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EXPONENTIAL DECAY OF SOLUTIONS TO A CLASS OF FOURTH-ORDER NONLINEAR HYPERBOLIC EQUATIONS MODELING THE OSCILLATIONS OF SUSPENSION BRIDGES

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Abstract. This paper is concerned with a class of fourth-order nonlinear hyperbolic equations subject to free boundary conditions that can be used to describe the nonlinear dynamics of suspension bridges.

Keywords: fourth-order nonlinear hyperbolic equations, weak solutions, exponential decay, a family of potential wells.

Mathematics Subject Classification: 35L35, 35D30, 35B40.

1. INTRODUCTION

In this paper, we study the following fourth-order nonlinear hyperbolic equation

$$u_{tt} + \Delta^2 u + au + \mu u_t = |u|^{p-2}u, \ (x, y, t) \in \Omega \times (0, \infty) \tag{1.1}$$

with initial conditions

$$u(x, y, 0) = u_0(x, y), \ u_t(x, y, 0) = u_1(x, y), \ (x, y) \in \Omega \tag{1.2}$$

and free boundary conditions (see [18, Section 2.5])

$$\begin{cases} u(0, y, t) = u_{xx}(0, y, t) = u(\pi, y, t) = u_{xx}(\pi, y, t) = 0, \\ \quad (y, t) \in (-l, l) \times (0, \infty), \\ u_{yy}(x, \pm l, t) + \sigma u_{xx}(x, \pm l, t) = 0, \ (x, t) \in (0, \pi) \times (0, \infty), \\ u_{yyy}(x, \pm l, t) + (2 - \sigma) u_{xxy}(x, \pm l, t) = 0, \ (x, t) \in (0, \pi) \times (0, \infty), \end{cases} \tag{1.3}$$

where $\Omega = (0, \pi) \times (-l, l) \subset \mathbb{R}^2$, $\mu > 0$, $2 < p < \infty$, $\sigma \in (0, \frac{1}{2})$, and $a = a(x, y)$ is a sign-changing and bounded measurable function.

Problem (1.1)–(1.3) can be used to describe the nonlinear dynamics of suspension bridges (see [4, 19]). The open rectangular plate $\Omega = (0, \pi) \times (-l, l)$ represents the roadway of a suspension bridge, and the edges $x = 0, \pi$ connect with the ground while the edges $y = \pm l$ are free. The unknown function u represents the vertical displacement of the plate Ω.

A one-dimensional simply supported beam suspended by the hangers was suggested as a model for suspension bridges in [7, 11, 12]. But if one models a suspension bridge by a beam, there is no way to highlight the torsional oscillations. Moreover, a reliable model for suspension bridges should be nonlinear and it should have enough degrees of freedom to display torsional oscillations. There have been some studies on the nonlinear behavior of suspension bridges (see e.g. [3, 5, 6, 15]). Ferrero and Gazzola [4] suggested

$$u_{tt} + \Delta^2 u + h(x, y, u) + \mu u_t = f, \tag{1.4}$$

subject to (1.2) and (1.3), where $h(x, y, u)$ is the restoring force due to the hangers of suspension bridges, and f is the external force. They investigated the existence, uniqueness and asymptotic behavior of weak solutions to the problem (1.4), (1.2), (1.3). Their main results showed that if $f \in L^2(\Omega)$ is independent of t, then the unique global solution to the problem (1.4), (1.2), (1.3) converges to the stationary solution as time tends to infinity. Subsequently, Wang [19] studied the local existence, global existence, uniqueness, polynomial decay and finite time blow-up of weak solutions to the problem (1.1)–(1.3) with $E(0) < d$, so-called the initial energy is less than the potential well depth. Recently, Xu *et al.* [22] considered the following fourth-order hyperbolic equation with nonlinear damping

$$u_{tt} + \Delta^2 u + au + \mu |u_t|^{q-2} u_t = |u|^{p-2} u, \tag{1.5}$$

where $2 < q < p < \infty$. They obtained the local existence, global existence, uniqueness, polynomial decay and finite time blow-up of weak solutions to the problem (1.5), (1.2), (1.3) with the subcritical initial energy $E(0) < d$ and the critical initial energy $E(0) = d$. Moreover, in the case $E(0) > 0$, they derived the finite time blow-up of weak solutions to the problem (1.1)–(1.3). Mohammed et al. [13] considered the existence, asymptotic boundary estimates and uniqueness of large solutions to fully nonlinear equations in bounded domains. Baraket and Radulescu [2] studied two classes of nonhomogeneous elliptic problems with Dirichlet boundary condition and involving a fourth-order differential operator with variable exponent and power-type nonlinearities and established the existence of a nontrivial weak solution in the case of a small perturbation of the right-hand side.

In the present paper, we focus on the decay rates of weak solutions to the problem (1.1)–(1.3). Our main results show that certain norms of solutions can decay exponentially to zero as time tends to infinity, which complements the existing results on the asymptotic behavior of solutions to the problem (1.1)–(1.3). Our main technical tools are a family of potential wells (see [8, 10, 20, 21]), which include the classical potential well as a special case. One advantage of introducing a family of potential wells is that they can provide more accurate estimates for the Nehari functional, which the classical potential well can not do.

