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THE TAYLOR TRANSFORMATION HYBRID METHOD APPLIED FOR SOLVING THE STEFAN PROBLEM

Abstract. The paper presents the analytic-numerical hybrid method using, among others, the Taylor transformation, thanks to which the solution of the Stefan problem is replaced by the solution of a nonlinear system of equations.

1. Introduction

A very important task in simulation of the continuous casting process is a possibly exact reconstruction of the temperature field $u(t, x)$ determining, in turn, an essential element of the continuous casting process, that is the interface location [2–4]. Furthermore, location of the interface determines the thickness of a skin, that is the thickness of the solidified layer. Precise retrieval of this parameter is of great importance because when the skin layer increases too slowly, it may generate the financial losses caused by the leakage of liquid metal, whereas when it increases to quickly, it may cause the cracks of the ingot in consequence of the thermal stresses.

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In this paper there is presented an analytic-numerical method for determining an approximate solution of the selected class of problems possible to be replaced by the one-phase problem of solidifying plate with the unknown and varying in time boundary of a region, in which the solution is sought.

Simulation of the heat conduction process, with the solid – liquid phase transitions taken into account, requires the solution of the parabolic type partial differential equations. Moreover, the desire of determining the moving interface location needs, very often, to apply some advanced mathematical methods. Mathematical models, invented for such kind of problems, are usually, in stage of their creation and implementation as well, quite complex from the computational and programming point of view. For solving such kind of problems one can use various approaches, for example the Adomian decomposition method [5–7], methods based on the genetic algorithms and other biologically inspired optimization algorithms [8, 12] and some other methods as well [9–11]. Therefore in this paper we propose to solve the considered problem in a quite different way, by using the, so called, Taylor transformation hybrid method consisted in connecting the Taylor transformation method with the finite difference method. This will be not a typical approach since we do not intend to seek directly the temperature field $u(t,x)$, we will just look for the temperature fields $u_l(x)$ in the discretized time layers t_l.

In other words, the procedure for solving the investigated problem, proposed in this paper, is based on the expansion into the Taylor series [1] of the appropriate functions generated by using the functions $u(t,x)$ discretized with respect to variable t and, in consequence, on the solution of some nonlinear system of equations.

2. The Taylor transformation

Before defining the mathematical model of discussed problem, let us describe shortly the idea of Taylor transformation. We assume that only the functions of real variable t, defined in some region $T \subset \mathbb{R}$, which can be expanded into the Taylor series, will be considered here. These functions will be called as the originals or the transformable functions and will be denoted by the small letters of Latin alphabet, for example f, u, v, and so on. Thus, if function f is the original then the following equality holds true

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k, \qquad t \in T, \tag{1}$$

where $\alpha \in T$ denotes a point, in the neighbourhood of which function f is expanded into the Taylor series.

Each original f corresponds to function F of nonnegative integer arguments $k = 0, 1, 2, \ldots$, according to formula

$$F(k) = \frac{f^{(k)}(\alpha)}{k!}, \qquad k = 0, 1, 2, \ldots \tag{2}$$

Function F will be called as the image of function f, whereas T will be named as the T-function of function f or the transform of function f and the transformation itself will be called as the Taylor transformation or the Taylor transform.

An obvious fact is that if we have the T-function, we can, according to formula (2), find the corresponding original in the form of its expansion into the Taylor series, that is

$$f(t) = \sum_{k=0}^{\infty} F(k)(t-\alpha)^k, \qquad t \in T. \tag{3}$$

Transformation (2), associating the original with its image, will be called as the direct transformation. Whereas transformation (3), associating the image with the corresponding original, will be named as the inverse transformation. The connection between these two transformations, more precisely, their mutual correspondence, will be denoted with symbol $\rightleftharpoons$ and described in the following way

$$f(t) = \sum_{k=0}^{\infty} F(k)(t-\alpha)^k \quad \rightleftharpoons \quad F(k) = \frac{f^{(k)}(\alpha)}{k!}, \qquad k = 0, 1, 2, \ldots \tag{4}$$

Apart from notation (4), there exists also, similarly like in case of the integral transformations [1], the alternative notation for Taylor transformation, that is

$$F(k) = \mathcal{T}\left[f(t); k\right], \tag{5}$$

for the direct transformation and

$$f(t) = \mathcal{T}^{-1}\left[F(k); t\right], \tag{6}$$

for the inverse transformation, where $\mathcal{T}$ and $\mathcal{T}^{-1}$ are the symbols of respective transformations.

