closed curve. For $u \in H_0^1(\Omega) \setminus \{0\}$, we define the functional

$$J(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dy + \frac{1}{2} \int_{\Omega} hu^2 dy + \frac{1}{2+\delta} \int_{\Omega} bu^{2+\delta} dy - \frac{1}{2^*_\sigma} \int_{\Omega} \rho_\Gamma^{-\sigma} |u|^{2^*_\sigma} dy. \tag{3.1}$$

We let $\eta \in \mathcal{C}_c^\infty(F_{y_0}(Q_{2r}))$ be such that

$$0 \leq \eta \leq 1 \quad \text{and} \quad \eta \equiv 1 \quad \text{in } Q_r.$$

For $\varepsilon > 0$, we consider $u_\varepsilon : \Omega \to \mathbb{R}$ given by

$$u_\varepsilon(y) := \varepsilon^{\frac{2-N}{2}} \eta(F_{y_0}^{-1}(y)) w\left(\varepsilon^{-1} F_{y_0}^{-1}(y)\right). \tag{3.2}$$

In particular, for every $x = (t,z) \in \mathbb{R} \times \mathbb{R}^{N-1}$, we have

$$u_\varepsilon(F_{y_0}(x)) := \varepsilon^{\frac{2-N}{2}} \eta(x) \theta(|t|/\varepsilon, |z|/\varepsilon). \tag{3.3}$$

It is clear that $u_\varepsilon \in H_0^1(\Omega)$. Then we have the following proposition.

Proposition 3.1. *For all $N \geq 4$, we have*

$$J(u_\varepsilon) = S_{N,\sigma} + \varepsilon^{2-\frac{\delta(N-2)}{2}} b(y_0) \int_{\mathbb{R}^N} w^{\delta+2} dx + o\left(\varepsilon^{2-\frac{\delta(N-2)}{2}}\right), \tag{3.4}$$

as $\varepsilon \to 0$.

The proof of Proposition 3.1 is divided in two parts, Lemma 3.2 and Lemma 3.3 below. For that we set

$$J_1(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} hu^2 dx - \frac{1}{2^*_\sigma} \int_{\Omega} \rho_\Gamma^{-\sigma} |u|^{2^*_\sigma} dx,$$

the following is due to the second author and Fall [9].

Lemma 3.2. *We have*

$$J_1\left(u_\varepsilon\right) = S_{N,\sigma} + \begin{cases} O(\varepsilon^2) & \text{for all } N \geq 5, \\ O(\varepsilon^2 |\log(\varepsilon)|) & \text{for all } N = 4. \end{cases} \tag{3.5}$$

We finish the proof by the following lemma.

Lemma 3.3. *We have*

$$\begin{cases} \int\limits_\Omega b u_\varepsilon^{2+\delta} dx = \varepsilon^{2-\frac{\delta(N-2)}{2}} b(y_0) \int\limits_{\mathbb{R}^N} w^{\delta+2} dx + O\left(\varepsilon^2\right) & \text{for } N \geq 4, \\ \int\limits_\Omega b u_\varepsilon^{2+\delta} dx = \varepsilon^{2-\frac{\delta}{2}} b(y_0) \int\limits_{Q_{r/\varepsilon}} w^{\delta+2} dx + O\left(\varepsilon^2\right) & \text{for } N = 3 \text{ and } \delta \leq 1, \\ \int\limits_\Omega b u_\varepsilon^{2+\delta} dx = \varepsilon^{2-\frac{\delta}{2}} b(y_0) \int\limits_{\mathbb{R}^N} w^{\delta+2} dx + O\left(\varepsilon^{1+\frac{\delta}{2}}\right) & \text{for } N = 3 \text{ and } \delta > 1 \end{cases}$$

as $\varepsilon \to 0$.