This paper is organized as follows. In Section 2, we display some notations, definitions and lemmas related to the problem (1.1)–(1.3). Moreover, we present our main results on the problem (1.1)–(1.3). In Section 3, we establish the global existence and uniqueness of solutions with the subcritical initial energy $E(0) < d$ and the critical initial energy $E(0) = d$. Although the global existence and uniqueness of solutions to the problem (1.1)–(1.3) with $E(0) < d$ has been proved in [19], our proof is different. In Section 4, we prove the exponential decay of solutions with $E(0) < d$ and $E(0) = d$.

2. PRELIMINARIES AND MAIN RESULTS

Throughout this paper, for the sake of simplicity, we denote

$$\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}, \quad \|\cdot\| := \|\cdot\|_2, \quad (u,v) := \int\limits_\Omega uv\,\mathrm{d}x\mathrm{d}y$$

by [16] and according to [4],

$$H_*^2(\Omega) = \{u \in H^2(\Omega) | u = 0 \text{ on } \{0,\pi\} \times (-l,l)\}$$

is a Hilbert space with the inner product

$$(u,v)_* := (u,v)_{H_*^2(\Omega)} = \int\limits_\Omega (\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}))\mathrm{d}x\mathrm{d}y$$

and the norm

$$\|u\|_* := \|u\|_{H_*^2(\Omega)} = \left(\int\limits_\Omega |\Delta u|^2 \mathrm{d}x\mathrm{d}y + 2(1-\sigma)\int\limits_\Omega (u_{xy}^2 - u_{xx}u_{yy})\mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}},$$

which is equivalent to $\|\cdot\|_{H^2(\Omega)}$ for $\sigma \in \left(0, \frac{1}{2}\right)$. Here, in terms of [1, Theorem 4.15], $\|\cdot\|_{H^2(\Omega)}$ can be defined by $(\|\nabla^2 \cdot\|^2 + \|\cdot\|^2)^{\frac{1}{2}}$. Moreover, according to [19], there holds the following Sobolev embedding inequality.

Lemma 2.1 ([19]). *Assume that $1 \le q < \infty$. Then, for any $u \in H_*^2(\Omega)$, there holds*

$$\|u\|_q \le S_q \|u\|_*,$$

where

$$S_q = \left(\frac{\pi}{2l} + \frac{\sqrt{2}}{2}\right)(2\pi l)^{\frac{q+2}{2q}} \left(\frac{1}{1-\sigma}\right)^{\frac{1}{2}}.$$

Lemma 2.2 ([19]). *Assume that $-\Lambda_1 < a_1 \le a \le a_2$, where $\{\Lambda_i\}_{i=1}^{\infty}$ is the eigenvalue sequence to the eigenvalue problem*

$$\begin{cases} \Delta^2 u = \Lambda u, \ (x,y) \in \Omega, \\ u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0, \ y \in (-l,l), \\ u_{yy}(x,\pm l) + \sigma u_{xx}(x,\pm l) = u_{yyy}(x,\pm l) + (2-\sigma) u_{xxy}(x,\pm l) = 0, \ x \in (0,\pi), \end{cases}$$

and $\Lambda_1 < 1$. Then, for any $u \in H_*^2(\Omega)$, there holds

$$A_1 \|u\|_*^2 \le \|u\|_*^2 + (au,u) \le A_2 \|u\|_*^2,$$

where

$$A_1 = \begin{cases} 1 + \dfrac{a_1}{\Lambda_1}, & a_1 < 0, \\ 1, & a_1 \ge 0, \end{cases}$$

and

$$A_2 = \begin{cases} 1, & a_2 < 0, \\ 1 + \dfrac{a_2}{\Lambda_1}, & a_2 \ge 0. \end{cases}$$

Definition 2.3 (Weak solutions)**.** A function $u \in L^\infty(0,T; H_*^2(\Omega))$ with $u_t \in L^\infty(0,T; L^2(\Omega))$ is called a weak solution to the problem (1.1)–(1.3) in $\Omega \times [0,T)$, provided $u(0) = u_0$ in $H_*^2(\Omega)$, $u_t(0) = u_1$ in $L^2(\Omega)$, and

$$\begin{aligned} &(u_t(t),v) + \int_0^t (u(\tau),v)_* \,\mathrm{d}\tau + \int_0^t (au(\tau),v)\,\mathrm{d}\tau + \mu(u(t),v) \\ &= \int_0^t (|u(\tau)|^{p-2}u(\tau),v)\,\mathrm{d}\tau + (u_1,v) + \mu(u_0,v) \end{aligned} \tag{2.1}$$

for all $v \in H_*^2(\Omega)$ and $t \in (0,T)$.