Following the taken notational convention, for example for function $f(t) = e^t$ and $\alpha = 0$ we have

$$F(k) = \mathcal{T}[f(t); k] = \mathcal{T}[e^t; k] = \mathcal{T}\left[\sum_{k=0}^{\infty} \frac{t^k}{k!}; k\right] = \frac{1}{k!}, \qquad k = 0, 1, 2, \ldots$$

Whereas in case of the inverse transform for the above function we write

$$f(t) = \mathcal{T}^{-1}[F(k); t] = \mathcal{T}^{-1}\left[\frac{1}{k!}; t\right] = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t.$$

The Taylor transformation possesses a number of useful properties, thanks to which the usage of this tool is quite simple. Here we present few of the most important properties [1]:

$$f(t) = u(t) \pm v(t) \rightleftharpoons F(k) = U(k) \pm V(k), \tag{7}$$

$$f(t) = c \cdot u(t) \rightleftharpoons F(k) = c \cdot U(k), \tag{8}$$

$$f(t) = u(t)v(t) \rightleftharpoons F(k) = \sum_{r=0}^{k} U(r)V(k-r), \tag{9}$$

$$f(t) = u'(t) \rightleftharpoons F(k) = (k+1)U(k+1), \tag{10}$$

$$f(t) = u''(t) \rightleftharpoons F(k) = (k+1)(k+2)U(k+2), \tag{11}$$

$$f(t) = \frac{u(t)}{v(t)} \rightleftharpoons F(k) = \frac{1}{V(0)}\left[U(k) - \sum_{r=0}^{k-1} F(r)V(k-r)\right]. \tag{12}$$

3. Formulation of the problem

Let us define now the mathematical model of discussed problem. Firstly we assume that the conduction is the most essential element determining transport of the heat (which is the consequence of the assumption that temperature of the system is constant in the initial moment of time and it is equal to the phase transition temperature $\bar{u}$). Second assumption concerns the material property – we consider the material as the ideal eutectic which allows us to suppose that the phase transition occurs in the given and constant (as mentioned before) temperature $\bar{u}$. Moreover we assume that the material parameters γ, λ, c, κ and a (density, thermal conductivity, specific heat, latent heat of fusion and thermal diffusivity coefficient, respectively) do not depend on temperature nor on the phase. Addi-

tionally, if we assume that the heat transport is symmetric and one-dimensional, that is the investigated object is a plate with the same cooling conditions from both of its sides, then we can eliminate the liquid phase from our considerations and the one-phase Stefan problem, posed in this way, will be described by the following system of equations:

— the heat conduction equation describing the temperature field in the solid phase

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}, \quad x \in (s(t), x_{max}), \quad t \in (0, t_{max}], \tag{13}$$

— the boundary condition of the first kind on the heat exchange surface

$$u(t, x_{max}) = \varphi(t), \quad t \in (0, t_{max}], \tag{14}$$

— the energy balance conditions in the interface

$$u(t, s(t)) = \bar{u}, \quad t \in [0, t_{max}], \tag{15}$$

$$\lambda\frac{\partial u(t, s(t))}{\partial x} = \kappa s'(t), \quad t \in (0, t_{max}]. \tag{16}$$

In the above equations x denotes the spatial variable, t – time, x_{max} – half of the plate thickness, t_{max} – duration of the process (time of the plate solidification), f – varying in time temperature of the plate surface ($\varphi(t) \leq \bar{u}$), finally s denotes the function describing the varying in time interface location and we have $s(0) = x_{max}$.

4. Method of solution

In the first step of the proposed method of solving the discussed problem we discretize the time variable t, that is we divide the interval $[0, t_{max}]$ with the equidistant points $t_l = l \cdot \Delta t$, $l = 0, 1, \ldots, M$, where $\Delta t = \frac{t_{max}}{M}$, thanks to which the considered spatial-time object takes the layer structure with the layers $t_l \times [s(t_l), x_{max}]$, $l = 0, 1, \ldots, M$. On each of such created layers the temperature field function $u(t, x)$ is the function of one variable x because t is fixed (see Figure 1). Let us call such defined function as the function associated to the given layer and expand it into the Taylor series around the fixed x_{max}. So, according to (3), we have