Proof. We have

$$\int\limits_\Omega b(x) u_\varepsilon^{2+\delta} dx = \int\limits_{F_{y_0}(Q_r)} b(x) u_\varepsilon^{2+\delta} dx + \int\limits_{F_{y_0}(Q_{2r}) \setminus F_{y_0}(Q_r)} b(x) u_\varepsilon^{2+\delta} dx.$$

Since b is continuous and r is small, then by the change of variable formula $y = \frac{F(x)}{\varepsilon}$, we have

$$\begin{aligned} \int\limits_\Omega b(x) u_\varepsilon^{2+\delta} dx &= b(y_0) \varepsilon^{2-\frac{\delta(N-2)}{2}} \int\limits_{Q_{r/\varepsilon}} w^{2+\delta} dx \\ &\quad + O\left(\varepsilon^{4-\frac{\delta(N-2)}{2}} \int\limits_{Q_{r/\varepsilon}} |x|^2 w^{2+\delta} dx + \varepsilon^{2-\frac{\delta(N-2)}{2}} \int\limits_{Q_{2r/\varepsilon} \setminus Q_{r/\varepsilon}} w^{2+\delta} dx\right) \\ &= b(y_0) \varepsilon^{2-\frac{\delta(N-2)}{2}} \int\limits_{Q_{r/\varepsilon}} w^{2+\delta} dx \\ &\quad + O\left(\varepsilon^{4-\frac{\delta(N-2)}{2}} \int\limits_{Q_{r/\varepsilon}} |x|^2 w^{2+\delta} dx + \varepsilon^{2-\frac{\delta(N-2)}{2}} \int\limits_{Q_{2r/\varepsilon} \setminus Q_{r/\varepsilon}} w^{2+\delta} dx\right). \end{aligned}$$

Thanks to (2.20), we have

$$\varepsilon^{4-\frac{\delta(N-2)}{2}} \int\limits_{Q_{r/\varepsilon}} |x|^2 w^{2+\delta} dx + \varepsilon^{2-\frac{\delta(N-2)}{2}} \int\limits_{Q_{2r/\varepsilon} \setminus Q_{r/\varepsilon}} w^{2+\delta} dx = O(\varepsilon^2) \quad \text{for all } N \geq 3$$

and

$$\varepsilon^{2-\frac{\delta(N-2)}{2}} \int\limits_{Q_{2r/\varepsilon}\setminus Q_{r/\varepsilon}} w^{2+\delta} dx = O(\varepsilon^2) \quad \text{for all } N \geq 4.$$

We finish by noticing that, for $N = 3$, we have

$$\int\limits_{\mathbb{R}^N \setminus Q_{r/\varepsilon}} w^{2+\delta} dx = O(\varepsilon^{\delta-1}).$$

This then ends the proof of the lemma. □

4. EXISTENCE OF MINIMIZER FOR $\mu_h(\Omega, \Gamma, h, b)$ IN DIMENSION THREE

We consider the function

$$\mathcal{R} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \quad x \mapsto \mathcal{R}(x) = \frac{1}{|x|}$$

which satisfies

$$-\Delta \mathcal{R} = 0 \quad \text{in } \mathbb{R}^3 \setminus \{0\}. \tag{4.1}$$

We denote by G the solution to the equation

$$\begin{cases} -\Delta_x G(y, \cdot) + hG(y, \cdot) = 0 & \text{in } \Omega \setminus \{y\}, \\ G(y, \cdot) = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

and satisfying

$$G(x, y) = \mathcal{R}(x - y) + O(1) \quad \text{for } x, y \in \Omega \text{ and } x \neq y. \tag{4.3}$$

We note that G is proportional to the Green function of $-\Delta + h$ with zero Dirichlet data.

We let $\chi \in C_c^\infty(-2, 2)$ with $\chi \equiv 1$ on $(-1, 1)$ and $0 \leq \chi < 1$. For $r > 0$, we consider the cylindrical symmetric cut-off function

$$\eta_r(t, z) = \chi\left(\frac{|t| + |z|}{r}\right) \quad \text{for every } (t, z) \in \mathbb{R} \times \mathbb{R}^2. \tag{4.4}$$

It is clear that

$$\eta_r \equiv 1 \quad \text{in } Q_r, \quad \eta_r \in H_0^1(Q_{2r}), \quad |\nabla \eta_r| \leq \frac{C}{r} \quad \text{in } \mathbb{R}^3.$$

For $y_0 \in \Omega$, we let $r_0 \in (0, 1)$ such that

$$y_0 + Q_{2r_0} \subset \Omega. \tag{4.5}$$

We define the function $M_{y_0} : Q_{2r_0} \to \mathbb{R}$ given by

$$M_{y_0}(x) := G(y_0, x + y_0) - \eta_r(x)\frac{1}{|x|} \quad \text{for every } x \in Q_{2r_0}. \tag{4.6}$$