Now we are in a position to define the total energy associated with the problem (1.1)–(1.3)

$$E(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|u(t)\|_*^2 + \frac{1}{2}(au(t),u(t)) - \frac{1}{p}\|u(t)\|_p^p,$$

the potential energy functional

$$J(u) = \frac{1}{2}\|u\|_*^2 + \frac{1}{2}(au,u) - \frac{1}{p}\|u\|_p^p$$

and the Nehari functional

$$I(u) = \|u\|_*^2 + (au, u) - \|u\|_p^p.$$

Thus, the Nehari manifold can be defined by

$$\mathcal{N} = \{u \in H_*^2(\Omega) \setminus \{0\} | I(u) = 0\}.$$

We introduce the potential well (see e.g. [9, 14, 17, 19, 22, 23])

$$\mathcal{W} = \{u \in H_*^2(\Omega) \,|\, I(u) > 0, J(u) < d\} \cup \{0\}$$

and its closure

$$\overline{\mathcal{W}} = \{u \in H_*^2(\Omega) \,|\, I(u) \geq 0, J(u) \leq d\},$$

where the depth of potential well

$$d = \inf_{u \in \mathcal{N}} J(u).$$

The main results of this paper are as follows.

Theorem 2.4 (Global existence)**.** *Let $-\wedge_1 < a_1 \leq a \leq a_2$, $u_0 \in H_*^2(\Omega)$ and $u_1 \in L^2(\Omega)$.*

(i) *Assume that $E(0) < d$, and $I(u_0) > 0$ or $\|u_0\|_* = 0$. Then the problem* (1.1)–(1.3) *admits a unique global solution $u(t) \in \mathcal{W}$ for all $t \in [0, \infty)$. Moreover,*

$$E(t) + \mu \int_0^t \|u_\tau(\tau)\|^2 \,\mathrm{d}\tau \leq E(0). \tag{2.2}$$

(ii) *Assume that $E(0) = d$ and $I(u_0) \geq 0$. Then the problem* (1.1)–(1.3) *admits a unique solution $u(t) \in \overline{\mathcal{W}}$ for all $t \in [0, \infty)$ that satisfies* (2.2)*.*

Theorem 2.5 (Exponential decay)**.** *Let $-\wedge_1 < a_1 \leq a \leq a_2$, $u_0 \in H_*^2(\Omega)$ and $u_1 \in L^2(\Omega)$.*

(i) *Assume that $0 < E(0) < d$, and $I(u_0) > 0$ or $\|u_0\|_* = 0$. Then there exist constants $\alpha, \beta > 0$ such that*

$$\|u_t(t)\|^2 + \|u(t)\|_*^2 \leq \alpha e^{-\beta t} \tag{2.3}$$

for all $t \in [0, \infty)$.

(ii) *Assume that $E(0) = d$ and $I(u_0) \geq 0$. Then there exists a constant $\tilde{t} > 0$ such that* (2.3) *remains valid for the solution $u(t) \in \mathcal{W}$ for all $t \in [\tilde{t}, \infty)$.*

3. PROOF OF THEOREM 2.4

Proof of Theorem 2.4. (i) Let $\{w_j\}_{j=1}^{\infty}$ be the eigenfunctions of the eigenvalue problem in Lemma 2.2. Then, according to [4, Theorem 3.4], $\{w_j\}_{j=1}^{\infty}$ is an orthogonal basis of $H_*^2(\Omega)$ and an orthonormal basis of $L^2(\Omega)$. We construct the approximate solutions to the problem (1.1)–(1.3)

$$u_n(t) = \sum_{j=1}^{n} \xi_{jn}(t) w_j, \quad n = 1, 2, \ldots,$$

satisfying

$$\begin{aligned}&(u_{ntt}(t), w_j) + (u_n(t), w_j)_* + (a u_n(t), w_j) + \mu(u_{nt}(t), w_j)\\ &= (|u_n(t)|^{p-2} u_n(t), w_j), \quad j = 1, 2, \ldots, n,\end{aligned} \tag{3.1}$$

with

$$\begin{cases} u_n(0) = \displaystyle\sum_{j=1}^{n} \xi_{jn}(0) w_j \to u_0 \ \text{ in } H_*^2(\Omega), \\ u_{nt}(0) = \displaystyle\sum_{j=1}^{n} \xi'_{jn}(0) w_j \to u_1 \ \text{ in } L^2(\Omega). \end{cases} \tag{3.2}$$

Multiplying (3.1) by $\xi'_{jn}(t)$ and summing for j, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} E_n(t) + \mu \|u_{nt}(t)\|^2 = 0, \tag{3.3}$$

where

$$E_n(t) = \frac{1}{2}\|u_{nt}(t)\|^2 + \frac{1}{2}\|u_n(t)\|_*^2 + \frac{1}{2}(a u_n(t), u_n(t)) - \frac{1}{p}\|u_n(t)\|_p^p. \tag{3.4}$$

Therefore, by integrating (3.3) with respect to τ from 0 to t, we obtain

$$E_n(t) + \mu \int_0^t \|u_{n\tau}(\tau)\|^2 \,\mathrm{d}\tau = E_n(0), \quad \forall t \in [0, \infty). \tag{3.5}$$

We now claim that

$$u_n(t) \in \mathcal{W} \tag{3.6}$$

for all $t \in [0, \infty)$ and sufficiently large n.

