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ANALYSIS OF DEFLECTION IN VISCO-THERMOELASTIC BEAM RESONATORS SUBJECTED TO HARMONIC LOADING

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This paper analyses the transverse deflection in a homogeneous, isotropic, visco-thermoelastic beam when subjected to harmonic load. The ends of the beam are considered at different boundary conditions (both axial ends clamped, both axial ends simply supported and left end clamped and right end free). The deflection has been studied by using the Laplace transform. Numerical computation of analytical expression of deflection obtained after Inverse Laplace transform has been done using MATLAB software. The graphical observations have been discussed under various boundary conditions for different values of time and length. The above work has applications in design of resonators.

Keywords: visco-thermoelastic, beam, loading, Laplace transform.

1. Introduction

Viscoelastic materials such as plastic materials and polymer science have received great interest due to numerous applications in modern engineering structures, in which materials are under high temperature. Lord and Shulman [1] formulated the theory of thermoelasticity which incorporated the coupling between temperature and strain rate. Christensen [2] discussed the stress-strain constitutive relations and described thermoviscoelastic stress. Drozdov [3] derived a model for thermoviscoelastic materials which takes into consideration the changes in elastic moduli and relaxation times.

Guo [4] studied the effect of a thermoelastic coupling on the wave characteristics such as the frequency ratio and non-dimensional frequency for micro-machined beam resonators. Sun [5] analysed the influence of a thermoelastic coupling on deflection amplitudes, thermal moment amplitudes for micro-scale beam resonators. Sun [6] studied the out of plane vibrations of a circular plate resonator under the effects of thermoelastic damping. Yanping and Yilong [7] applied the neural network method to study the static deflection in micro-cantilever elastic beam subjected to transverse loading.

Sharma and Grover [8] derived analytical expressions for the thermoelastic damping and frequency shift in transverse vibrations of a homogenous isotropic, thermoelastic thin beam with voids, based on the Euler–Bernoulli theory under clamped and simply supported boundary conditions. Grover [9] derived expressions for transverse vibrations of a homogenous, isotropic, thermally conducting Kelvin-Voigt type viscothermoelastic thin beam with variable thickness.

Guo *et al.* [10] analysed the thermoelastic damping using dual-phase-lagging model and studied the effects of the beam height and aspect ratio. Sharma *et al.* [11] analysed the wave characteristics under the effects of temperature, rotation, viscosity and thermal relaxation time in an elastic medium. Sharma and Kaur [12] analysed the transverse deflection and thermal moment of transverse vibrations in an isotropic, thermo-elastic beam under the action of harmonic concentrated load. Sharma and Kaur [13] studied the dynamic response of a homogeneous, transversely isotropic, thermoelastic micro-beam resonator under the action of time varying load and clamped-clamped conditions at axial ends. Partap and Chugh [14] investigated the

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flexural vibrations of homogeneous isotropic micropolar microstretch thermoelastic thin beam resonators under the influence of time harmonic load at different boundary conditions of clamped-clamped, simply supported-simply supported or clamped-free. Thakare *et al.* [15] analysed the effect of inhomogeneity on thermal and mechanical behaviour in the two dimensional nonhomogeneous thick hollow cylinder in the context of fractional order derivative.

In this paper, an attempt has been made to study the dynamic response of a homogeneous isotropic viscothermoelastic beam under the action of harmonic loading. The Laplace transform technique has been used twice with respect to time and space domain. The analytical solution for clamped-clamped, simply supported-simply supported and cantilever-free beams has been evaluated using the method of residues. MATLAB software has been used for representing the results graphically for comparison.

2. Primary equations

In this paper, a homogeneous isotropic, viscothermoelastic beam has been considered which is initially at uniform temperature κ_0 and is undeformed. The basic equation of motion has been considered in the Cartesian coordinate system and is given by

$$\sigma_{ij,j} = \rho \frac{\partial^2 \nu_i}{\partial t^2} . \tag{2.1}$$

In the context of Lord Shulman [1] model of generalised thermoelasticity, the equation of heat conduction along with the constitutive relations, in the absence of heat sources and body forces, which govern the displacement vector $\nu = (\nu_1, \nu_2, \nu_3)$ and temperature change $\kappa(x,y,z,t)$ at time t are given as

$$\sigma_{ij} = \Lambda_v \delta_{ij} e_{kk} + 2\mu_v e_{ij} - \beta_v \kappa \delta_{ij} , \tag{2.2}$$

$$K\nabla^2\kappa = \rho C_e \left(\frac{\partial \kappa}{\partial t} + t_0 \frac{\partial^2 \kappa}{\partial t^2} \right) + \beta_v \kappa_0 \left(\frac{\partial}{\partial t} + t_0 \frac{\partial^2}{\partial t^2} \right) \nabla . \nu \tag{2.3}$$

where

$$\Lambda_v = \Lambda \left(1 + \epsilon_0 \frac{\partial}{\partial t} \right), \mu_v = \mu \left(1 + \epsilon_1 \frac{\partial}{\partial t} \right),$$

$$\beta_v = \beta \left(1 + \beta_0 \frac{\partial}{\partial t} \right), \qquad \beta_0 = \left(3\Lambda\epsilon_0 + 2\mu\epsilon_1 \right) \frac{\epsilon_\kappa}{\beta} . \tag{2.4}$$

3. Modelling of beam structure

We consider a small flexural deflection of a homogeneous isotropic, viscothermoelastic beam of the following dimensions: length $L(0 \le x \le L)$, width $b\left(-\frac{b}{2} \le y \le \frac{b}{2} \right)$, and thickness $h\left(-\frac{h}{2} \le z \le \frac{h}{2} \right)$. In equilibrium, the beam is under zero stress, zero strain and also kept at stable temperature κ_0. In accordance with Euler-Bernoulli assumptions, any plane cross-section, initially normal to the axis of the beam remains flat and normal after deformation. The displacement vector ν and temperature function κ are given as

$$\nu_1 = -z \frac{\partial D}{\partial x}, \qquad \nu_2 = 0, \qquad \nu_3 = D(x,t), \qquad \kappa = \kappa(x,z,t) . \tag{3.1}$$

Now by substituting Eq.(3.1) in Eqs (2.2) and (2.3), we get the following set of equations.

$$\sigma_{ij} = (\Lambda + 2\mu)\left(-z\frac{\partial^2 D}{\partial x^2}\right) + (\Lambda\epsilon_0 + 2\mu\epsilon_1)\left(-z\frac{\partial^3 D}{\partial t\partial x^2}\right) - \beta\left(\kappa + \beta_0\frac{\partial\kappa}{\partial t}\right), \tag{3.2}$$

$$K\left(\frac{\partial^2\kappa}{\partial x^2} + \frac{\partial^2\kappa}{\partial z^2}\right) = \rho C_e\left(\frac{\partial\kappa}{\partial t} + t_0\frac{\partial^2\kappa}{\partial t^2}\right) - \beta_v z\kappa_0\left(\frac{\partial^3 D}{\partial x^2\partial t} + t_0\frac{\partial^4 D}{\partial x^2\partial t^2}\right). \tag{3.3}$$