It follows from (4.3) that $M_{y_0} \in L^\infty(Q_{r_0})$. By (4.2) and (4.1),

$$|-\Delta M_{y_0}(x) + h(x)M_{y_0}(x)| \le \frac{C}{|x|} = C\mathcal{R}(x) \quad \text{for every } x \in Q_{r_0},$$

whereas $\mathcal{R} \in L^p(Q_{r_0})$ for every $p \in (1,3)$. Hence by elliptic regularity theory, $M_{y_0} \in W^{2,p}(Q_{r_0/2})$ for every $p \in (1,3)$. Therefore by Morrey's embdding theorem, we deduce that

$$\|M_{y_0}\|_{C^{1,\varrho}(Q_{r_0/2})} \le C \quad \text{for every } \varrho \in (0,1). \tag{4.7}$$

In view of (1.6), the mass of the operator $-\Delta + h$ in Ω at the point $y_0 \in \Omega$ is given by

$$\mathbf{m}(y_0) = M_{y_0}(0). \tag{4.8}$$

We recall that the positive ground state solution w satisfies

$$-\Delta w = |z|^{-\sigma} w^{2^*_\sigma - 1} \quad \text{in } \mathbb{R}^3, \tag{4.9}$$

where $x = (t,z) \in \mathbb{R} \times \mathbb{R}^2$. In addition by (2.20), we have

$$\frac{C_1}{1+|x|} \le w(x) \le \frac{C_2}{1+|x|} \quad \text{in } \mathbb{R}^3. \tag{4.10}$$

The following result will be crucial in the sequel.

Lemma 4.1. *Consider the function $v_\varepsilon : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ given by*

$$v_\varepsilon(x) = \varepsilon^{-1} w\left(\frac{x}{\varepsilon}\right).$$

Then there exists a constant $\boldsymbol{c} > 0$ and a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ (still denoted by ε) such that

$$v_\varepsilon(x) \to \frac{\boldsymbol{c}}{|x|} \quad \textit{for almost every } x \in \mathbb{R}^3$$

and

$$v_\varepsilon(x) \to \frac{\boldsymbol{c}}{|x|} \quad \textit{for every } x \in \mathbb{R}^3 \setminus \{z = 0\}. \tag{4.11}$$

For a proof, see for instance [9, Lemma 5.1].

Next, given $y_0 \in \Gamma \subset \Omega \subset \mathbb{R}^3$, we let r_0 as defined in (4.5). For $r \in (0, r_0/2)$, we consider $F_{y_0} : Q_r \to \Omega$ (see Section 2) parameterizing a neighborhood of y_0 in Ω, with the property that $F_{y_0}(0) = y_0$. For $\varepsilon > 0$, we consider $u_\varepsilon : \Omega \to \mathbb{R}$ given by

$$u_\varepsilon(y) := \varepsilon^{-1/2}\eta_r(F_{y_0}^{-1}(y))w\left(\frac{F_{y_0}^{-1}(y)}{\varepsilon}\right).$$

We can now define the test function $\Psi_\varepsilon : \Omega \to \mathbb{R}$ by

$$\Psi_\varepsilon\,(y) = u_\varepsilon(y) + \varepsilon^{1/2}\mathbf{c}\,\eta_{2r}(F_{y_0}^{-1}(y))M_{y_0}(F_{y_0}^{-1}(y)). \tag{4.12}$$

It is plain that $\Psi_\varepsilon \in H_0^1(\Omega)$ and

$$\Psi_\varepsilon\,(F_{y_0}(x)) = \varepsilon^{-1/2}\eta_r(x)w\left(\frac{x}{\varepsilon}\right) + \varepsilon^{1/2}\mathbf{c}\,\eta_{2r}(x)M_{y_0}(x) \qquad \text{for every } x \in \mathbb{R}^N.$$

The main result of this section is contained in the following result.