Indeed, if $\|u_0\|_* = 0$, then $u_0 \in \mathcal{W}$. If $I(u_0) > 0$, then, from $E(0) < d$, i.e.,

$$\frac{1}{2}\|u_1\|^2 + J(u_0) < d,$$

it follows that $J(u_0) < d$. Hence, $u_0 \in \mathcal{W}$. Thus, $u_n(0) \in \mathcal{W}$ for sufficiently large n due to (3.2). As a result, assertion (3.6) follows as desired. Arguing by contradiction, we suppose that there would exist a $t_0 > 0$ such that $u_n(t_0) \in \partial\mathcal{W}$, i.e., $I(u_n(t_0)) = 0$ and $\|u_n(t_0)\|_* \neq 0$, or $J(u_n(t_0)) = d$. In terms of (3.4), (3.5) and (3.2), we get

$$\frac{1}{2}\|u_{nt}(t)\|^2 + J(u_n(t)) < d, \tag{3.7}$$

for all $t \in [0, \infty)$ and sufficiently large n. This tells us that $J(u_n(t_0)) = d$ is impossible. On the other hand, if $I(u_n(t_0)) = 0$ and $\|u_n(t_0)\|_* \neq 0$, then we get $J(u_n(t_0)) \geq d$ by the definition of d, which contradicts (3.7).

From (3.4)–(3.6), (3.2), Lemma 2.2 and

$$J(u_n(t)) = \frac{p-2}{2p}\big(\|u_n(t)\|_*^2 + (au_n(t), u_n(t))\big) + \frac{1}{p}I(u_n(t)),$$

we deduce that

$$\frac{1}{2}\|u_{nt}(t)\|^2 + \frac{p-2}{2p}A_1\|u_n(t)\|_*^2 + \mu\int_0^t \|u_{n\tau}(\tau)\|^2\,\mathrm{d}\tau < d$$

for all $t \in [0, \infty)$ and sufficiently large n. Moreover,

$$\||u_n(t)|^{p-2}u_n(t)\|_r^r \leq S_p^p\|u_n(t)\|_*^p < S_p^p\left(\frac{2pd}{(p-2)A_1}\right)^{\frac{p}{2}},$$

where $r = \frac{p}{p-1}$. Therefore, there exist u and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ and we shall not repeat, such that as $n \to \infty$,

$$u_n \rightharpoonup u \text{ weakly star in } L^\infty(0,\infty; H_*^2(\Omega)), \tag{3.8}$$

$$u_n \to u \text{ a.e. in } \Omega\times[0,\infty) \text{ and strongly in } L^p(\Omega) \text{ for each } t > 0, \tag{3.9}$$

$$u_{nt} \rightharpoonup u_t \text{ weakly star in } L^\infty(0,\infty; L^2(\Omega)) \text{ and weakly in } L^2(0,\infty; L^2(\Omega)), \tag{3.10}$$

$$|u_n|^{p-2}u_n \rightharpoonup |u|^{p-2}u \text{ weakly star in } L^\infty(0,\infty; L^r(\Omega)).$$

Integrating (3.1) with respect to t, we get

$$\begin{aligned}(u_{nt}(t), w_j) &+ \int_0^t (u_n(\tau), w_j)_*\,\mathrm{d}\tau + \int_0^t (au_n(\tau), w_j)\,\mathrm{d}\tau + \mu(u_n(t), w_j) \\ &= \int_0^t (|u_n(\tau)|^{p-2}u_n(\tau), w_j)\,\mathrm{d}\tau + (u_{nt}(0), w_j) + \mu(u_n(0), w_j).\end{aligned}$$

For fixed j, taking $n \to \infty$, we get

$$\begin{aligned}(u_t(t), w_j) &+ \int_0^t (u(\tau), w_j)_*\,\mathrm{d}\tau + \int_0^t (au(\tau), w_j)\,\mathrm{d}\tau + \mu(u(t), w_j) \\ &= \int_0^t (|u(\tau)|^{p-2}u(\tau), w_j)\,\mathrm{d}\tau + (u_1, w_j) + \mu(u_0, w_j).\end{aligned}$$

By virtue of (3.2), we have $u(0) = u_0$ in $H^2_*(\Omega)$ and $u_t(0) = u_1$ in $L^2(\Omega)$. Therefore, u is a global solution to the problem (1.1)–(1.3) in the sense of Definition 2.3.

In addition, from (3.8)–(3.10), (3.5) and (3.2), we deduce that

$$\begin{aligned}
&\frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|u(t)\|_*^2 + \frac{1}{2}(au(t), u(t)) + \mu\int_0^t \|u_\tau(\tau)\|^2\,\mathrm{d}\tau \\
&\le \liminf_{n\to\infty}\left(\frac{1}{2}\|u_{nt}(t)\|^2 + \frac{1}{2}\|u_n(t)\|_*^2 + \frac{1}{2}(au_n(t), u_n(t)) + \mu\int_0^t \|u_{n\tau}(\tau)\|^2\,\mathrm{d}\tau\right) \\
&= \liminf_{n\to\infty}\left(E_n(0) + \frac{1}{p}\|u_n(t)\|_p^p\right) \\
&= E(0) + \frac{1}{p}\|u(t)\|_p^p
\end{aligned}$$

for all $t \in [0, \infty)$. Thus, there holds (2.2). By (2.2) and the similar arguments to the proof of assertion (3.6), we have $u(t) \in \mathcal{W}$ for all $t \in [0, \infty)$.

