THE METHOD OF BALANCING THE PRODUCTION AND CONSUMPTION MODEL IN THE CASE OF INDIVISIBLE ARTICLES

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Abstract. In this article, a detailed case of the unbalanced production-consumption, namely a model, which is used in the case of indivisible articles, is presented. Also, the method of balancing this model is given, relying on introducing a certain ordering relation in the consumer's set.

Keywords: *balancing, model, indivisible articles*

Introduction

In article [1] the unsustainable production and consumption model was defined, in which occur $n \geq 2$ contractors, $m \geq 1$ producers and $n - m \geq 1$ consumers of a certain good. The model assumes that total demand exceeds total supply in a fixed period of time, what we write in the form of so-called uncorrected n-dimensional supply-demand vector, fulfilling the conditions

1) $p_i > 0$ for every $i < m$

2) $p_i < 0$ for every $m + 1 \leq i \leq n$

3) $\sum_{i=1}^{n} p_i < 0$

The general case was discussed in [1-4] articles. A detailed case are so-called indivisible articles, for example: washing machines, television sets etc. (you cannot buy 1.5 washing machine).

In this case, instead of uncorrected supply-demand vector, we consider so called discrete vector, ie. uncorrected supply-demand vector, of which every coordinate is an integer.

Let there be vectors:

a) discrete vector $\vec{\mathbf{p}} = (p_1, \ldots, p_n)$, where $p_i > 0$ for $i = 1, 2, \ldots m < n$ (p_i - demand of i^{th} contractor), $p_j < 0$ for $j = m+1, \ldots n$ (p_j - supply of j^{th} contractor), so that $p_1 + p_2 + \ldots + p_n < 0$,

b) maximum concession vector $\vec{\mathbf{u}} = (u_1, \ldots u_n)$, where ($u_i \in \mathrm{N} \cup \{0\}$, u_i - the size of the maximum possible concession of i^{th} contractor), so that $(p_1 + u_1) + (p_2 + u_2) + \ldots + (p_n + u_n) \geq 0$,

c) vector $\vec{\mathbf{a}} = (a_1, a_2, \ldots, a_n)$, a_i - concession weight of i^{th} contractor.

Taking $\vec{\mathbf{p}}^{\mathbf{0}} = \mathbf{p}$, $\vec{\mathbf{a}}^{\mathbf{0}} = \vec{\mathbf{a}}$ for $s \geq 1$ we define the size of

$$r^s = -\left(p_1^{s-1} + \ldots + p_n^{s-1}\right)$$

$$N^s = \left\{ i \leq n : \sum_{j \leq s} a_i^{j-1} r^j \leq u_i \right\}$$

$$\vec{\mathbf{a}}^s = \left(a_1^s, \ldots, a_n^s\right) \text{ where } a_i^s = \begin{cases} 0 & \text{for} \quad i \notin N^s \\ a_i^{s-1}\left(B^{s-1}\right)^{-1} & \text{for} \quad i \in N^s \end{cases}$$

where $B^{s-1} = \sum_{j \in N^{s-1}} a_j^{s-1}$

$$\overline{N}^s = N^{s-1} \backslash N^s \text{ where } N^o = \{1, \ldots, n\}$$

$$\vec{\mathbf{p}}^s = \left(p_1^s, \ldots, p_n^s\right) \text{ where } p_i^s = \begin{cases} p_i^{s-1} + r^s \cdot a_i^{s-1} & \text{for} \quad i \in N^s \\ p_i + u_i & \text{for} \quad i \in \overline{N}^s \\ p_i^{s-1} & \text{for} \quad i \in \bigcup_{j \langle\ s} \overline{N}^j \end{cases}$$

The iteration process, whose properties were described in [2], we finish for s^{th} iteration, such that $r^s = 0$. Then vector $\vec{\mathbf{p}}^{s-1}$ balances the model. Let $\vec{\mathbf{x}} = (x_1, \ldots, x_n)$ be any vector. By $\mathrm{E}(\vec{\mathbf{x}})$ we understand vector $\left[\mathrm{E}(x_1), \ldots, \mathrm{E}(x_n)\right]$ where $\mathrm{E}(x_i)$ - integer's total part x_i. Because $\vec{\mathbf{p}}^{s-1} = \vec{\mathbf{p}}^s$, continuing in this article, by the overall vector correction vector $\vec{\mathbf{p}}^0$ we adopt vector $\vec{\mathbf{p}}^s$.