$$u_l(x) = \sum_{k=0}^{\infty} U(l, k)(x - x_{max})^k, \quad l = 0, 1, 2, \ldots, M, \tag{17}$$

where $U(l,k)$, $k = 0, 1, 2, \ldots$ are the values of the image of function $u_l(x)$. Next we take the N-th partial sum of such received series and we obtain in this way some approximation of this function, that is

$$u_l(x) = \sum_{k=0}^{\infty} U(l,k)(x - x_{max})^k \approx \sum_{k=0}^{N} U(l,k)(x - x_{max})^k, \quad l = 0, 1, \ldots, M \quad (18)$$

where the coefficients $U(l,k)$ – values of T-function U – are unknown for now, except the values $U(0,k)$, $k = 0, 1, \ldots, N$, which, in view of (2), result directly from condition (15), so we have $U(0,k) = 0$, $k = 1, 2, \ldots, N$ and $U(0,0) = \bar{u}$.

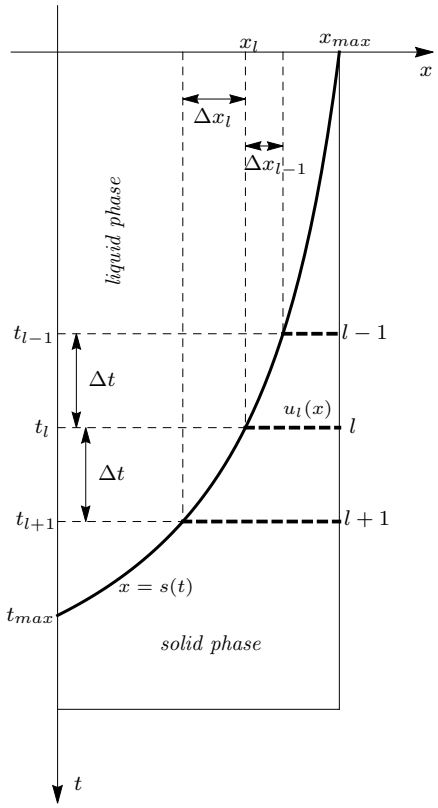

Fig. 1. Geometric interpretation of the discussed problem

By including the introduced discretization and by replacing the differentiation operator on the left hand side of equation (13) with the forward difference operator,

the equation (13) can be written in the form

$$\frac{u_l(x) - u_{l-1}(x)}{\Delta t} = a\frac{d^2u_l(x)}{dx^2}, \quad l = 1, 2, \ldots, M-1, \tag{19}$$

hence, after taking the properties (7), (8) and (11) into account, we obtain the relations for the unknown values of T-function U:

$$U(l, k+2) = \frac{U(l,k) - U(l-1,k)}{a\Delta t(k+1)(k+2)}, \tag{20}$$

for $k = 0, 1, \ldots, N-2$ and $l = 1, 2, \ldots, M$. Moreover, by using the relation (14) and relation (2) we get the succeeding, for $k = 0$, relations for the unknown values of T-function U:

$$U(l, 0) = u_l(x_{max}) = \varphi(l\Delta t), \tag{21}$$

where $l = 1, 2, \ldots, M$.

Applying the condition (15) and property (3) we obtain the next series of relations

$$\sum_{k=0}^{N} U(l,k)(x_l - x_{max})^k = \bar{u}, \tag{22}$$

for $l = 1, 2, \ldots, M$. Whereas from condition (16), by replacing the differentiation operator with the forward difference operator and by including property (10), after simple transformations we get

$$\sum_{k=0}^{N} kU(l,k)(x_l - x_{max})^{k-1} = \frac{\kappa(x_l - x_{l-1})}{\lambda\Delta x}, \tag{23}$$

for $l = 1, 2, \ldots, M$. We have in relations (22) and (23) the other unknowns – the unknown values x_l, $l = 1, 2, \ldots, M$ (x_0 is known and equal to x_{max} – see Figure 1).

Thus we have $M \cdot N$ unknown elements which are the values of $T-$function $U(l,k)$, $k = 1, 2, \ldots, N$, $l = 1, 2, \ldots, M$ and the next M unknown values x_l, $l = 1, 2, \ldots, M$. However, by using relation (20) and the known values of U we can make the unknowns $U(l,k)$, $k = 2, 3, \ldots, M$, $l = 1, 2, \ldots, M$ dependent on the unknowns $U(l,1)$, $l = 1, 2, \ldots, M$. In result we get $2M$ unknown elements: $U(l,1)$, $l = 1, 2, \ldots, M$ and x_l, $l = 1, 2, \ldots, M$. But we have also $2M$ equations (relations (22) and (23)), thus, by solving them, we obtain the values of sought unknowns.