Next we prove the uniqueness of solutions. Suppose that u and $\bar{u}$ are two solutions to the problem (1.1)–(1.3). Set

$$\tilde{u} = \bar{u} - u,$$

$$\hat{u}(t) = \begin{cases} -\int_t^s \tilde{u}(\tau)\,\mathrm{d}\tau, & t \le s, \\ 0, & t > s, \end{cases} \qquad s \in (0, T]$$

and

$$\check{u}(t) = \int_0^t \tilde{u}(\tau)\,\mathrm{d}\tau.$$

Then

$$\begin{aligned}
&-\int_0^s (\tilde{u}_t(t), \hat{u}_t(t))\,\mathrm{d}t + \int_0^s (\tilde{u}(t), \hat{u}(t))_*\,\mathrm{d}t + \int_0^s (a\tilde{u}(t), \hat{u}(t))\,\mathrm{d}t + \mu\int_0^s (\tilde{u}_t(t), \hat{u}(t))\,\mathrm{d}t \\
&= \int_0^s (|\bar{u}(t)|^{p-2}\bar{u}(t) - |u(t)|^{p-2}u(t), \hat{u}(t))\,\mathrm{d}t.
\end{aligned}$$

Taking into account $\tilde{u}(t) = \hat{u}_t(t)$ and $\hat{u}(0) = -\check{u}(s)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|\tilde{u}(s)\|^2 + \frac{1}{2}\|\check{u}(s)\|_*^2 + \frac{1}{2}(a\check{u}(s), \check{u}(s)) + \mu\int_0^s \|\tilde{u}(t)\|^2\,\mathrm{d}t \\
&= \int_0^s (|u(t)|^{p-2}u(t) - |\bar{u}(t)|^{p-2}\bar{u}(t), \hat{u}(t))\,\mathrm{d}t.
\end{aligned}$$

Hence, by mean value inequality, Hölder's inequality, Minkowski's inequality, Lemmas 2.1 and 2.2, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\|\tilde{u}(s)\|^2 + \frac{A_1}{2}\|\check{u}(s)\|_*^2 \\
&\leq C_1 \int_0^s \int_\Omega (|u(t)|^{p-2} + |\bar{u}(t)|^{p-2})|\tilde{u}(t)||\hat{u}(t)|\,\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&\leq C_1 \int_0^s (\|u(t)\|_{2p-2}^{p-2} + \|\bar{u}(t)\|_{2p-2}^{p-2})\|\tilde{u}(t)\|\|\hat{u}(t)\|_{2p-2}\,\mathrm{d}t \\
&\leq C_2 \int_0^s \|\tilde{u}(t)\|\|\hat{u}(t)\|_*\,\mathrm{d}t.
\end{aligned}$$

Since $\hat{u}(t) = \check{u}(t) - \check{u}(s)$, it follows from Cauchy's inequality with $\epsilon = \frac{A_1}{4}$ that

$$\begin{aligned}
&\frac{1}{2}\|\tilde{u}(s)\|^2 + \frac{A_1}{2}\|\check{u}(s)\|_*^2 \\
&\leq C_2 \int_0^s \|\tilde{u}(t)\|(\|\check{u}(t)\|_* + \|\check{u}(s)\|_*)\,\mathrm{d}t \\
&\leq \frac{A_1}{4}\|\check{u}(s)\|_*^2 + C_3 \int_0^s (\|\tilde{u}(t)\|^2 + \|\check{u}(t)\|_*^2)\,\mathrm{d}t
\end{aligned}$$

for some $C_3 > 0$. Thus there exists a $C_4 > 0$ such that

$$\|\tilde{u}(s)\|^2 + \|\check{u}(s)\|_*^2 \leq C_4 \int_0^s (\|\tilde{u}(t)\|^2 + \|\check{u}(t)\|_*^2)\,\mathrm{d}t,$$

which, together with Gronwall's inequality, gives $\tilde{u} = 0$, i.e., $u = \bar{u}$.

(ii) We divide the proof of (ii) into two cases.