Also, the flexural moment of cross section $M(x,t)$ is represented as

$$M(x,t) = \int_{-h/2}^{h/2} b\sigma_{xx} z dz\,.$$

Using Eq.(3.2) we get

$$M(x,t) = (\Lambda + 2\mu)\frac{\partial^2 D}{\partial x^2}I + (\Lambda\epsilon_0 + 2\mu\epsilon_1)\frac{\partial^3 D}{\partial t\partial x^2}I + \beta\left(M_\kappa + \beta_0\frac{\partial M_\kappa}{\partial t}\right) \tag{3.4}$$

where $I = \dfrac{bh^3}{12}$ and $M_\kappa = \int_{-\frac{h}{2}}^{\frac{h}{2}} b\kappa z dz$ represent the moments of inertia of the cross section and of the beam due to thermal effects, respectively. Now taking up the equation of transverse motion of the beam

$$\frac{\partial^2 M}{\partial x^2} + \rho A\frac{\partial^2 D}{\partial t^2} = q(x,t) \tag{3.5}$$

where $A = bh$ represents the area of the cross-section and $q(x,t)$ represents harmonic loading on beam, so the equation of motion of the beam reduces to

$$(\Lambda + 2\mu)I\frac{\partial^4 D}{\partial x^4} + (\Lambda\epsilon_0 + 2\mu\epsilon_1)I\frac{\partial^5 D}{\partial t\partial x^4} + \beta\left(\frac{\partial^2 M_\kappa}{\partial x^2} + \beta_0\frac{\partial^3 M_\kappa}{\partial t\partial x^2}\right) + \rho A\frac{\partial^2 D}{\partial t^2} = q(x,t). \tag{3.6}$$

Considering non-dimensional quantities

$$x' = \frac{x}{L}, \quad D' = \frac{D}{h}, \quad z' = \frac{z}{h}, \quad t' = \frac{c_1}{L}t, \quad t_0' = \frac{c_1}{L}t_0, \quad \kappa' = \frac{\kappa}{\kappa_0},$$

in Eqs (3.3) and (3.6), we get

$$\frac{1}{12A_R^2}\left(\frac{\partial^4 D}{\partial x^4} + \frac{\delta_1^2 c_1}{L}\frac{\partial^5 D}{\partial t\partial x^4}\right) + \overline{\beta}\left(\frac{\partial^2 M_\kappa}{\partial x^2} + \frac{c_1\beta_0}{L}\frac{\partial^3 M_\kappa}{\partial t\partial x^2}\right) + \frac{\partial^2 D}{\partial t^2} = q \tag{3.7}$$

where

$$M_{\kappa} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \kappa z \, dz ,$$

$$\left(\frac{\partial^2 \kappa}{\partial x^2} + A_R^2 \frac{\partial^2 \kappa}{\partial z^2}\right) = \frac{\delta C_e c_1 L}{K}\left(\frac{\partial \kappa}{\partial t} + t_0 \frac{\partial^2 \kappa}{\partial t^2}\right) - \frac{z h^2 c_1 \beta}{LK}\left(1 + \frac{\beta_0 c_1}{L}\frac{\partial}{\partial t}\right)\left(\frac{\partial^3 D}{\partial x^2 \partial t} + t_0 \frac{\partial^4 D}{\partial x^2 \partial t^2}\right) \tag{3.8}$$

where

$$A_R = \frac{L}{h}, \quad c_1^2 = \frac{\Lambda + 2\mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}, \quad c_3^2 = \frac{\Lambda \epsilon_0 + 2\mu \epsilon_1}{\rho},$$
$$\delta^2 = \frac{c_2^2}{c_1^2}, \quad \delta_1^2 = \frac{c_3^2}{c_1^2}, \quad \bar{\beta} = \frac{\beta \kappa_0}{\rho c_1^2}, \quad q' = \frac{qL^2}{Ah\rho c_1^2}. \tag{3.9}$$

Ignoring the primes for the sake of convenience.

4. Initial and boundary conditions

A beam whose edges are either CC, SS or CF, where CC, SS, CF stand for clamped-clamped, simply supported-simply supported, clamped-free respectively, has been considered and the following conditions have been taken into account.

Initial conditions are as follows:

$$D(x,0) = \left(\frac{\partial D(x,t)}{\partial t}\right)_{t=0} = 0, \quad \left(\frac{\partial^2 D(x,t)}{\partial t^2}\right)_{t=0} = k(const.),$$

$$\kappa(x,z,0) = \left(\frac{\partial \kappa(x,z,t)}{\partial t}\right)_{t=0} = 0.$$

Boundary conditions are considered as

Case I: For CC beam

$$D(0,t) = \left(\frac{\partial D(x,t)}{\partial x}\right)_{x=0} = 0, \; D(1,t) = \left(\frac{\partial D(x,t)}{\partial x}\right)_{x=1} = 0 . \tag{4.1}$$

Case II: For SS beam

$$D(0,t) = \left(\frac{\partial^2 D(x,t)}{\partial x^2}\right)_{x=0} = 0, \; D(1,t) = \left(\frac{\partial^2 D(x,t)}{\partial x^2}\right)_{x=1} = 0 . \tag{4.2}$$

Case III: For CF beam

$$D(0,t)=\left(\frac{\partial D(x,t)}{\partial x}\right)_{x=0}=0,\ \left(\frac{\partial^2 D(x,t)}{\partial x^2}\right)_{x=1}=\left(\frac{\partial^3 D(x,t)}{\partial x^3}\right)_{x=1}=0\,. \tag{4.3}$$

5. Laplace transform approach

We apply the Laplace transform to Eqs (3.7) and (3.8) with respect to the time domain, defined as

$$W(x,s)=\int_0^\infty e^{-st}D(x,t)\,dt, \quad \text{and} \quad \Theta(x,z,s)=\int_0^\infty e^{-st}\kappa(x,z,t)\,dt\,,$$

$$\frac{1}{12A_R^2}\left(1+\frac{\delta_1^2 c_1 s}{L}\right)\frac{\partial^4 W}{\partial x^4}+\overline{\beta}\left(1+\frac{c_1\beta_0 s}{L}\right)\frac{\partial^2 M_\Theta}{\partial x^2}+s^2W=Q\,, \tag{5.1}$$

$$\left(\frac{\partial^2\Theta}{\partial x^2}+A_R^2\frac{\partial^2\Theta}{\partial z^2}\right)=\frac{\rho C_e c_1 L s\gamma_0}{K}\Theta-\frac{zh^2c_1\beta\gamma_0\gamma_1 s}{LK}\frac{\partial^2 W}{\partial x^2}\,, \tag{5.2}$$

$$1+st_0=\gamma_0,\ 1+\frac{sc_1\beta_0}{L}=\gamma_1,\ M_\Theta=\int_{-h/2}^{h/2}\Theta z\,dz\,, \tag{5.3}$$

$Q(x,s)$ is the Laplace transform of load $q(x,t)$.
Under the conditions that no heat flows through upper and lower surfaces of the beam

$$\frac{\partial\Theta}{\partial z}=0 \quad \text{at} \quad z=\pm\frac{1}{2}.$$

The solution of Eq.(5.2) is

$$\Theta(x,z,s)=\frac{h^2\beta\gamma_1}{C_eL^2\rho}\left(z-\frac{\sin pz}{p\cos\left(\frac{p}{2}\right)}\right)\frac{\partial^2W}{\partial x^2} \quad \text{where} \quad p^2=-\frac{\rho C_e c_1 L s\gamma_0}{KA_R^2}\,. \tag{5.4}$$