The proposed approach requires to explain two more issues: how the above described system of equations is solved and how do we know the value M which

depends on the unknown value t_{max}, which, in turn, is connected with the value M. It appears that these questions are strictly joined: let us notice that instead of solving the full system of $2M$ equations, we can solve this system sequentially, by solving each time, for the successive $l = 1, 2, \ldots, M$, the system of two equations with two variables – we select each time one equation from relations (22) and one equation from relations (23), respectively. Thanks to this approach, in the numerical method of solving such reduced system we select as the zero approximation (starting point) the solution of the previous system and the value M will be determined by the found value x_l – if it differs from zero less than the fixed $\varepsilon > 0$, then we take that the solidification process has terminated which implies the termination of computing.

5. Example

Let the sought temperature field function has the form $u(t, x) = 4 - e^{t+x-1}$. Additionally, let us take that $x_{max} = 1$, $\lambda = \kappa = a = 1$ and let us fix the value Δt. Under these assumptions we know that $\varphi(t) = 4 - e^t$, $s(t) = 1 - t$, hence $\bar{u} = 3$.

In the successive figures there are presented the selected reconstructions of the interface location for the varying values of elements responsible for the density of discretization Δt and for the number N of terms in expansions into the Taylor series of functions $u_l(x)$, $l = 1, 2, \ldots, M$, and also the plots of temperature fields in the time layers t_l (the exact temperatures $u(t_l, x)$ and approximate $u_l(x)$ – obtained on the basis of formula (18)) for the successive $l = 1, 2, \ldots, M$. The cases have been selected so that they confirm the effectiveness of investigated method. We show that for the given density of time variable t discretization, more precisely – for the fixed Δt, one can choose the number N so that the reconstruction of the interface location would be of the expected good quality. We also show that when the number N increases, the quality of reconstruction improves as well. The successive pairs of plots are the following:

— first pair of plots (Figures 2, 4, 6, 8, 10):

  — left hand side – interface location (solid line) and its reconstruction (points),

  — right hand side – values of the absolute errors $\delta(l)$ of this reconstruction (points) determined for the given t_l, $l = 1, 2, \ldots, M$, from relation $\delta_l = |s(x_l) - t_l| = |(1 - x_l) - t_l|$, where x_l denote the values computed

from the system of equations (22)–(23). For better illustration of results these points have been connected by the line segments;

— second pair of plots (Figures 3, 5, 7, 9, 11):

  — left hand side – M pairs of plots of the temperature fields (the exact ones – solid line and calculated from formula (18) – dashed line) in the selected layers t_l,

  — right hand side – plots of the absolute errors δu_l of reconstructing the temperature fields $u_l(x)$ in the selected layers t_l, determined from relation $|u_l(x) - u(t_l, x)|$.

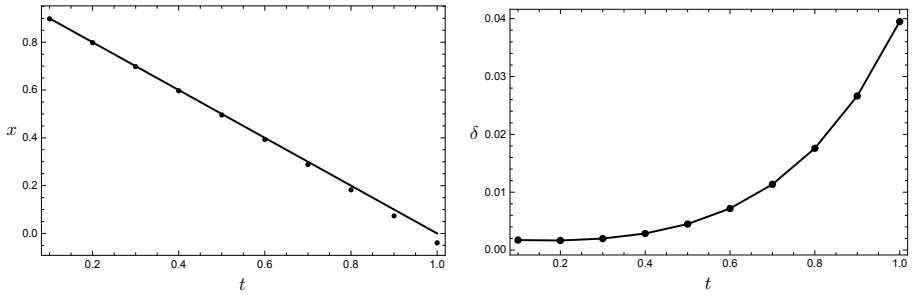

Fig. 2. Reconstruction of the solidification front and absolute errors of this reconstruction in layers t_l, $l = 1, 2, \ldots, M$, for $N = 3$, $\Delta t = 0.1$ and $M = 10$