Case 1. $\|u_0\|_* \neq 0$. Let $\lambda_m = 1 - \frac{1}{m}$, $u_{m0} = \lambda_m u_0$, $m = 2, 3, \ldots$ We consider the problem (1.1), (1.3) with the following initial conditions

$$u(x, y, 0) = u_{m0}(x, y), \;\; u_t(x, y, 0) = u_1(x, y). \tag{3.11}$$

From $I(u_0) \geq 0$,

$$J(\lambda u) = \frac{1}{2}\lambda^2\|u\|_*^2 + \frac{1}{2}\lambda^2(au, u) - \frac{1}{p}\lambda^p\|u\|_p^p$$

and

$$I(\lambda u) = \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u),$$

it is simpleness to verify that there exists a unique $\lambda_0 = \lambda_0(u_0) \geq 1$ such that $J(\lambda u)$ is strictly increasing for $\lambda \in [0, \lambda_0]$ and takes the maximum at $\lambda = \lambda_0$. Hence $J(u_{m0}) < J(u_0)$ and $I(u_{m0}) > 0$. Moreover,

$$J(u_{m0}) = \frac{p-2}{2p}\left(\|u_{m0}\|_*^2 + (au_{m0}, u_{m0})\right) + \frac{1}{p} I(u_{m0}) > 0.$$

We further obtain

$$E_m(0) = \frac{1}{2}\|u_1\|^2 + J(u_{m0}) > 0$$

and

$$E_m(0) < \frac{1}{2}\|u_1\|^2 + J(u_0) = E(0) = d.$$

Hence, we conclude from (i) that, for each m, problem (1.1), (1.3), (3.11) admits a unique global solution $u_m(t) \in \mathcal{W}$ satisfying

$$(u_{mt}(t), v) + \int_0^t (u_m(\tau), v)_* \,\mathrm{d}\tau + \int_0^t (au_m(\tau), v)\,\mathrm{d}\tau + \mu(u_m(t), v)$$
$$= \int_0^t (|u_m(\tau)|^{p-2} u_m(\tau), v)\,\mathrm{d}\tau + (u_1, v) + \mu(u_{m0}, v)$$

and

$$E_m(t) + \mu \int_0^t \|u_{m\tau}(\tau)\|^2 \,\mathrm{d}\tau \leq E_m(0).$$

Consequently,

$$\frac{1}{2}\|u_{mt}(t)\|^2 + \frac{p-2}{2p} A_1 \|u_m(t)\|_*^2 + \mu \int_0^t \|u_{m\tau}(\tau)\|^2 \,\mathrm{d}\tau < d.$$

By the similar arguments to the proof of (i), problem (1.1)–(1.3) admits a unique global solution $u(t) \in \overline{\mathcal{W}}$ satisfying (2.2).

Case 2. $\|u_0\|_* = 0$. It is obvious that $J(u_0) = 0$ in this case. Thus

$$E(0) = \frac{1}{2}\|u_1\|^2.$$

Let $\lambda_m = 1 - \dfrac{1}{m}$ and $u_{m1} = \lambda_m u_1$, $m = 2, 3, \ldots$, and consider the problem (1.1), (1.3) with the following initial conditions

$$u(x, y, 0) = u_0(x, y),\ \ u_t(x, y, 0) = u_{m1}(x, y). \tag{3.12}$$

Note that

$$0 < E_m(0) = \frac{1}{2}\|u_{m1}\|^2 < E(0).$$

We conclude from (i) that, for each m, problem (1.1), (1.3), (3.12) admits a unique global solution $u_m(t) \in \mathcal{W}$. The remainder of the proof is the same as that in Case 1.

This completes the proof of Theorem 2.4. □

4. PROOF OF THEOREM 2.5

We introduce a family of potential wells

$$\mathcal{W}_\delta = \{u \in H^2_*(\Omega) \mid I_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\}, \quad \delta \in \left(0, \frac{p}{2}\right),$$

where the depth of family of potential wells

$$d(\delta) = \inf_{u \in \mathcal{N}_\delta} J(u),$$

the δ-Nehari manifold

$$\mathcal{N}_\delta = \{u \in H^2_*(\Omega) \setminus \{0\} \mid I_\delta(u) = 0\}$$

and the δ-Nehari functional

$$I_\delta(u) = \delta\|u\|^2_* + \delta(au, u) - \|u\|^p_p.$$

Proposition 4.1.

$$d(\delta) \geq \frac{p - 2\delta}{2p}\delta^{\frac{2}{p-2}} A_1^{\frac{p}{p-2}} S_p^{-\frac{2p}{p-2}},$$

where $\delta \in \left(0, \frac{p}{2}\right)$.

Proof. Let $u \in \mathcal{N}_\delta$, then

$$\delta\|u\|^2_* + \delta(au, u) = \|u\|^p_p,$$

which, together with Lemmas 2.1 and 2.2, gives

$$\delta A_1\|u\|^2_* \leq S^p_p\|u\|^p_*,$$

i.e.,

$$\|u\|_* \geq \delta^{\frac{1}{p-2}} A_1^{\frac{1}{p-2}} S_p^{-\frac{p}{p-2}}.$$

Note that

$$J(u) \geq \frac{p - 2\delta}{2p} A_1\|u\|^2_* + \frac{1}{p} I_\delta(u).$$

Hence, Proposition 4.1 follows from the definition of $d(\delta)$. □

Clearly, when $\delta = 1$, we have $d(\delta) = d$ and $\mathcal{W}_\delta = \mathcal{W}$.