Using Eq.(5.3) to find M_Θ and differentiating twice with respect to x, we get

$$\frac{\partial^2 M_\Theta}{\partial x^2}=\frac{h^2\beta\gamma_1}{12C_eL^2\rho}\left(1+f(p)\right)\frac{\partial^4 W}{\partial x^4} \quad \text{where} \quad f(p)=\frac{24}{p^3}\left(\frac{p}{2}-\tan\left(\frac{p}{2}\right)\right). \tag{5.5}$$

Using Eq.(5.5) in (5.1), we obtain

$$F_s\frac{\partial^4 W}{\partial x^4}+s^2W=Q \qquad \text{where} \qquad F_s=\frac{1}{12A_R^2}\left(1+\frac{\delta_1^2 c_1 s}{L}+\left(1+f(p)\right)\frac{\beta\bar{\beta}\gamma_1^2}{\rho C_e}\right),$$

$$\frac{\partial^4 W}{\partial x^4}-\zeta^4 W=\frac{Q}{F_s} \qquad \text{where} \qquad \zeta^4=-\frac{s^2}{F_s}. \tag{5.6}$$

Considering harmonic loading on the beam $q(x,t)=q_0\sin\omega t$, we get

$$Q(s,t)=\frac{q_0\omega}{s^2+\omega^2}.$$

Applying the Laplace transform with respect to the space domain defined as $\bar{W}(\xi,s)=\int\limits_0^\infty e^{-\xi x}W(x,s)dx$ Eq.(5.6) reduces to

$$\left[\xi^4\bar{W}-\xi^3W(0,s)-\xi^2W'(0,s)-\xi^3W''(0,s)-W'''(0,s)\right]-\zeta^4\bar{W}=\frac{q_0\omega}{F_s\left(s^2+\omega^2\right)\xi}. \tag{5.7}$$

Using the boundary conditions at $x=0$ defined by Eqs (4.1)-(4.3) and applying the inverse Laplace transform with respect to the space domain.

Case I

$$W=\frac{a_1C(\zeta x)}{2\zeta^2}+\frac{a_2S(\zeta x)}{2\zeta^3}+\frac{q_0\omega}{F_s\left(s^2+\omega^2\right)}\frac{\bar{C}(\zeta x)-2}{2\zeta^4}. \tag{5.8}$$

Case II

$$W=\frac{a_3\bar{S}(\zeta x)}{2\zeta}+\frac{a_4S(\zeta x)}{2\zeta^3}+\frac{q_0\omega}{F_s\left(s^2+\omega^2\right)}\frac{\bar{C}(\zeta x)-2}{2\zeta^4}. \tag{5.9}$$

Case III

$$W=\frac{a_5C(\zeta x)}{2\zeta^2}+\frac{a_6S(\zeta x)}{2\zeta^3}+\frac{q_0\omega}{F_s\left(s^2+\omega^2\right)}\frac{\bar{C}(\zeta x)-2}{2\zeta^4} \tag{5.10}$$

where

$$C(\zeta x)=\cosh(\zeta x)-\cos(\zeta x), \quad S(\zeta x)=\sinh(\zeta x)-\sin(\zeta x),$$

$$\bar{C}(\zeta x)=\cosh(\zeta x)+\cos(\zeta x), \quad \bar{S}(\zeta x)=\sinh(\zeta x)+\sin(\zeta x).$$

Using the boundary conditions at $x = 1$ defined by Eqs (4.1)-(4.3), a set of non-homogeneous linear equations is obtained and the condition for existence of an infinite solutions is

Case I $\cos\zeta\cosh\zeta = 1$, (5.11)

Case II $\sin\zeta\sinh\zeta = 0$, (5.12)

Case III $\cos\zeta\cosh\zeta = -1$. (5.13)

The respective roots of the Eqs (5.11)-(5.13) are given by

$$\begin{aligned}
&Case\,I: \zeta_1 = 4.730, \quad \zeta_2 = 7.8532, \quad \zeta_k = \left(k + \frac{1}{2}\right)\pi, \quad k \geq 3, \\
&Case\,II: \zeta_1 = 3.1416, \quad \zeta_2 = 6.2832, \quad \zeta_k = k\pi, \quad k \geq 3, \\
&Case\,III: \zeta_1 = 1.8751, \quad \zeta_2 = 4.6941, \quad \zeta_k = \left(k - \frac{1}{2}\right)\pi, \quad k \geq 3,
\end{aligned} \tag{5.14}$$

and solutions for three cases of boundary conditions are given by

Case I

$$W = \frac{q_0\omega}{2\zeta^4 F_s\left(s^2 + \omega^2\right)}\left(\frac{A_1(\zeta)C(\zeta x) + B_1(\zeta)S(\zeta x) + G_1(\zeta)\left(\bar{C}(\zeta x) - 2\right)}{G_1(\zeta)}\right), \tag{5.15}$$

Case II

$$W = \frac{q_0\omega}{4\zeta^4 F_s\left(s^2 + \omega^2\right)}\left(\frac{A_2(\zeta)\bar{S}(\zeta x) + B_2(\zeta)S(\zeta x) + 2G_2(\zeta)\left(\bar{C}(\zeta x) - 2\right)}{G_2(\zeta)}\right), \tag{5.16}$$

Case III

$$W = \frac{q_0\omega}{2\zeta^4 F_s\left(s^2 + \omega^2\right)}\left(\frac{A_3(\zeta)C(\zeta x) + B_3(\zeta)S(\zeta x) + G_3(\zeta)\left(\bar{C}(\zeta x) - 2\right)}{G_3(\zeta)}\right) \tag{5.17}$$

where

$$A_1(\zeta) = \cosh\zeta - \cos\zeta - \sinh\zeta\sin\zeta,\ B_1(\zeta) = \sinh\zeta(\cos\zeta - 1) + \sin\zeta(\cosh\zeta - 1),$$

$$A_2(\zeta) = \sinh\zeta(1 - \cos\zeta) + \sin\zeta(1 - \cosh\zeta),$$

$$B_2(\zeta) = \sinh\zeta(\cos\zeta - 1) + \sin\zeta(1 - \cosh\zeta),$$

$$A_3(\zeta)=\sinh\zeta\sin\zeta, B_3(\zeta)=-(\cosh\zeta\sin\zeta+\sinh\zeta\cos\zeta),$$

$$G_1(\zeta)=1-\cosh\zeta\cos\zeta, G_2(\zeta)=\sinh\zeta\sin\zeta, G_3(\zeta)=\cosh\zeta\cos\zeta+1.$$