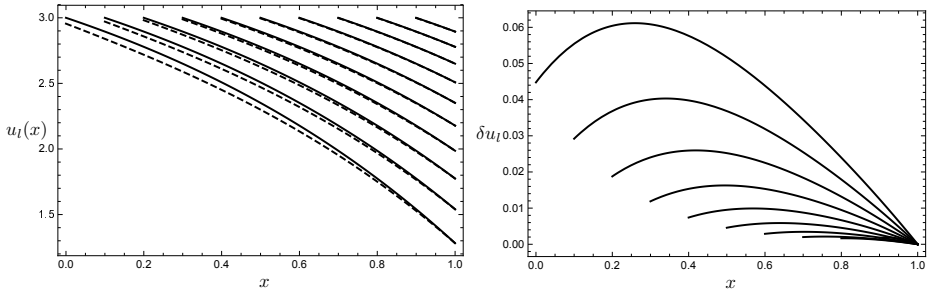

Fig. 3. Reconstruction of the temperature fields and absolute errors of this reconstruction in layers t_l, $l = 1, 2, \ldots, M$, for $N = 3$, $\Delta t = 0.1$ and $M = 10$

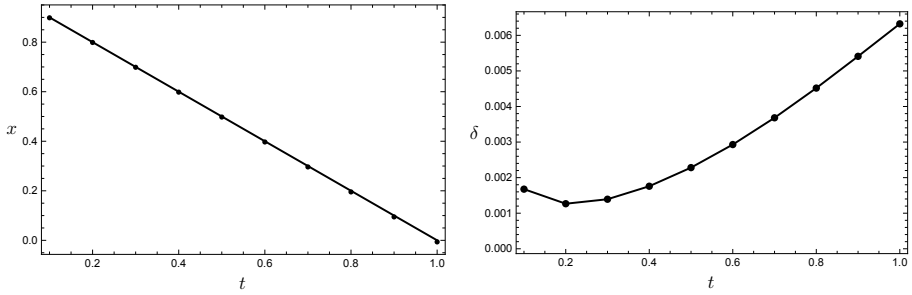

Fig. 4. Reconstruction of the solidification front and absolute errors of this reconstruction in layers t_l, $l = 1, 2, \ldots, M$, for $N = 6$, $\Delta t = 0.1$ and $M = 10$

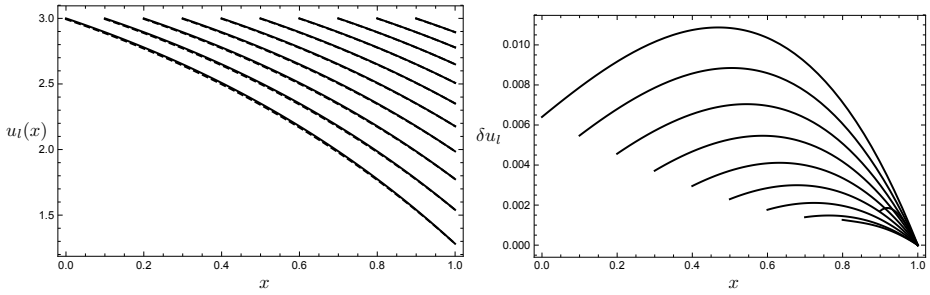

Fig. 5. Reconstruction of the temperature fields and absolute errors of this reconstruction in layers t_l, $l = 1, 2, \ldots, M$, for $N = 6$, $\Delta t = 0.1$ and $M = 10$

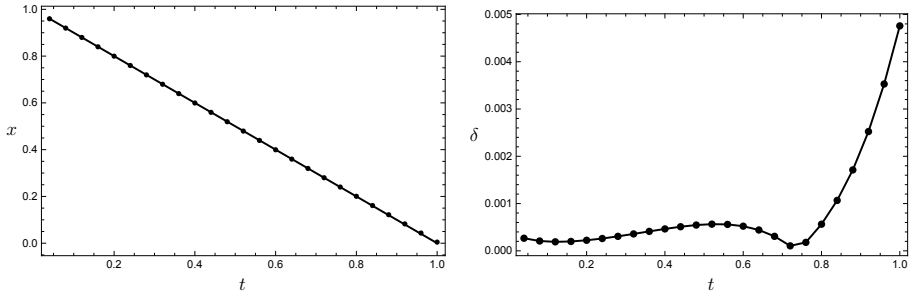

Fig. 6. Reconstruction of the solidification front and absolute errors of this reconstruction in layers t_l, $l = 1, 2, \ldots, M$, for $N = 4$, $\Delta t = 0.04$ and $M = 25$