If $\mathrm{E}\left(p_1^s\right)+\mathrm{E}\left(p_2^s\right)+\ldots+\mathrm{E}\left(p_n^s\right)=0$ then vector $\mathrm{E}\left(\vec{\mathrm{p}}^s\right)$ balances the model in case of indivisible articles. Assume that $t=\mathrm{E}\left(p_1^s\right)+\ldots+\mathrm{E}\left(p_n^s\right)>0$, which means that supply on certain good exceeds demand. In order to achieve sustainable modem, some producers should reduce the size of production (supply reduction) or, chosen consumers increase demand. In this situation, you may prefer consumer and use a second variant of balancing. It therefore remains to determine which consumer and by how much the demand should be increased. The solution can be as follows: add by one unit of good for those consumers who the most resigned.

1. The balancing of model

It is, therefore necessary to prove that:
a) has remained integer of units goods to distribute among consumers,
b) the number of consumers, which can increase demand by one unit of good (in this case, the border in output demand), is not smaller than number of units of goods, which is left to distribute.

It will be proved by the statement below.

Statement 1.

If:

$M_1=\left\{i\in N^0: m<i\leq n \text{ and } \mathrm{E}(p_i^s)-1\geq p_i\right\}$ and $t>0$ then the cardinality of the set M_1 is bigger than t and t is an integer.

Proof.

$t=\sum_{i=1}^{n}E\left(p_i^s\right)$ is, obviously, an integer as a sum of integers.

For every $i\leq m$ there is an inequality of $p_i^s\geq 0$, or $E\left(p_i^s\right)-p_i^s\leq 0$, and for every $i>m$ there is an inequality $p_i^s\leq 0$, or $0\leq E\left(p_i^s\right)-p_i^s<1$. Therefore:

$$t=\sum_{i=1}^{n}E\left(p_i^s\right)=\sum_{i=1}^{n}E\left(p_i^s\right)-\sum_{i=1}^{n}p_i^s=$$

$$=\sum_{i=1}^{m}\left(E\left(p_i^s\right)-p_i^s\right)+\sum_{i=m+1}^{n}\left(E\left(p_i^s\right)-p_i^s\right)<\sum_{i=m+1}^{n}\left(E\left(p_i^s\right)-p_i^s\right)\leq\#\overline{M}_1$$

where: $\overline{M}_1=\left\{i\in N^0: m<i\leq n \text{ and } E\left(p_i^s\right)\neq p_i^s\right\}$.

It remains to be presented, that $\overline{M}_1 \subset M_1$.

Let $i \in \overline{M}_1$, or $E\left(p_i^s\right) \neq p_i^s,\ p_i^s \geq p_i^0$, therefore:

$p_i^0 \leq p_i^s < E\left(p_i^s\right)$, or $p_i^0 < E\left(p_i^s\right)$, but $p_i^0,\ E\left(p_i^s\right)$ belong to the set of integers, so there is inequality $p_i^o \leq E\left(p_i^s\right) - 1$, therefore $i \in M_1$.

If $\overline{M}_1 \subset M_1$, that $\#\overline{\mathrm{M}}_1 \leq \#\mathrm{M}_1$ but $\#\overline{\mathrm{M}}_1 > t$, hence $\#\mathrm{M}_1 > t$ ($\#\mathrm{M}_1$ - is the cardinality of the set M_1), which ends the proof of the above statement.

Statement 1 shows that if $t > 0$, then deducting t coordinates from the M_1 set of vector $\left[E\left(p_1^s\right),\ldots,E\left(p_n^s\right)\right]$ by one, then a vector, which balances the model, will be obtained. It creates the question, which of t coordinates from the M_1 set, should deducted by one, in other words, to which of the consumers an additional unit of good should be assigned. The following solution is being proposed: increase the demand by one unit for those consumers who are the closest to its limits. This distance is determined by number of $x_i = \left(p_i + u_i - E\left(p_i^s\right) + 1\right)u_i^{-1}$.

The $100x_i$ number expresses the percentage of concession remaining i^{th} consumer, if their demand will verify to the amount equal to $\mathrm{E}\left(p_i^s\right) - 1$. In order to indicate a concrete consumer, to which an additional unit of good in M_1 set will be assigned, an ordering relations is introduced.

Statement 2.

If for every $i, j \in \mathrm{M}_1,\ i\ \mathrm{R}\, j \Leftrightarrow \begin{cases} x_i < x_j, & x_i \neq x_j \\ i \geq j, & x_i = x_j \end{cases}$ then R relation is ordering relations in M_1 set.

Proof

It should be presented that relation R is:

a) reversible, ie. for every $i \in \mathrm{M}_1$ and R_i,

b) weakly symmetric, ie. for every $i, j \in \mathrm{M}_1\ \ i\ \mathrm{R}\, j$ and $j\ \mathrm{R}\ i \Leftrightarrow i = j$,

c) transitive, ie. for every $i, j, k \in \mathrm{M}_1\ \ i\ \mathrm{R}\, j$ and $j\ \mathrm{R}\ k \Leftrightarrow i\ \mathrm{R}\ k$,

d) consistent, ie. for every $i, j \in \mathrm{M}_1\ \ i\ \mathrm{R}\, j$ or $j\ \mathrm{R}\ i$.