Taking the inverse Laplace transform with respect to the time domain using the method of residues defined as

$$D(x,t)=\Sigma\,Residues\quad\text{of}\quad e^{st}W(x,s), \tag{5.18}$$

we can write

$$r=\sqrt{1+\omega^2 t_0^2},\ R=\frac{1}{A_R}\sqrt{\frac{\rho C_e L c\omega r}{K}},\ \theta=\tan^{-1}\left(\frac{1}{\omega t_0}\right),$$

$$f_R=\frac{12\cos\theta}{R^2}-\frac{24}{R^3}\left[\frac{\cos\left(\frac{3\theta}{2}\right)\sin(R\cos\theta)+\sin\left(\frac{3\theta}{2}\right)\sinh(R\sin\theta)}{\cos(R\cos\theta)+\cosh(R\sin\theta)}\right],$$

$$f_I=\frac{12\sin\theta}{R^2}-\frac{24}{R^3}\left[\frac{\sin\left(\frac{3\theta}{2}\right)\sin(R\cos\theta)-\cos\left(\frac{3\theta}{2}\right)\sinh(R\sin\theta)}{\cos(R\cos\theta)+\cosh(R\sin\theta)}\right],$$

$$\zeta_R=\sqrt[4]{12}\,\sqrt{A_R\omega}\left[1-\frac{\bar{\beta}\beta}{4\rho C_e}\left(\left(1-\left(\frac{tc\beta_0}{L}\right)^2\right)\right)(1+f_R)-\frac{2tc\beta_0}{L}f_I\right],$$

$$\zeta_I=\sqrt[4]{12}\,\sqrt{A_R t}\left[-\frac{\delta_1^2 ct}{4L}-\frac{\bar{\beta}\beta}{4\rho C_e}\left(\left(1-\left(\frac{tc\beta_0}{L}\right)^2\right)\right)f_I+\frac{2tc\beta_0}{L}(1+f_R)\right]$$

where f_R, f_I are the real and imaginary parts of $f(p)$ and ζ_R, ζ_I are the real and imaginary parts of ζ, respectively, at $s=\pm\iota\omega$.

Case I

$s=0$ is removable singularity, residue=0. $s=\pm\iota\omega$ are simple poles and their residues are conjugates of each other, so the sum of residues is twice the real part of residue of $e^{st}W(x,s)$ at $s=\iota\omega$. The sum of residues at $s=\pm\iota\omega$ is

$$\frac{q_0}{2\omega^2}\left(\sin(\omega t)\left(\frac{(T(P+R)+U(Q+S))}{T^2+U^2}+V\right)+\cos(\omega t)\left(\frac{T(Q+S)-U(P+R)}{T^2+U^2}+Y\right)\right) \tag{5.19}$$

where

$$P=\Big[\big(\cosh(\zeta_R x)\cos(\zeta_I x)-\cos(\zeta_R x)\cosh(\zeta_I x)\big)\big(\cosh\zeta_R\cos\zeta_I-\cos\zeta_R\cosh\zeta_I+$$
$$-\sinh\zeta_R\cos\zeta_I\sin\zeta_R\cosh\zeta_I+\cosh\zeta_R\sin\zeta_I\cos\zeta_R\sinh\zeta_I\big)\Big]-\Big[\big(\sinh(\zeta_R x)\sin(\zeta_I x)+$$
$$+\sin\zeta_R x\sinh(\zeta_I x)\big)\big(\sinh\zeta_R\sin\zeta_I+\sin\zeta_R\sinh\zeta_I-\sinh\zeta_R\cos\zeta_I\cos\zeta_R\sinh\zeta_I+$$
$$-\cosh\zeta_R\sin\zeta_I\sin\zeta_R\cosh\zeta_I\big)\Big],$$

$$Q=\Big[\big(\sinh(\zeta_R x)\sin(\zeta_I x)+\sin(\zeta_R x)\sinh(\zeta_I x)\big)\big(\cosh\zeta_R\cos\zeta_I-\cos\zeta_R\cosh\zeta_I+$$
$$-\sinh\zeta_R\cos\zeta_I\sin\zeta_R\cosh\zeta_I+\cosh\zeta_R\sin\zeta_I\cos\zeta_R\sinh\zeta_I\big)\Big]+\Big[\big(\cosh(\zeta_R x)\cos(\zeta_I x)+$$
$$-\cos(\zeta_R x)\cosh(\zeta_I x)\big)\big(\sinh\zeta_R\sin\zeta_I+\sin\zeta_R\sinh\zeta_I-\sinh\zeta_R\cos\zeta_I\cos\zeta_R\sinh\zeta_I+$$
$$-\cosh\zeta_R\sin\zeta_I\sin\zeta_R\cosh\zeta_I\big)\Big],$$

$$R=\Big[\big(\sinh(\zeta_R x)\cos(\zeta_I x)-\sin(\zeta_R x)\cosh(\zeta_I x)\big)\big(\sinh\zeta_R\cos\zeta_I(\cos\zeta_R\cosh\zeta_I-1)+$$
$$+\cosh\zeta_R\sin\zeta_I\sin\zeta_R\sinh\zeta_I+\sin\zeta_R\cosh\zeta_I(\cosh\zeta_R\cos\zeta_I-1)+$$
$$-\cos\zeta_R\sinh\zeta_I\sinh\zeta_R\sin\zeta_I\big)\Big]-\Big[\big(\cosh(\zeta_R x)\sin(\zeta_I x)-\cos(\zeta_R x)\sinh(\zeta_I x)\big)\times$$
$$\times\big((\cosh\zeta_R\sin\zeta_I)(\cos\zeta_R\cosh\zeta_I-1)-(\sinh\zeta_R\cos\zeta_I\sin\zeta_R\sinh\zeta_I)+$$
$$+(\sin\zeta_R\cosh\zeta_I\sinh\zeta_R\sin\zeta_I)+(\cos\zeta_R\sinh\zeta_I)(\cosh\zeta_R\cos\zeta_I-1)\Big],$$

$$S=\Big[\big(\cosh(\zeta_R x)\sin(\zeta_I x)-\cos(\zeta_R x)\sinh(\zeta_I x)\big)\big(\sinh\zeta_R\cos\zeta_I(\cos\zeta_R\cosh\zeta_I-1)+$$
$$+\cosh\zeta_R\sin\zeta_I\sin\zeta_R\sinh\zeta_I+\sin\zeta_R\cosh\zeta_I(\cosh\zeta_R\cos\zeta_I-1)+$$
$$-\cos\zeta_R\sinh\zeta_I\sinh\zeta_R\sin\zeta_I\big)\Big]+\Big[\big(\sinh(\zeta_R x)\cos(\zeta_I x)-\sin(\zeta_R x)\cosh(\zeta_I x)\big)\times$$
$$\times\big((\cosh\zeta_R\sin\zeta_I)(\cos\zeta_R\cosh\zeta_I-1)-(\sinh\zeta_R\cos\zeta_I\sin\zeta_R\sinh\zeta_I)+$$
$$+(\sin\zeta_R\cosh\zeta_I\sinh\zeta_R\sin\zeta_I)+(\cos\zeta_R\sinh\zeta_I)(\cosh\zeta_R\cos\zeta_I-1)\big)\Big],$$

$$T=1-\cos\zeta_R\cosh\zeta_I\cosh\zeta_R\cos\zeta_I-\sin\zeta_R\sinh\zeta_I\sinh\zeta_R\sin\zeta_I,$$

$$U=\sin\zeta_R\sinh\zeta_I\cosh\zeta_R\cos\zeta_I-\cos\zeta_R\cosh\zeta_I\sinh\zeta_R\sin\zeta_I,$$

$$V=\cosh(\zeta_R x)\cos(\zeta_I x)+\cos(\zeta_R x)\cosh(\zeta_I x)-2,$$

$$Y=\sinh(\zeta_R x)\sin(\zeta_I x)-\sin(\zeta_R x)\sinh(\zeta_I x).$$

Singularities corresponding to $G_I(\zeta)=0$ given by Eq.(5.14) are simple poles. Using Eq.(5.6),