6. Conclusion

The examples, presented above, confirm the effectiveness of proposed method. It appears that for the given discretization one can choose the appropriate number of terms in series (17) so that the obtained reconstruction would be of satisfying

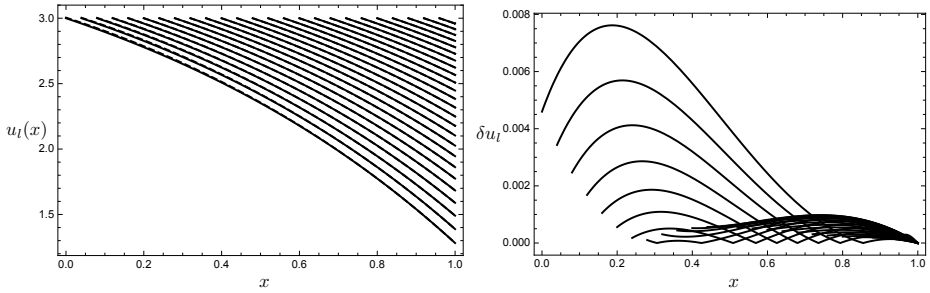

Fig. 7. Reconstruction of the temperature fields and absolute errors of this reconstruction in layers t_l, $l = 1, 2, \ldots, M$, for $N = 4$, $\Delta t = 0.04$ and $M = 25$

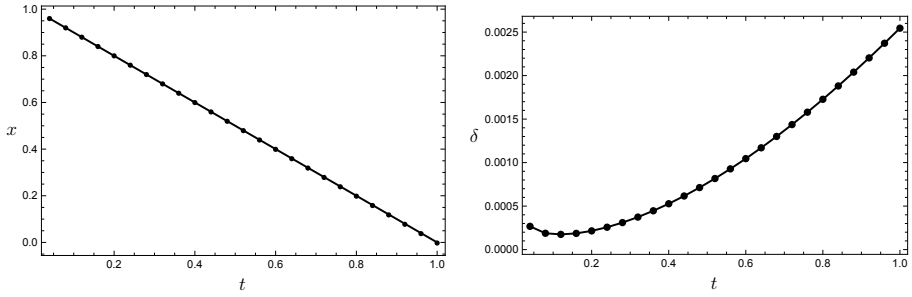

Fig. 8. Reconstruction of the solidification front and absolute errors of this reconstruction in layers t_l, $l = 1, 2, \ldots, M$, for $N = 8$, $\Delta t = 0.04$ and $M = 25$

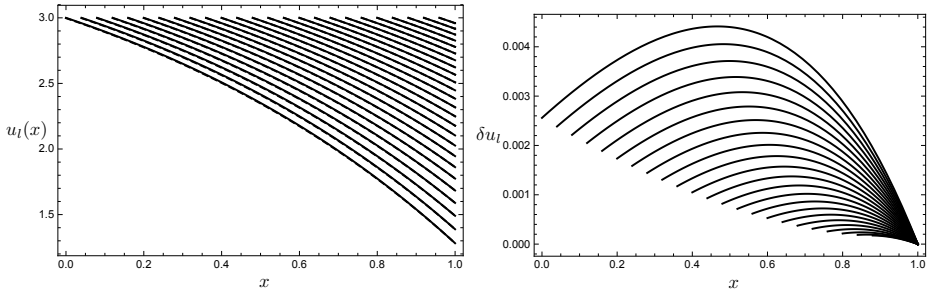

Fig. 9. Reconstruction of the temperature fields and absolute errors of this reconstruction in layers t_l, $l = 1, 2, \ldots, M$, for $N = 8$, $\Delta t = 0.04$ and $M = 25$

quality. The reconstruction errors behave rationally – when the number of terms in the series increases, the error decreases. Also when the density of variable t discretization is greater, the error becomes respectively smaller and, similarly like for the lower value of parameter N, by increasing the number of terms in the series we get the reconstruction of good quality.

