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POINTWISE COMPLETENESS AND POINTWISE DEGENERACY OF POSITIVE FRACTIONAL DESCRIPTOR CONTINUOUS–TIME LINEAR SYSTEMS WITH REGULAR PENCILS

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Pointwise completeness and pointwise degeneracy of positive fractional descriptor continuous-time linear systems with regular pencils are addressed. Conditions for pointwise completeness and pointwise degeneracy of the systems are established and illustrated by an example.

Keywords: pointwise completeness, pointwise degeneracy, descriptor system, fractional system, positive system.

1. Introduction

Descriptor (singular) linear systems have been considered in many papers and books (Bru *et al.*, 2003; 2002; Choundhury, 1972; Campbell *et al.*, 1976; Dai, 1989; Guang-Ren Duan, 2010; Kaczorek, 2013; 2011a; 2014b; 2004; 1992; 2014c; 2011c; 2011d; 2011e; Virnik, 2008). The eigenvalue and invariant assignment by state and output feedbacks was investigated by Kaczorek (2004; 2011d), along with minimum energy control of descriptor linear systems (Kaczorek, 2014b). In positive systems inputs, state variables and outputs take only nonnegative values (Farina and Rinaldi, 2000; Kaczorek, 2002). Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive fractional linear systems and selected problems in the theory of fractional systems were addressed by Kaczorek (2011d).

Descriptor standard positive linear systems by the use of the Drazin inverse were discussed by Bru *et al.* (2003; 2002), Campbell *et al.* (1976) and Kaczorek (2013; 2011a; 2014a; 2011c; 2011e), who also applied the shuffle algorithm to check the positivity of descriptor linear systems (Kaczorek, 2011a). The stability of positive descriptor systems was investigated by Virnik (2008). Reduction and decomposition of descriptor fractional discrete-time linear systems were considered by Kaczorek (2011c), who also introduced a new class of descriptor fractional linear discrete-time systems (Kaczorek, 2011e). Pointwise completeness and pointwise degeneracy for standard and fractional linear systems were investigated by Busłowicz (2008), Busłowicz *et al.* (2006), Choundhury (1972), Kaczorek and Busłowicz (2009), Kaczorek (2011b; 2009; 2010), Olbrot (1972), Popov (1972), Trzasko *et al.* (2007) and Weiss (1970). The Drazin inverse of matrices was applied to find solutions of the state equations of fractional descriptor continuous-time linear systems with regular pencils by Kaczorek (2014a).

In this paper, pointwise completeness and pointwise degeneracy of positive fractional descriptor continuous-time linear systems with regular pencils will be addressed.

The paper is organized as follows. In Section 2 some definitions, lemmas and theorems concerning the positive fractional descriptor continuous-time linear systems are recalled. The main result of the paper is presented in Section 3, where the conditions for the pointwise completeness and pointwise degeneracy of fractional descriptor continuous-time linear systems with regular pencils are established and illustrated with an example. Concluding remarks are given in Section 4.

The following notation will be used: $\mathbb{R}$, the set of real numbers; $\mathbb{R}^{n\times m}$, the set of $n\times m$ real matrices and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$; $\mathbb{R}_+^{n\times m}$, the set of $n\times m$ matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$; M_n, the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries); I_n, the $n\times n$ identity matrix, ker A (im A), the kernel (image) of the matrix A.

2. Preliminaries

Consider the autonomous fractional descriptor continuous-time linear system

$$E\,{}_0D_t^\alpha x(t) = Ax(t), \quad 0<\alpha<1, \tag{1}$$

where α is the fractional order, $x(t)\in\mathbb{R}^n$ is the state vector, $E, A\in\mathbb{R}^{n\times n}$, and

$${}_0D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-\tau)^\alpha}\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau \tag{2}$$

is the Caputo definition of the α-th ($\alpha\in\mathbb{R}$) order derivative of $x(t)$, and

$$\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}\,\mathrm{d}t \tag{3}$$

is the Euler gamma function.