$$s=\pm\iota\zeta_k^2\sqrt{F_s}=\pm\iota s_k\ ,s_k=s_0\left(1+\frac{\delta_I^2 cs_0}{2L}+\frac{\bar{\beta}\beta}{2\rho C_e}\left(1+\frac{s_o\beta_0 c}{L}\right)^2\left(1+f_{p0}\right)\right)$$

where

$$s_0=\frac{\zeta_k^2}{2\sqrt{3}A_R}\ ,P^2=\frac{\rho C_e cLs_0\gamma_0}{KA_R^2}\ ,1+f_{p0}=1-\frac{12}{P^2}+\frac{24\tanh\left(\frac{P}{2}\right)}{P^3}\ ,$$

the sum of the residues at $s=\pm\iota s_k$ is equal

$$\begin{aligned}&\left[\frac{4F_s q_0\omega\cos(s_k t)}{\zeta_k\left(\omega^2-s_k^2\right)}\left(\frac{A_1(\zeta_k)C(\zeta_k x)+B_1(\zeta_k)S(\zeta_k x)+\left(\bar{C}(\zeta_k x)-2\right)G_1(\zeta_k)}{(\sin\zeta_k\cosh\zeta_k-\cos\zeta_k\sinh\zeta_k)}\right)\right]\times\\&\times\left[-2s_kF_s+s_k^2\left(\frac{\delta_1^2c_1}{L}\right)+\frac{2\beta\beta_0\bar{\beta}c_1\left(1+\frac{s_k\beta_0c_1}{L}\right)(1+f(p))}{\rho C_e L}+\right.\\&\left.+\beta\bar{\beta}c_1L\left(1+\left(\frac{s_k\beta_0c_1}{L}\right)^2\right)(1+2s_kt_0)K^{-1}A_R^{-2}\left(\left(\frac{12\left(1+\sec^2\left(\frac{p}{2}\right)\right)}{p^4}\right)-\left(\frac{36\tanh\left(\frac{p}{2}\right)}{p^5}\right)\right)\right]^{-1}.\end{aligned}\tag{5.20}$$

Case II

$s=0$ is removable singularity, residue=0. $s=\pm\iota\omega$ are simple poles and their residues are conjugates of each other, so the sum of residues is twice the real part of residue of $e^{st}W(x,s)\,at\,s=\iota\omega$. The sum of residues at $s=\pm\iota\omega$ is equal

$$\begin{aligned}&\frac{q_0}{4\omega^2}\left(\sin(\omega t)\left(\frac{\left(T(P+R)+U(Q+S)\right)}{T^2+U^2}+2V\right)+\right.\\&\left.+\cos(\omega t)\left(\frac{T(Q+S)-U(P+R)}{T^2+U^2}+2Y\right)\right)\end{aligned}\tag{5.21}$$

where

$$\begin{aligned}P=&\Big[\left(\sinh(\zeta_R x)\cos(\zeta_I x)+\sin(\zeta_R x)\cosh(\zeta_I x)\right)\left(\sinh\zeta_R\cos\zeta_I(1-\cos\zeta_R\cosh\zeta_I)+\right.\\&+\sin\zeta_R\cosh\zeta_I(1-\cosh\zeta_R\cos\zeta_I)-\cosh\zeta_R\sin\zeta_I\sin\zeta_R\sinh\zeta_I+\\&\left.+\cos\zeta_R\sinh\zeta_I\sinh\zeta_R\sin\zeta_I\right)\Big]-\Big[\left(\cosh(\zeta_R x)\sin(\zeta_I x)+\cos(\zeta_R x)\sinh(\zeta_I x)\right)\times\\&\times\left(\sinh\zeta_R\cos\zeta_I\sin\zeta_R\sinh\zeta_I+\cosh\zeta_R\sin\zeta_I(1-\cos\zeta_R\cosh\zeta_I)+\right.\\&\left.-\sin\zeta_R\cosh\zeta_I\sinh\zeta_R\sin\zeta_I+\cos\zeta_R\sinh\zeta_I(1-\cosh\zeta_R\cos\zeta_I)\right)\Big],\end{aligned}$$

$$Q=\Big[\big(\cosh(\zeta_R x)\sin(\zeta_I x)+\cos(\zeta_R x)\sinh(\zeta_I x)\big)\big(\sinh\zeta_R\cos\zeta_I\,(1-\cos\zeta_R\cosh\zeta_I)+$$
$$+\sin\zeta_R\cosh\zeta_I\,(1-\cosh\zeta_R\cos\zeta_I)-\cosh\zeta_R\sin\zeta_I\sin\zeta_R\sinh\zeta_I+$$
$$+\cos\zeta_R\sinh\zeta_I\sinh\zeta_R\sin\zeta_I\big)\Big]+\Big[\big(\sinh(\zeta_R x)\cos(\zeta_I x)+\sin(\zeta_R x)\cosh(\zeta_I x)\big)\times$$
$$\times\big(\sinh\zeta_R\cos\zeta_I\sin\zeta_R\sinh\zeta_I+\cosh\zeta_R\sin\zeta_I\,(1-\cos\zeta_R\cosh\zeta_I)$$
$$-\sin\zeta_R\cosh\zeta_I\sinh\zeta_R\sin\zeta_I+\cos\zeta_R\sinh\zeta_I\,(1-\cosh\zeta_R\cos\zeta_I)\big)\Big],$$

$$R=\Big[\big(\sinh(\zeta_R x)\cos(\zeta_I x)+\sin(\zeta_R x)\cosh(\zeta_I x)\big)\big(\sinh\zeta_R\cos\zeta_I\,(\cos\zeta_R\cosh\zeta_I-1)+$$
$$+\sin\zeta_R\cosh\zeta_I\,(1-\cosh\zeta_R\cos\zeta_I)+\cosh\zeta_R\sin\zeta_I\sin\zeta_R\sinh\zeta_I+$$
$$+\cos\zeta_R\sinh\zeta_I\sinh\zeta_R\sin\zeta_I\big)\Big]-\Big[\big(-\sinh\zeta_R\cos\zeta_I\sin\zeta_R\sinh\zeta_I+$$
$$+\cosh\zeta_R\sin\zeta_I\,(\cos\zeta_R\cosh\zeta_I-1)-\sin\zeta_R\cosh\zeta_I\sinh\zeta_R\sin\zeta_I+$$
$$+\cos\zeta_R\sinh\zeta_I\,(1-\cosh\zeta_R\cos\zeta_I)\big)\big(\cosh(\zeta_R x)\sin(\zeta_I x)-\cos(\zeta_R x)\sinh(\zeta_I x)\big)\Big],$$

$$S=\Big[\big(\cosh(\zeta_R x)\sin(\zeta_I x)-\cos(\zeta_R x)\sinh(\zeta_I x)\big)\big(\sinh\zeta_R\cos\zeta_I\,(\cos\zeta_R\cosh\zeta_I-1)+$$
$$+\sin\zeta_R\cosh\zeta_I\,(1-\cosh\zeta_R\cos\zeta_I)+\cosh\zeta_R\sin\zeta_I\sin\zeta_R\sinh\zeta_I+$$
$$+\cos\zeta_R\sinh\zeta_I\sinh\zeta_R\sin\zeta_I\big)\Big]+\Big[\big(\sinh(\zeta_R x)\cos(\zeta_I x)+\sin(\zeta_R x)\cosh(\zeta_I x)\big)\times$$
$$\times\big(-\sinh\zeta_R\cos\zeta_I\sin\zeta_R\sinh\zeta_I+\cosh\zeta_R\sin\zeta_I\,(\cos\zeta_R\cosh\zeta_I-1)+$$
$$-\sin\zeta_R\cosh\zeta_I\sinh\zeta_R\sin\zeta_I+\cos\zeta_R\sinh\zeta_I\,(1-\cosh\zeta_R\cos\zeta_I)\big)\Big],$$