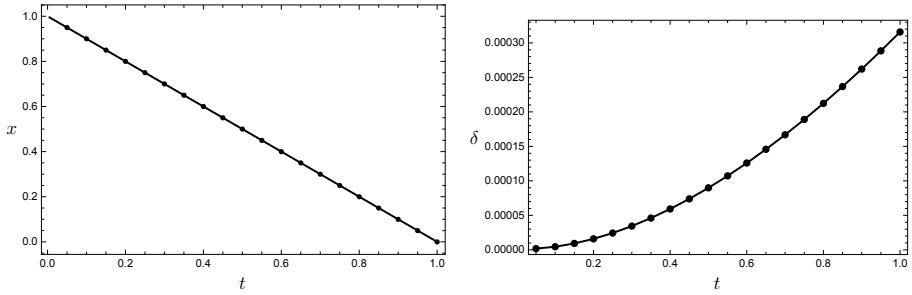

Fig. 10. Reconstruction of the solidification front and absolute errors of this reconstruction in layers t_l, $l = 10, 20, \ldots, M$, for $N = 12$, $\Delta t = 0.005$ and $M = 200$

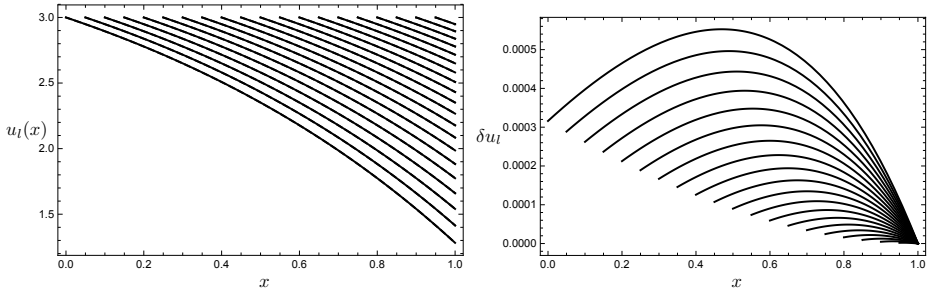

Fig. 11. Reconstruction of the temperature fields and absolute errors of this reconstruction in layers t_l, $l = 10, 20, \ldots, M$, for $N = 15$, $\Delta t = 0.005$ and $M = 200$

References

1. Grzymkowski R.: *Taylor transformation and its applications.* Jacek Skalmierski Computer Studio, Gliwice 2015 (in Polish).
2. Grzymkowski R., Hetmaniok E., Pleszczyński M.: *Analytic-numerical method of determining the freezing front location.* Arch. Foundry Eng. **11**, no. 3 (2011), 75–80.
3. Grzymkowski R., Hetmaniok E., Pleszczyński M., Słota D.: *A certain analytical method used for solving the Stefan problem.* Therm. Sci. **17** (2013), 635–642.
4. Grzymkowski R., Pleszczyński M., Hetmaniok E.: *Problem of the moving boundary in continuous casting solved by the analytic-numerical method.* Arch. Foundry Eng. **13**, no. 1 (2013), 33–38.
5. Grzymkowski R., Pleszczyński M., Słota D.: *Application of the Adomian decomposition method for solving the heat equation in the cast-mould heterogeneous domain.* Arch. Foundry Eng. **9**, no. 4 (2009), 57–62.

6. Grzymkowski R., Pleszczyński M., Słota D.: *The two-phase Stefan problem solved by the Adomian decomposition method*. Proceedings of the 15th IASTED International Conference Applied Simulation and Modelling, ACTA Press, Rhodos, 2006, 511–516.
7. Hetmaniok E., Słota D., Wituła R., Zielonka A.: *Comparison of the Adomian decomposition method and the variational iteration method in solving the moving boundary problem*. Comput. Math. Appl. **61** (2011), 1931–1934.
8. Hetmaniok E., Słota D., Zielonka A., Wituła R.: *Comparison of ABC and ACO Algorithms Applied for Solving the Inverse Heat Conduction Problem*. Lect. Notes Comput. Sc. **7269** (2012), 249–257.
9. Majchrzak E., Mochnacki B., Dziewoński M., Jasiński M.: *Identification of boundary heat flux on the continuous casting surface*. Arch. Foundry Eng. **8**, no. 4 (2008), 105–110.
10. Mendakiewicz J., Piasecka Belkhayat A., Szopa, R.: *Modeling of the Stefan Problem Using the BEM*. Solidification of Metals and Alloys **2**, no. 44 (2000), 223–228.
11. Mochnacki B., Pawlak E.: *Identification of boundary condition on the contact surface of continuous casting mould*. Arch. Foundry Eng. **7** no. 4 (2007), 202–206.
12. Słota D.: *Solving the inverse Stefan design problem using genetic algorithms*. Inverse Probl. Sci. En. **16** (2008), 829–846.