Proposition 4.2. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ and $\boldsymbol{c}$ be the sequence and the number given by Lemma 4.1. Then there exists $r_0, n_0 > 0$ such that for every $r \in (0, r_0)$ and $n \geq n_0$*

$$\begin{cases} J(\Psi_\varepsilon) = S_{3,\sigma} - \varepsilon_n\pi^2\boldsymbol{m}(y_0)\boldsymbol{c}^2 + \dfrac{\varepsilon_n^{2-\frac{\delta}{2}}}{2+\delta}\displaystyle\int_{Q_{r/\varepsilon}} w^{2+\delta}dx + \mathcal{O}_r(\varepsilon_n) & \text{for } \delta \leq 1, \\ J(\Psi_\varepsilon) = S_{3,\sigma} - \varepsilon_n\pi^2\boldsymbol{m}(y_0)\boldsymbol{c}^2 + \dfrac{\varepsilon_n^{2-\frac{\delta}{2}}}{2+\delta}\displaystyle\int_{\mathbb{R}^3} w^{2+\delta}dx + \mathcal{O}_r(\varepsilon_n) & \text{for } \delta > 1 \end{cases}$$

for some numbers $\mathcal{O}_r(\varepsilon_n)$ satisfying

$$\lim_{r\to 0}\lim_{n\to\infty} \varepsilon_n^{-1}\mathcal{O}_r(\varepsilon_n) = 0.$$

The proof of this proposition will be separated into two steps given by Lemma 4.3 and Lemma 4.4 below. To alleviate the notations, we will write ε instead of ε_n and we will remove the subscript y_0, by writing M and F in the place of M_{y_0} and F_{y_0}, respectively. We define

$$\begin{aligned} \widetilde{\eta}_r(y) &:= \eta_r(F^{-1}(y)), \\ V_\varepsilon(y) &:= v_\varepsilon(F^{-1}(y)), \\ \widetilde{M}_{2r}(y) &:= \eta_{2r}(F^{-1}(y))M(F^{-1}(y)), \end{aligned}$$

where $v_\varepsilon(x) = \varepsilon^{-1}w\left(\frac{x}{\varepsilon}\right)$. With these notations, (4.12) becomes

$$\Psi_\varepsilon(y) = u_\varepsilon(y) + \varepsilon^{\frac{1}{2}}\mathbf{c}\,\widetilde{M}_{2r}(y) = \varepsilon^{\frac{1}{2}}V_\varepsilon(y) + \varepsilon^{\frac{1}{2}}\mathbf{c}\,\widetilde{M}_{2r}(y). \tag{4.13}$$

We first consider the numerator in (4.2).

Lemma 4.3. *We have*

$$J_1(\Psi_\varepsilon) = S_{3,\sigma} - \varepsilon\pi^2\boldsymbol{c}^2\boldsymbol{m}(y_0) + \mathcal{O}_r(\varepsilon),$$

as $\varepsilon \to 0$.

For a proof, see for instance [9, Proposition 5.3]. The following result together with the previous lemma provides the proof of Proposition 4.2.

Lemma 4.4. *We have*

$$\int_{\Omega} |\Psi_\varepsilon|^{2+\delta} dy = \varepsilon^{2-\frac{\delta}{2}} b(y_0) \int_{Q_{r/\varepsilon}} w^{2+\delta} dx + o\left(\varepsilon^{2-\frac{\delta}{2}}\right),$$

as $\varepsilon \to 0$.

Proof. Since $\delta > 0$, by the Taylor expansion we have

$$\begin{aligned}\int_{\Omega} |\Psi_\varepsilon|^{2+\delta} dy &= \int_{\Omega} |u_\varepsilon + \varepsilon^{1/2} \widetilde{M}_{2r}|^{2+\delta} dy \\ &= \int_{\Omega} |u_\varepsilon|^{2+\delta} dy + O\Big(\varepsilon^{1/2} \int_{\Omega} |u_\varepsilon|^{1+\delta} |\widetilde{M}_{2r}| dy \\ &\quad + \int_{\Omega} |u_\varepsilon|^{\delta} |\widetilde{M}_{2r}|^2 dy + \int_{\Omega} |\widetilde{M}_{2r}|^{2+\delta} dy\Big).\end{aligned} \tag{4.14}$$