$$T=\sinh\zeta_R\cosh\zeta_I\sin\zeta_R\cos\zeta_I-\cosh\zeta_R\sinh\zeta_I\cos\zeta_R\sin\zeta_I\,,$$

$$U=\sinh\zeta_R\sinh\zeta_I\cos\zeta_R\cos\zeta_I+\cosh\zeta_R\cosh\zeta_I\sin\zeta_R\sin\zeta_I\,,$$

$$V=\cosh(\zeta_R x)\cos(\zeta_I x)+\cos(\zeta_R x)\cosh(\zeta_I x)-2\,,$$

$$Y=\sinh(\zeta_R x)\sin(\zeta_I x)-\sin(\zeta_R x)\sinh(\zeta_I x)\,.$$

Singularities corresponding to $G_2(\zeta)=0$ given by Eq.(5.14) are simple poles. Using Eq.(5.6),

$$s=\pm\iota\zeta_k^2\sqrt{F_s}=\pm\iota s_k\ ,\qquad s_k=s_0\left(1+\frac{\delta_1^2 cs_0}{2L}+\frac{\bar{\beta}\beta}{2\rho C_e}\left(1+\frac{s_o\beta_0 c}{L}\right)^2\left(1+f_{p0}\right)\right)$$

where

$$s_0=\frac{\zeta_k^2}{2\sqrt{3}A_R}\ ,\qquad P^2=\frac{\rho C_e cLs_0\gamma_0}{KA_R^2}\ ,\qquad 1+f_{p0}=1-\frac{12}{P^2}+\frac{24\tanh\left(\dfrac{P}{2}\right)}{P^3}\,,$$

the sum of the residues at $s=\pm\iota s_k$ is equal

$$
\begin{aligned}
&=\left[\frac{2F_s q_0 \omega\cos(s_k t)}{\zeta_k\left(\omega^2-s_k^2\right)}\left(\frac{A_2(\zeta_k)\bar{S}(\zeta_k x)+B_2(\zeta_k)S(\zeta_k x)+2\left(\bar{C}(\zeta_k x)-2\right)G_2(\zeta_k)}{(\sinh\zeta_k\cos\zeta_k+\cosh\zeta_k\sin\zeta_k)}\right)\right]\times\\
&\times\left[-2s_k F_s+s_k^2\left(\frac{\delta_1^2 c_1}{L}\right)+\frac{2\beta\beta_0\bar{\beta}c_1\left(1+\frac{s_k\beta_0 c_1}{L}\right)(1+f(p))}{\rho C_e L}+\right.\\
&\left.+\beta\bar{\beta}c_1 L\left(1+\left(\frac{s_k\beta_0 c_1}{L}\right)^2\right)(1+2s_k t_0)K^{-1}A_R^{-2}\left(\left(\frac{12\left(1+\sec^2\left(\frac{p}{2}\right)\right)}{p^4}\right)-\left(\frac{36\tanh\left(\frac{p}{2}\right)}{p^5}\right)\right)\right]^{-1}.
\end{aligned}
\tag{5.22}
$$

Case III

$s=0$ is removable singularity, Residue=0. $s=\pm\iota\omega$ are simple poles and their residues are conjugates of each other, so the sum of residues is twice the real part of residue of $e^{st}W(x,s)\, at\, s=\iota\omega$. The sum of residues at $s=\pm\iota\omega$ is equal

$$
\begin{aligned}
&\frac{q_0}{2\omega^2}\left(\sin(\omega t)\left(\frac{(T(P+R)+U(Q+S))}{T^2+U^2}+V\right)+\right.\\
&\left.+\cos(\omega t)\left(\frac{T(Q+S)-U(P+R)}{T^2+U^2}+Y\right)\right),
\end{aligned}
\tag{5.23}
$$

$$
\begin{aligned}
P=&\left[\left(\cosh(\zeta_R x)\cos(\zeta_I x)-\cos(\zeta_R x)\cosh(\zeta_I x)\right)(\sinh\zeta_R\cos\zeta_I\sin\zeta_R\cosh\zeta_I+\right.\\
&\left.-\cosh\zeta_R\sin\zeta_I\sinh\zeta_I\cos\zeta_R)\right]-\left[\left(\sinh(\zeta_R x)\sin(\zeta_I x)+\sin(\zeta_R x)\sinh(\zeta_I x)\right)\times\right.\\
&\left.\times(\sinh\zeta_R\cos\zeta_I\cos\zeta_R\sinh\zeta_I+\cosh\zeta_R\sin\zeta_I\sin\zeta_R\cosh\zeta_I)\right],
\end{aligned}
$$

$$
\begin{aligned}
Q=&\left[\left(\sinh(\zeta_R x)\sin(\zeta_I x)+\sin(\zeta_R x)\sinh(\zeta_I x)\right)(\sinh\zeta_R\cos\zeta_I\sin\zeta_R\cosh\zeta_I+\right.\\
&\left.-\cosh\zeta_R\sin\zeta_I\sinh\zeta_I\cos\zeta_R)\right]+\left[\left(\cosh(\zeta_R x)\cos(\zeta_I x)-\cos(\zeta_R x)\cosh(\zeta_I x)\right)\times\right.\\
&\left.\times(\sinh\zeta_R\cos\zeta_I\cos\zeta_R\sinh\zeta_I+\cosh\zeta_R\sin\zeta_I\sin\zeta_R\cosh\zeta_I)\right],
\end{aligned}
$$

$$
\begin{aligned}
R=&-\left[\left(\sinh(\zeta_R x)\cos(\zeta_I x)-\sin(\zeta_R x)\cosh(\zeta_I x)\right)\times\left((\cosh\zeta_R\cos\zeta_I\sin\zeta_R\cosh\zeta_I+\right.\right.\\
&-\sinh\zeta_R\sin\zeta_I\cos\zeta_R\sinh\zeta_I)+(\sinh\zeta_R\cos\zeta_I\cos\zeta_R\cosh\zeta_I+\\
&\left.+\cosh\zeta_R\sin\zeta_I\sin\zeta_R\sinh\zeta_I)\right]+\left[(\cosh\zeta_R\cos\zeta_I\cos\zeta_R\sinh\zeta_I+\right.\\
&+\sinh\zeta_R\sin\zeta_I\sin\zeta_R\cosh\zeta_I-\sinh\zeta_R\cos\zeta_I\sin\zeta_R\sinh\zeta_I+\cosh\zeta_R\sin\zeta_I\cos\zeta_R\cosh\zeta_I)\times\\
&\left.\times\left(\cosh(\zeta_R x)\,sin(\zeta_I x)-\cos(\zeta_R x)\sinh(\zeta_I x)\right)\right],
\end{aligned}
$$