It is assumed that $\det E = 0$, but the pencil (E, A) of (1) is regular, i.e.,

$$\det[Es-A]\neq 0 \quad \text{for some } s\in\mathbb{C}. \tag{4}$$

Assuming that, for some chosen $c\in\mathbb{C}$, $\det[Ec-A]\neq 0$ and premultiplying (1) by $[Ec-A]^{-1}$, we obtain

$$\bar{E}\,{}_0D_t^\alpha x(t) = \bar{A}x(t), \tag{5a}$$

where

$$\bar{E} = [Ec-A]^{-1}E, \quad \bar{A} = [Ec-A]^{-1}A. \tag{5b}$$

Note that Eqns. (1) and (5a) have the same solution $x(t)$.

Definition 1. (*Campbell et al., 1976; Kaczorek, 1992*) The smallest nonnegative integer q is called the *index* of the matrix $\bar{E}\in\mathbb{R}^{n\times n}$ if

$$\operatorname{rank}\bar{E}^q = \operatorname{rank}\bar{E}^{q+1}. \tag{6}$$

Definition 2. (*Campbell et al., 1976; Kaczorek, 1992*) A matrix $\bar{E}^D$ is called the *Drazin inverse* of $\bar{E}\in\mathbb{R}^{n\times n}$ if it satisfies the conditions

$$\bar{E}\bar{E}^D = \bar{E}^D\bar{E}, \tag{7a}$$

$$\bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D, \tag{7b}$$

$$\bar{E}^D\bar{E}^{q+1} = \bar{E}^q, \tag{7c}$$

where q is the index of $\bar{E}$ defined by (6).

The Drazin inverse $\bar{E}^D$ of a square matrix $\bar{E}$ always exists and is unique (Campbell *et al.*, 1976; Kaczorek, 1992). If $\det\bar{E}\neq 0$, then $\bar{E}^D = \bar{E}^{-1}$. Some methods for computation of the Drazin inverse are given by Kaczorek (1992).

Lemma 1. (Campbell *et al.*, 1976; Kaczorek, 1992; 2014a) *The matrices $\bar{E}$ and $\bar{A}$ defined by (5b) satisfy the following equalities:*

$$\bar{A}\bar{E} = \bar{E}\bar{A}, \quad \bar{A}^D\bar{E} = \bar{E}\bar{A}^D, \tag{8a}$$

$$\bar{E}^D\bar{A} = \bar{A}\bar{E}^D, \quad \bar{A}^D\bar{E}^D = \bar{E}^D\bar{A}^D, \tag{8b}$$

$$\ker\bar{A}\cap\ker\bar{E} = \{0\}, \tag{8c}$$

$$\bar{E} = T\begin{bmatrix} J & 0\\ 0 & N\end{bmatrix}T^{-1}, \tag{8d}$$

$$\bar{E}^D = T\begin{bmatrix} J^{-1} & 0\\ 0 & 0\end{bmatrix}T^{-1}, \tag{8e}$$

$\det T\neq 0$, $J\in\mathbb{R}^{n_1\times n_1}$, *is nonsingular*, $N\in\mathbb{R}^{n_2\times n_2}$ *is nilpotent*, $n_1+n_2=n$,

$$(I_n-\bar{E}\bar{E}^D)\bar{A}\bar{A}^D = I_n-\bar{E}\bar{E}^D, \tag{8f}$$

$$(I_n-\bar{E}\bar{E}^D)(\bar{E}\bar{A}^D)^q = 0. \tag{8g}$$

Theorem 1. (Kaczorek, 2014a) *The solution of Eqn. (1) is given by*

$$x(t) = \Phi_0(t)\bar{E}\bar{E}^D w, \tag{9a}$$

where

$$\Phi_0(t) = \sum_{k=0}^\infty \frac{(\bar{E}^D\bar{A})^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{9b}$$

and the vector $w\in\mathbb{R}^n$ is arbitrary.