Using Hölder's inequality and (2.9), we have

$$\begin{aligned}\int_{F(Q_{4r})} |\eta u_\varepsilon|^{\delta} \left(\varepsilon^{1/2} \widetilde{M}_r\right)^2 dy &\le \varepsilon \|u_\varepsilon\|^{\delta}_{L^{2+\delta}(F(Q_{4r}))} \|\widetilde{M}_{2r}\|^2_{L^{2+\delta}(F(Q_{4r}))} \\ &= \varepsilon^{4-\frac{\delta}{2}} \|w\|^{\delta}_{L^{2+\delta}(Q_{4r}; \sqrt{|g|})} \|\widetilde{M}_{2r}\|^2_{L^{2+\delta}(F(Q_{4r}))} \\ &\le \varepsilon^{4-\frac{\delta}{2}} \|\widetilde{M}_{2r}\|^2_{L^{2+\delta}(F(Q_{4r}))} = o(\varepsilon),\end{aligned} \tag{4.15}$$

Since $\delta > 0$, by (4.7), we easily get

$$\int_{F(Q_{4r})} |\varepsilon^{1/2} \widetilde{M}_{2r}|^{2+\delta} dy = O(\varepsilon^{1+\frac{\delta}{2}}) = o(\varepsilon). \tag{4.16}$$

By (4.14), (4.16), (4.15) and Lemma 3.3, it results

$$\begin{aligned}\int_{\Omega} |\Psi_\varepsilon|^{2+\delta} dy &= \int_{F(Q_r)} |u_\varepsilon|^{2+\delta} dy + O\left(\varepsilon^{1/2} \int_{F(Q_r)} |u_\varepsilon|^{1+\delta} \widetilde{M}_{2r} dy\right) + o(\varepsilon) \\ &= \varepsilon^{2-\frac{\delta}{2}} b(y_0) \int_{Q_{r/\varepsilon}} w^{\delta+2} dx + O\left(\varepsilon^{1/2} \int_{F(Q_r)} |u_\varepsilon|^{1+\delta} \widetilde{M}_{2r} dy\right) + o(\varepsilon).\end{aligned}$$

We define

$$B_\varepsilon(x) := M(\varepsilon x) \sqrt{|g_\varepsilon|}(x) = M(\varepsilon x) \sqrt{|g|}(\varepsilon x).$$

Then by the change of variable $y = \frac{F(x)}{\varepsilon}$ in the above identity and recalling (2.9), then by oddness, we have

$$\varepsilon^{1/2} \int_{\Omega} |u_\varepsilon|^{1+\delta} |\widetilde{M}_{2r}| dy = O\left(\varepsilon^{3-\delta/2} \int_{Q_{r/\varepsilon}} |w|^{1+\delta} dx \right) = O\left(\varepsilon^{3-\delta/2}\right).$$

Therefore

$$\int_{\Omega} |\Psi_\varepsilon|^{2+\delta} dy = \varepsilon^{2-\frac{\delta}{2}} b(y_0) \int_{Q_{r/\varepsilon}} w^{2+\delta} dx + o\left(\varepsilon^{2-\frac{\delta}{2}}\right),$$

as $\varepsilon \to 0$. This then ends the proof. □

5. PROOFS OF THEOREM 1.1 AND THEOREM 1.2

It is well known in the literature that if

$$\mu_\sigma(\Omega, \Gamma, h, b) < S_{N,\sigma}, \tag{5.1}$$

then $\mu_\sigma(\Omega, \Gamma, h, b)$ is achieved by a positive function $u \in H^1_0(\Omega)$. For a similar result, we refer to the works of [9, 14, 22] and references therein. Therefore, the proofs of Theorem 1.1 and Theorem 1.2 are direct consequences of Propositions 3.1 and 4.2, and inequality (5.1) above.

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Idowu Esther Ijaodoro
idowu.e.ijaodoro@aims-senegal.org
https://orcid.org/0000-0002-9550-9090

African Institute for Mathematical Sciences in Senegal
KM 2, Route de Joal, B.P. 14 18, Mbour, Senegal

El Hadji Abdoulaye Thiam (corresponding author)
elhadjiabdoulaye.thiam@univ-thies.sn
https://orcid.org/0000-0003-1206-8312

Université de Thies
UFR des Sciences et Techniques
Département de Mathématiques, Thies, Senegal

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