$$S=-\Big[\big(\cosh(\zeta_R x)\sin(\zeta_I x)-\cos(\zeta_R x)\sinh(\zeta_I x)\big)\big((\cosh\zeta_R\cos\zeta_I\sin\zeta_R\cosh\zeta_I+$$
$$-\sinh\zeta_R\sin\zeta_I\cos\zeta_R\sinh\zeta_I)+(\sinh\zeta_R\cos\zeta_I\cos\zeta_R\cosh\zeta_I+$$
$$+\cosh\zeta_R\sin\zeta_I\sin\zeta_R\sinh\zeta_I)\Big]-\Big[(\cosh\zeta_R\cos\zeta_I\cos\zeta_R\sinh\zeta_I+$$
$$+\sinh\zeta_R\sin\zeta_I\sin\zeta_R\cosh\zeta_I-\sinh\zeta_R\cos\zeta_I\sin\zeta_R\sinh\zeta_I+\cosh\zeta_R\sin\zeta_I\cos\zeta_R\cosh\zeta_I)\times$$
$$\times\big(\sinh(\zeta_R x)\cos(\zeta_I x)-\sin(\zeta_R x)\cosh(\zeta_I x)\big)\Big],$$

$$T=\cosh\zeta_R\cos\zeta_I\cos\zeta_R\cosh\zeta_I+\sinh\zeta_R\sin\zeta_I\sin\zeta_R\sin\zeta_I+1,$$

$$U=-\cosh\zeta_R\cos\zeta_I\sin\zeta_R\sinh\zeta_I+\sinh\zeta_R\sin\zeta_I\cos\zeta_R\cosh\zeta_I,$$

$$V=\cosh(\zeta_R x)\cos(\zeta_I x)+\cos(\zeta_R x)\cosh(\zeta_I x)-2,$$

$$Y=\sinh(\zeta_R x)\sin(\zeta_I x)-\sin(\zeta_R x)\sinh(\zeta_I x).$$

Singularities corresponding to $G_3(\zeta)=0$ given by Eq.(5.14) are simple poles. Using equation (5.6),

$$s=\pm\imath\zeta_k^2\sqrt{F_s}=\pm\imath s_k,\qquad s_k=s_0\left(1+\frac{\delta_I^2 cs_0}{2L}+\frac{\bar{\beta}\beta}{2\rho C_e}\left(1+\frac{s_o\beta_0 c}{L}\right)^2\left(1+f_{p0}\right)\right)$$

where

$$s_0=\frac{\zeta_k^2}{2\sqrt{3}A_R},\qquad P^2=\frac{\rho C_e cLs_0\gamma_0}{KA_R^2},\qquad 1+f_{p0}=1-\frac{12}{P^2}+\frac{24\tanh\left(\frac{P}{2}\right)}{P^3},$$

the sum of the residues at $s=\pm\imath s_k$ is equal

$$\left[\frac{4F_s q_0\omega\cos(s_k t)}{\zeta_k\left(\omega^2-s_k^2\right)}\left(\frac{A_3(\zeta_k)C(\zeta_k x)+B_3(\zeta_k)S(\zeta_k x)+\left(\bar{C}(\zeta_k x)-2\right)G_3(\zeta_k)}{(\sinh\zeta_k\cos\zeta_k-\cosh\zeta_k\sin\zeta_k)}\right)\right]\times$$
$$\times\left[-2s_kF_s+s_k^2\left(\frac{\delta_I^2 c_I}{L}\right)++\frac{2\beta\beta_0\bar{\beta}c_I\left(1+\frac{s_k\beta_0 c_I}{L}\right)(1+f(p))}{\rho C_e L}+\right. \tag{5.24}$$
$$\left.+\beta\bar{\beta}c_I L\left(1+\left(\frac{s_k\beta_0 c_I}{L}\right)^2\right)(1+2s_kt_0)K^{-1}A_R^{-2}\left(\left(\frac{12\left(1+\sec^2\left(\frac{p}{2}\right)\right)}{p^4}\right)-\left(\frac{36\tanh\left(\frac{p}{2}\right)}{p^5}\right)\right)\right]^{-1},$$

6. Numerical results and graphical explanations

Consider a viscothermoelastic solid like magnesium with the physical specifications as given below:

$$C_e = 1.04\times\frac{10^3\,J}{Kg}deg, \qquad \kappa_0 = 298^o\,K, \qquad \epsilon_0 = \epsilon_1 = 0.779\times10^{-9},$$

$$\epsilon_\kappa = 25\times10^{-6}, \qquad \beta = 2.68\times10^{6}, \qquad q_0 = 2\times10^{-7}.$$

The frequency ω is $0.1076Hz$. Dimensions of the beam are taken as $L = 200\mu m$, $b = 35\mu m$ and $h = 30\mu m$.

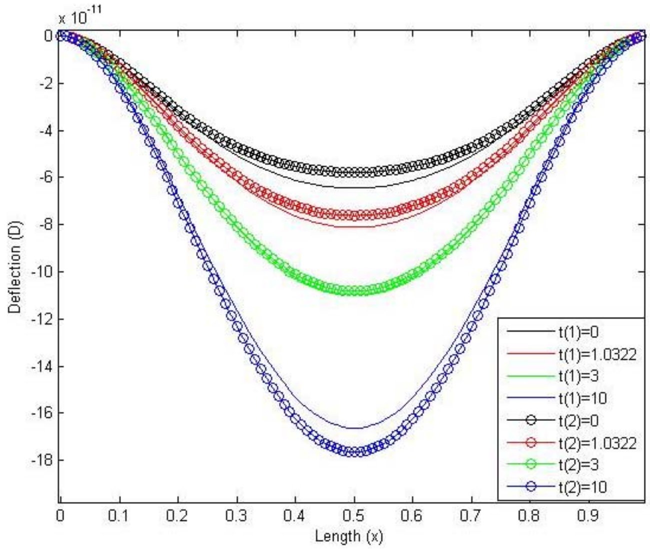

Fig.1. Deflection(D) in CC Viscothermoelastic beam with length (x) at different times for the first and second mode.

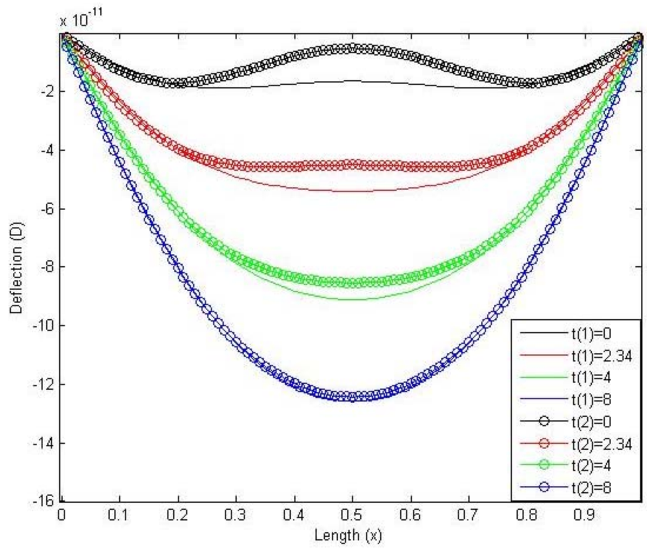

Fig.2. Deflection(D) in SS Viscothermoelastic beam with length (x) at different times for the first and second mode.

The non-dimensional value of relaxation time for CC, SS, CF beams are computed from relation $t_0 = s_0^{-1}$. So the values are given as $t_0 = 1.0322, 2.34, 6.5683$ for the first mode and $t_0 = 0.3744, 0.585, 1.0481$ for the second mode for the CC, SS and CF beam, respectively. Non-dimensional deflection has been evaluated using Eqs (5.18)-(5.24).