Lemma 4.2. *Under the conditions of* (i) *in Theorem 2.4, $u_n(t) \in \mathcal{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$, where (δ_1, δ_2) is the maximal interval such that $d(\delta) > E(0)$. Moreover,*

$$I(u_n(t)) \geq (1-\delta_1)A_1\|u_n(t)\|_*^2,$$

for all $t \in [0, \infty)$ and sufficiently large n.

Proof. From (3.5) we have

$$\frac{1}{2}\|u_{nt}(t)\|^2 + J(u_n(t)) \leq E_n(0) < d(\delta), \quad \forall t \in [0,\infty), \;\; \delta \in (\delta_1, \delta_2)$$

for sufficiently large n. Thus, by the similar arguments to the proof of [10, Theorem 3.1], we infer that $u_n(t) \in \mathcal{W}_\delta$ for all $\delta \in (\delta_1, \delta_2)$. Hence $I_\delta(u_n(t)) \geq 0$ for all $\delta \in (\delta_1, \delta_2)$, and so $I_{\delta_1}(u_n(t)) \geq 0$. Consequently,

$$\begin{aligned} I(u_n(t)) =&(1-\delta_1)(\|u_n(t)\|_*^2 + (au_n(t), u_n(t))) + I_{\delta_1}(u_n(t)) \\ \geq&(1-\delta_1)A_1\|u_n(t)\|_*^2. \end{aligned}$$

□

Proof of Theorem 2.5. (i) For the approximate solutions u_n given in the proof of (i) in Theorem 2.4, there holds (3.3). Multiplying (3.3) by $e^{\gamma t}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\gamma t}E_n(t)) + \mu e^{\gamma t}\|u_{nt}(t)\|^2 = \gamma e^{\gamma t}E_n(t), \tag{4.1}$$

where γ is a positive constant to be determined later. Integrating (4.1) with respect to t, we obtain

$$e^{\gamma t}E_n(t) + \mu\int_0^t e^{\gamma\tau}\|u_{n\tau}(\tau)\|^2\,\mathrm{d}\tau = E_n(0) + \gamma\int_0^t e^{\gamma\tau}E_n(\tau)\,\mathrm{d}\tau. \tag{4.2}$$

For the second term on the right side of (4.2), we deduce from (3.4) and Lemmas 2.2, 4.2 that

$$\begin{aligned} \int_0^t e^{\gamma\tau}E_n(\tau)\,\mathrm{d}\tau \leq&\frac{1}{2}\int_0^t e^{\gamma\tau}\left(\|u_{n\tau}(\tau)\|^2 + A_2\|u_n(\tau)\|_*^2\right)\,\mathrm{d}\tau \\ \leq&\frac{1}{2}\int_0^t e^{\gamma\tau}\left(\|u_{n\tau}(\tau)\|^2 + \frac{A_2}{(1-\delta_1)A_1}I(u_n(\tau))\right)\,\mathrm{d}\tau. \end{aligned}$$

Due to the fact that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(u_{nt}(t), u_n(t)) &= (u_{ntt}(t), u_n(t)) + \|u_{nt}(t)\|^2 \\ &= \|u_{nt}(t)\|^2 - \|u_n(t)\|_*^2 + \|u_n(t)\|_p^p \\ &\quad - (au_n(t), u_n(t)) - \mu(u_{nt}(t), u_n(t)) \\ &= \|u_{nt}(t)\|^2 - I(u_n(t)) - \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(t)\|^2, \end{aligned}$$

we further obtain

$$\begin{aligned}\int_0^t e^{\gamma\tau} E_n(\tau)\,\mathrm{d}\tau \le{}& \frac{(1-\delta_1)A_1 + A_2}{2(1-\delta_1)A_1} \int_0^t e^{\gamma\tau}\|u_{n\tau}(\tau)\|^2\,\mathrm{d}\tau \\ &- \frac{A_2}{2(1-\delta_1)A_1}\int_0^t e^{\gamma\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\Big((u_{n\tau}(\tau), u_n(\tau)) + \frac{\mu}{2}\|u_n(\tau)\|^2\Big)\,\mathrm{d}\tau.\end{aligned}\tag{4.3}$$

Note that

$$\begin{aligned}&-\int_0^t e^{\gamma\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\Big((u_{n\tau}(\tau), u_n(\tau)) + \frac{\mu}{2}\|u_n(\tau)\|^2\Big)\,\mathrm{d}\tau \\ &= (u_{nt}(0), u_n(0)) + \frac{\mu}{2}\|u_n(0)\|^2 \\ &\quad - e^{\gamma t}\Big((u_{nt}(t), u_n(t)) + \frac{\mu}{2}\|u_n(t)\|^2\Big) \\ &\quad + \gamma\int_0^t e^{\gamma\tau}\Big((u_{n\tau}(\tau), u_n(\tau)) + \frac{\mu}{2}\|u_n(\tau)\|^2\Big)\,\mathrm{d}\tau.\end{aligned}$$