From (9a) we have $x(0)=x_0=\bar{E}\bar{E}^D w$ and $x_0\in\operatorname{im}(\bar{E}\bar{E}^D)$, where 'im' denotes the image of $\bar{E}\bar{E}^D$.

Lemma 2. (Kaczorek, 2014a) *The matrix $\Phi_0(t)$ defined by (9b) is nonsingular for any matrix $A\in\mathbb{R}^{n\times n}$ and time $t\geq 0$.*

Theorem 2. (Kaczorek, 2014a) *Let*

$$P=\bar{E}\bar{E}^D \quad \text{and} \quad Q=\bar{E}^D\bar{A}. \tag{10}$$

Then we have

$$(i) \qquad P^k = P \quad \text{for} \quad k=2,3,\dots, \tag{11a}$$

$$(ii) \qquad PQ = QP = Q, \tag{11b}$$

$$(iii) \qquad P\Phi_0(t) = \Phi_0(t). \tag{11c}$$

Definition 3. (*Farina and Rinaldi, 2000; Kaczorek, 2011d*) The fractional descriptor system (1) is called (internally) *positive* if the state vector $x(t)\in\mathbb{R}_+^n$, $t\geq 0$, for all initial conditions $x_0\in\mathbb{R}_+^n$.

Theorem 3. (Farina and Rinaldi, 2000; Kaczorek, 2011d) *The fractional descriptor system (1) is (internally) positive if and only if*

$$\bar{E}^D\bar{A}\in M_n. \tag{12}$$

3. Pointwise completeness and pointwise degeneracy

Definition 4. The positive fractional descriptor system (1) is called *pointwise complete* for $t = t_f$ if for every final state $x_f \in \mathbb{R}^n_+$ there exists a vector of initial conditions $x_0 \in \mathrm{im}(\bar{E}\bar{E}^D) \in \mathbb{R}^n_+$ such that $x(t_f) = x_f \in \mathbb{R}^n_+$.

A matrix $A \in \mathbb{R}^{n\times n}$ is called *monomial* if in each row and in each column it has only one positive entry and the remaining entries are zero.

Theorem 4. *The positive fractional descriptor system (1) is pointwise complete for any $t = t_f$ and every final state $x_f \in \mathbb{R}^n$ belonging to the set*

$$x_f = \mathrm{im}[\Phi_0(t_f)x_0] \in \mathbb{R}^n_+ \tag{13}$$

if and only if $\Phi_0(t_f) \in \mathbb{R}^{n\times n}_+$ is a monomial matrix.

Proof. Substituting in (9a) $t = t_f$, we obtain

$$x_f = x(t_f) = \Phi_0(t_f)x_0 \tag{14}$$

and

$$x_0 = [\Phi_0(t_f)]^{-1}x_f, \tag{15}$$

since the matrix $\Phi_0(t)$ is monomial and

$$[\Phi_0(t_f)]^{-1} \in \mathbb{R}^{n\times n}_+.$$

■

Definition 5. The positive fractional descriptor system (1) is called *pointwise degenerated* in the direction *v* for $t = t_f$ if there exists a nonzero vector $v \in \mathbb{R}^n$ such that for all initial conditions $x_0 \in \mathrm{im}(\bar{E}\bar{E}^D) \in \mathbb{R}^n_+$ the solution of (1) satisfies the condition

$$v^T x_f = 0, \tag{16}$$

where T denotes the transpose.

Theorem 5. *The positive fractional descriptor system (1) is pointwise degenerated in the direction v defined by*

$$v^T\bar{E} = 0 \tag{17}$$

for any $t_f \geq 0$ and all initial conditions $x_0 \in \mathrm{im}(\bar{E}\bar{E}^D) \in \mathbb{R}^n_+$.