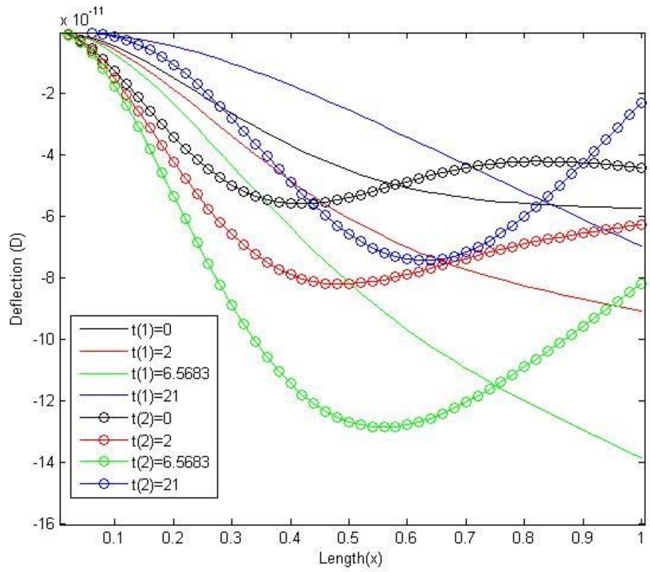

Fig.3. Deflection (D) in CF viscothermoelastic beam with length (x) at different times for the first and second mode.

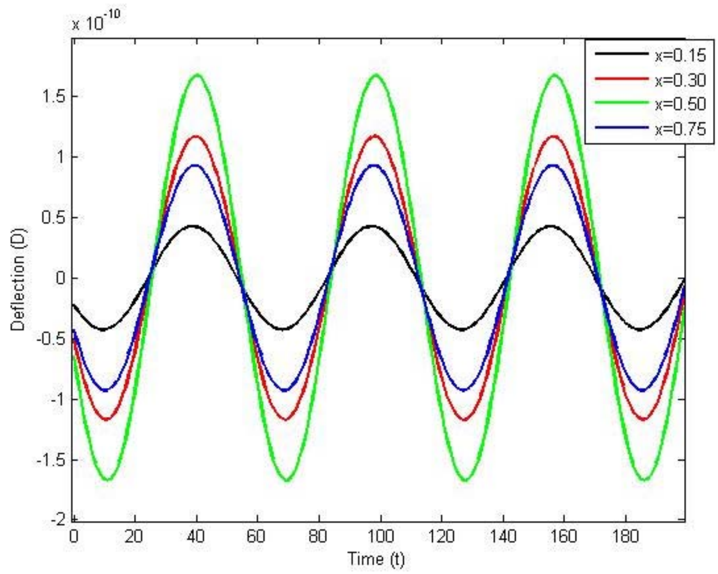

Fig.4. Deflection (D) in CC viscothermoelastic beam with time (t) at different lengths for the first mode.

Figures 1-3 represent the transition of deflection for a viscothermoelastic beam for different boundary conditions (CC, SS, CF) under the effect of harmonic load with respect to the length (x) at different time (t) for the first and second mode. From Figs 1-3 it has been observed that the magnitude of deflection increases with an increase of time except for a cantilever beam at $t = 21$. Also from Figs 1-2 it can be observed that the deflection curve is symmetrical about the middle point of the beam. Also, the deflections near the axial ends are more forceful for a clamped beam in comparison to a simple supported beam.

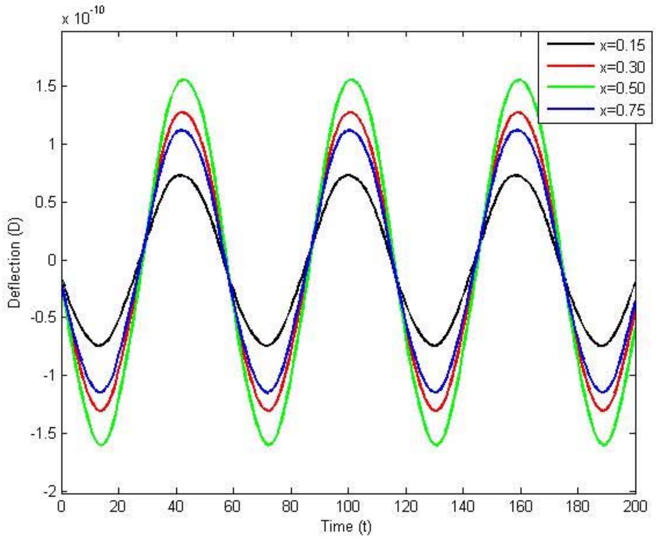

Fig.5. Deflection (D) in SS viscothermoelastic beam with time (t) at different lengths for the first mode.

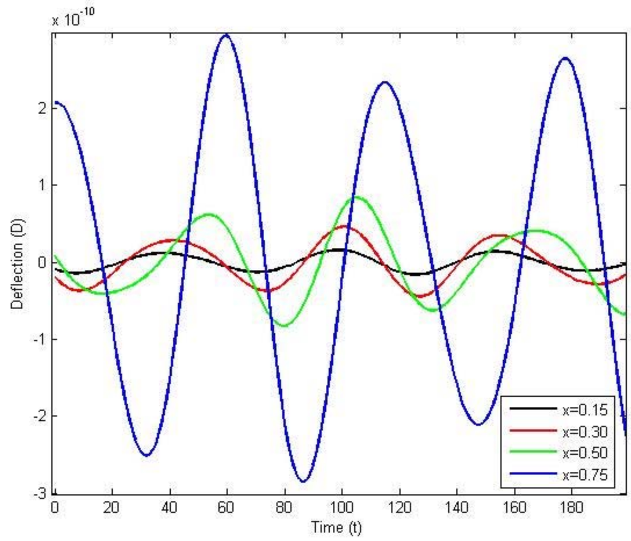

Fig.6. Deflection (D) in CF viscothermoelastic beam with time (t) at different lengths for the first mode.

Figures 4-6 depict the transition of deflection for a viscothermoelastic beam for different boundary conditions (CC, SS, CF) under the effect of harmonic load with respect to time (t) at various values of length (x) for the first mode. From Figs 4-5, it can be seen that maxima of deflection occur at middle spot of the beam. Whereas, with an increase in length, deflection also increases in the case of a cantilever beam (Fig.6). On analysing the magnitude of maximum value of deflection, it is observed that $D_{CC} \geq D_{CF} \geq D_{SS}$.

7. Conclusion

The dynamic response of a homogeneous isotropic viscothermoelastic beam under the action of harmonic loading has been studied. The Laplace transform technique has been used twice with respect to time and space domain. It is infered that

- With an increase in time, the deflecion also increases in the case of CC, SS, CF beams except for the CF beam at relaxation time $t = 21$,
- the deflection curve is symmetrical about the middle spot of the beam for the CC and SS beam,
- maxima of deflection occur at the middle spot of the beam

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Nomenclature

C_e – specific heat

K – thermal conductivity

t_0 – thermal relaxation time

δ_{ij} – Kronecker's delta function

ϵ_0, ϵ_1 – viscoelastic relaxation times

ϵ_κ – linear thermal expansion coefficient

Λ, μ – Lames parameters

ρ – density of medium

σ_{ij} – components of stress tensor

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