Thus it is easy to see from Cauchy's inequality that

$$\begin{aligned}&-\int_0^t e^{\gamma\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}\Big((u_{n\tau}(\tau), u_n(\tau)) + \frac{\mu}{2}\|u_n(\tau)\|^2\Big)\,\mathrm{d}\tau \\ &\le \frac{1}{2}(\|u_{nt}(0)\|^2 + (\mu+1)\|u_n(0)\|^2) \\ &\quad + \frac{1}{2}e^{\gamma t}(\|u_{nt}(t)\|^2 + (\mu+1)\|u_n(t)\|^2) \\ &\quad + \frac{\gamma}{2}\int_0^t e^{\gamma\tau}(\|u_{n\tau}(\tau)\|^2 + (\mu+1)\|u_n(\tau)\|^2)\,\mathrm{d}\tau.\end{aligned}\tag{4.4}$$

Moreover, from assertion (3.6), we get

$$\begin{aligned}E_n(t) &= \frac{1}{2}\|u_{nt}(t)\|^2 + J(u_n(t)) \\ &= \frac{1}{2}\|u_{nt}(t)\|^2 + \frac{1}{p}I(u_n(t)) + \frac{p-2}{2p}\big(\|u_n(t)\|_*^2 + (au_n(t), u_n(t))\big) \\ &\ge \frac{1}{2}\|u_{nt}(t)\|^2 + \frac{p-2}{2p}A_1\|u_n(t)\|_*^2.\end{aligned}\tag{4.5}$$

Then according to Lemma 2.1, there exists a constant $C_1 > 0$ such that

$$\frac{1}{2}(\|u_{nt}(t)\|^2 + (\mu+1)\|u_n(t)\|^2) \le C_1 E_n(t).\tag{4.6}$$

Therefore, we conclude from (4.2)–(4.4) and (4.6) that there exist constants $C_2, C_3 > 0$ such that

$$\begin{aligned} & e^{\gamma t}E_n(t) + \mu \int_0^t e^{\gamma\tau}\|u_{n\tau}(\tau)\|^2 \,\mathrm{d}\tau \\ \leq & \, C_2 E_n(0) + \frac{\gamma((1-\delta_1)A_1 + A_2)}{2(1-\delta_1)A_1} \int_0^t e^{\gamma\tau}\|u_{n\tau}(\tau)\|^2 \,\mathrm{d}\tau \\ & + C_3\gamma e^{\gamma t}E_n(t) + C_3\gamma^2 \int_0^t e^{\gamma\tau}E_n(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Choosing

$$\gamma < \min\left\{\frac{1}{2C_3}, \frac{2\mu(1-\delta_1)A_1}{(1-\delta_1)A_1 + A_2}\right\},$$

we have

$$e^{\gamma t}E_n(t) \leq 2C_2E_n(0) + 2C_3\gamma^2 \int_0^t e^{\gamma\tau}E_n(\tau)\,\mathrm{d}\tau.$$

Then it is easy to verify that

$$e^{\gamma t}E_n(t) \leq 2C_2E_n(0)e^{2C_3\gamma^2 t},$$

which gives

$$E_n(t) < 2C_2 d e^{-\beta t} \tag{4.7}$$

for sufficiently large n, where $\beta = \gamma - 2C_3\gamma^2$. From (4.5) we have

$$\|u_{nt}(t)\|^2 + \|u_n(t)\|_*^2 \leq C_4E_n(t)$$

for some $C_4 > 0$. This, combined with (3.8), (3.10) and (4.7), yields

$$\begin{aligned} \|u_t(t)\|^2 + \|u(t)\|_*^2 \leq & \liminf_{n\to\infty}(\|u_{nt}(t)\|^2 + \|u_n(t)\|_*^2) \\ \leq & 2C_2C_4 d e^{-\beta t} \end{aligned}$$

for all $t \in [0,\infty)$.

(ii) For the solution $u(t) \in \mathcal{W}$ to the problem (1.1)–(1.3), we have $I(u(t)) > 0$ for all $t \in [0,\infty)$. From (2.1) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_t(t), u(t)) = \|u_t(t)\|^2 - I(u(t)) - \mu(u_t(t), u(t)).$$

Hence, $\|u_t(t)\| > 0$, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \|u_\tau(\tau)\|^2 \,\mathrm{d}\tau > 0.$$

Consequently, there exists a constant $\tilde{t} > 0$ such that

$$\int_0^{\tilde{t}} \|u_\tau(\tau)\|^2 \, \mathrm{d}\tau > 0,$$

which, together with (2.2) and $E(0) = d$, yields

$$E(t) \le d - \mu \int_0^{\tilde{t}} \|u_\tau(\tau)\|^2 \, \mathrm{d}\tau, \quad \forall t \in [\tilde{t}, \infty).$$

Set

$$\tilde{d} := d - \mu \int_0^{\tilde{t}} \|u_\tau(\tau)\|^2 \, \mathrm{d}\tau.$$

Then $0 < E(t) \le \tilde{d} < d$ for all $t \in [\tilde{t}, \infty)$. Therefore, we infer from (i) that there holds (2.3) for all $t \in [\tilde{t}, \infty)$. □

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