Proof. Postmultiplying (17) by $\bar{E}^D w$ and using $x_0 = \bar{E}\bar{E}^D w$ and (16), we obtain

$$v^T\bar{E}\bar{E}^D w = v^T x_0 = 0. \tag{18}$$

Taking into account (9b), (14) and (17), we obtain

$$\begin{aligned} v^T x_f &= v^T\Phi_0(t_f)x_0 \\ &= \sum_{k=0}^{\infty} \frac{v^T(\bar{E}^D\bar{A})^k t_f^{k\alpha}}{\Gamma(k\alpha+1)} x_0 \\ &= v^T x_0 + \sum_{k=1}^{\infty} \frac{v^T\bar{E}\bar{E}^D(\bar{E}^D\bar{A})^k t_f^{k\alpha}}{\Gamma(k\alpha+1)} x_0 = 0, \end{aligned} \tag{19}$$

since (7b) and (17) hold. ■

Example 1. Consider the positive fractional descriptor system (1) with the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}. \tag{20}$$

The pencil of (20) is regular since

$$\det[Es - A] = \begin{vmatrix} s+1 & 0 \\ 0 & 2 \end{vmatrix} = 2(s+1) \neq 0 \tag{21}$$

and the assumption (4) is met.

Choosing $c = 1$, we obtain

$$\begin{aligned} \bar{E} &= [Ec - A]^{-1}E \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \\ \bar{A} &= [Ec - A]^{-1}A \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{22}$$

and

$$\bar{E}^D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}^D = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}. \tag{23}$$

In this case, the admissible initial conditions are

$$x_0 = \mathrm{im}(\bar{E}\bar{E}^D) = \mathrm{im}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x_{10} \\ 0 \end{bmatrix}, \tag{24}$$

and x_{10} is arbitrary.

Using (9b) and (23), we obtain the nonsingular matrix

$$\begin{aligned} \Phi_0(t) &= \sum_{k=0}^{\infty} \frac{(\bar{E}^D\bar{A})^k t^{k\alpha}}{\Gamma(k\alpha+1)} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{k=1}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \begin{bmatrix} -0.5 & 0 \\ 0 & 0 \end{bmatrix}^k \\ &= \begin{bmatrix} 1+\varphi(t) & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned} \tag{25a}$$

where

$$\varphi(t) = \sum_{k=1}^{\infty} \frac{(-0.5)^k t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{25b}$$

By Theorem 4 the positive fractional descriptor system (1) with (20) is pointwise complete for $t = t_f$ and every $x_f \in \mathbb{R}^2_+$ satisfying the condition (13). In this case, from (15) and (25) we have

$$\begin{aligned} x_0 &= [\Phi_0(t_f)]^{-1}x_f \\ &= \begin{bmatrix} [1+\varphi(t_f)]^{-1}x_{1f} \\ 0 \end{bmatrix} \in \mathrm{im}(\bar{E}\bar{E}^D) \\ &= \begin{bmatrix} x_{10} \\ 0 \end{bmatrix} \end{aligned} \tag{26}$$

if and only if

$$x_f \in \mathrm{im}(\bar{E}\bar{E}^D) = \begin{bmatrix} x_{1f} \\ 0 \end{bmatrix} \in \mathbb{R}^2_+,$$

where $x_{1f} > 0$ is arbitrary.

By Theorem 5 the system (1) with (20) is pointwise degenerated in the direction $v^T = [\ 0 \quad v_2\ ]$ for any v_2 since

$$v^T\bar{E} = [\ 0 \quad v_2\ ] \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} = [\ 0 \quad 0\ ]. \tag{27}$$

♦

4. Concluding remarks

The pointwise completeness and pointwise degeneracy of positive fractional descriptor continuous-time linear systems with regular pencils have been addressed. The conditions for pointwise completeness and the pointwise degeneracy of the systems have been established (Theorems 4 and 5). The discussion has been illustrated by a numerical example of a fractional descriptor system with regular pencil. The results can be extended to positive fractional descriptor linear systems with delays.

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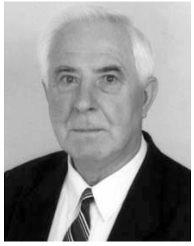

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