Ad a) $i\ \mathrm{R}\ i$, because $i \geq i$ and $x_i = x_i$.

Ad b) $i\ \mathrm{R}\, j$, ie. $x_i < x_j$ or $\left(x_i = x_j \text{ and } i \geq j\right)$,

$j\ \mathrm{R}\ i$, ie. $x_j < x_i$ or $\left(x_j = x_i \text{ and } j \geq i\right)$.

If $i\ \mathrm{R}\, j$ and $j\ \mathrm{R}\ i$, then $x_i = x_j,\ i \geq j,\ j \geq i$, therefore $x_i = x_j$ and $i = j$.

Ad c) $i\ \mathrm{R}\, j$, therefore $x_i < x_j$ or $\left(x_i = x_j \text{ and } i \geq j\right)$,

$j\ \mathrm{R}\ k$, therefore $x_j < x_k$ or $\left(x_j = x_k \text{ and } j \geq k\right)$.

If $x_i \neq x_j \neq x_k$, then $x_i < x_j < x_k$ or $x_i < x_k$, therefore i R k.
If $x_i = x_j \neq x_k$, then $x_i = x_j < x_k$ or $x_i < x_k$, therefore i R k.
If $x_i \neq x_j = x_k$, then $x_i < x_j = x_k$ or $x_i < x_k$, therefore i R k.
If $x_i = x_j = x_k$, then $i \geq j$ and or $j \geq k$ or $i \geq k$.
Therefore i R k.
Ad. d) Proof by contradiction.
If i R j does not occur, then $x_j > x_i$ or $\left(x_j = x_i \text{ and } i > j\right)$.
If $x_i > x_j$, a contradiction that $x_j > x_i$ or $x_i = x_j$.
If $x_j > x_i$, a contradiction that $x_i > x_j$ or $x_j = x_i$.
If $x_i = x_j$, a contradiction that $i = j$.
Must therefore take place one of the relations: i R j or j R i, which ends the proof of the above statement.
Ordered M_1 set, according R relations, determines by $\mathrm{M}_{1\mathrm{R}}$. Increasing demand to the first t^{th} consumers from the $\mathrm{M}_{1\mathrm{R}}$ set, by one unit, leads to sustainable model. In addition, supply and demand of all contractors will be expressed by integers. It will be presented in the statement below.

Statement 3.

If $t > 0$, $M_1^+ = \{i \in \mathrm{M}_{1\mathrm{R}} : i \leq s\}$,

$$d_i = \begin{cases} E(p_i^s) - 1 & \text{for} \quad i \in M_1^+ \\ E(p_i^s) & \text{for} \quad i \notin M_1^+ \end{cases}, \quad \vec{\mathbf{d}} = \left(d_1, \ldots, d_n\right)$$

then: for $i \leq n$ d_i is an integer and $d_1 + d_2 + \ldots + d_n = 0$.

Proof:

For $i \in M_1^+$ d_i is an integer as a difference of integers. For $i \notin M_1^+$ d_i is also an integer.

$$\sum_{i=1}^{n} d_i = \sum_{i \in M_1^+} d_i + \sum_{i \notin M_1^+} d_i = \sum_{i \in M_1^+} \left(\mathrm{E}(p_i^s) - 1\right) + \sum_{i \notin M_1^+} \mathrm{E}(p_i^s) = \sum_{i=1}^{n} \mathrm{E}(p_i^s) + \sum_{i \in M_1^+} (-1) = t - t = 0$$

Conclusions

Presented, in the above statement, the method of discrete vector correction is not just one, obviously, verification method of contractors' supply and demand. Having in mind various economic factors, the user can conduct a correction among any relation ordering the set of contractors, and among any assignment. Different measures of "distance" between vectors can also be used for this purpose.

References

[1] Ładyga M., Tkacz M., The unsustainable production and consumption model, Polish Journal of Management Studies, vol. 4, Czestochowa University of Technology, 2011.

[2] Ładyga M., Tkacz M., Balancing method of unsustainable production and consumption model, Scientific Research of the Institute of Mathematics and Computer Science 2011, 2(10), 135-145.

[3] Ładyga M., Tkacz M., The properties of method balancing the unsustainable production and consumption model, Scientific Research of the Institute of Mathematics and Computer Science 2012, 3(11), 105-109.

[4] Ładyga M., Tkacz M., The explicitness of vector balancing the unsustainable production and consumption model, Polish Journal of Management Studies, vol. 5, Czestochowa University of Technology, 2012, 261-265.