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ON EFFICIENCY AND DUALITY FOR A CLASS OF NONCONVEX NONDIFFERENTIABLE MULTIOBJECTIVE FRACTIONAL VARIATIONAL CONTROL PROBLEMS

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Abstract. In this paper, we consider the class of nondifferentiable multiobjective fractional variational control problems involving the nondifferentiable terms in the numerators and in the denominators. Under univexity and generalized univexity hypotheses, we prove optimality conditions and various duality results for such nondifferentiable multiobjective fractional variational control problems. The results established in the paper generalize many similar results established earlier in the literature for such nondifferentiable multiobjective fractional variational control problems.

Keywords: nondifferentiable multiobjective fractional variational control problem, efficient solution, optimality conditions, (generalized) univexity, Mond–Weir duality, Wolfe duality.

Mathematics Subject Classification: 65K10, 90C32, 90C46, 90C30, 90C26.

1. INTRODUCTION

The term multiobjective programming is used to denote a type of optimization problems where two or more objectives are to be minimized/maximized subject to certain constraints. Multiobjective variational control programming is an interesting subject that appears in diverse branches of operational research, for instance, in industrial process control, the control of production and inventory, information theory, impulsive control problems, heat exchange networking, biomedicine, flight control design, in the control of space structures, numerical analysis and many other areas of modern human activity. Therefore, investigation of optimality conditions and/or duality for multiobjective variational control programming problems has been one of the most attracting topics in the theory of nonlinear programming. In recent years, most of the optimality conditions and duality results that are available in the literature of continuous time programming have been established for such extremum problems involving generalized convex functions (see, for example, [1–4, 6, 7, 10, 12, 18, 19, 21, 23, 24, 36, 37, 39–41], and others). That is the case of nondifferentiable multiobjective fractional variational control problems, which cover a wide range of types of extremum problems studied in the literature. In this regard, the nondifferentiable multiobjective variational/control problems were investigated by Husain and Jain [15], Husain and Mattoo [16]. Mishra and Mukherjee [25] studied nondifferentiable multiobjective fractional problems under invexity. Under generalized invexity, Nahak and Nanda [34] obtained several duality results for multiobjective fractional control problems. Park and Jeong [35] faced duality for multiobjective fractional control problems under (F, ρ)-convexity. Later, Mititelu [26], Mititelu and Postolache [28], and Mititelu and Stancu-Minasian [29] got conditions on efficiency and duality results for multiobjective fractional variational problems, under (b, ρ)-quasiinvexity, for instance.

In the paper, the nondifferentiable multiobjective fractional variational control problem with equality and inequality restrictions is considered involving nondifferentiable terms in the numerators and the denominators of each objective function. Our aim in this paper is to provide optimality conditions and various duality results for such nonconvex nonsmooth multiobjective fractional continuous-time problems. In our approach, the usual convexity requirement for the involved functionals is relaxed to univexity and/or generalized univexity. Our definitions of univexity and generalized univexity are more general than those existing in the literature. Therefore, the optimality conditions and various duality theorems in the sense of Mond–Weir and in the sense of Wolfe established for the considered nondifferentiable multiobjective fractional variational control problem generalize and extend a number of results existing in the literature for such nonsmooth vector continuous-time optimization problems.

This work is organized as follows. In Section 2, we define a nondifferentiable multiobjective fractional variational control problem involving the nondifferentiable terms in the numerators and in the denominators of each objective function considered in the paper. Also, we introduce some denotations and present a number of definitions which will be needed in the sequel. In Section 3, we give the definition of univexity and the definitions of generalized univexity in the continuous vectorial case. In Section 4, we establish the sufficient optimality conditions for the considered nondifferentiable multiobjective fractional variational control problem. In order to prove these results, we use the definitions of univexity and generalized univexity introduced in the preceding section. Subsequently, in Sections 5 and 6, we apply the optimality conditions to formulate vector dual problems in the sense of Mond–Weir and in the sense of Wolfe, respectively, and we prove weak, strong, converse and strict converse duality theorems for nondifferentiable multiobjective fractional variational control problems involving the nondifferentiable terms in the numerators and in the denominators also under appropriate univexity and generalized univexity hypotheses. We derive duality results using also the so-called generalized Schwarz inequality. In Section 7, we conclude the results established in the paper. Moreover, we show that the results established in this paper for such nonconvex nondifferentiable multiobjective variational control problems are more general than those ones in a fairly large number of similar works.

2. PRELIMINARIES, NOTATIONS AND THE NONDIFFERENTIABLE MULTIOBJECTIVE FRACTIONAL VARIATIONAL CONTROL PROBLEM

The following convention for equalities and inequalities will be adopted in the paper.

For any $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$,
(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, \ldots, n$,
(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, \ldots, n$,
(iv) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

Let R^n be the n-dimensional Euclidean space and denote by $R^n_+ = \{y \in R^n : y \geqq 0\}$ and $R^n_{++} = \{y \in R^n : y > 0\}$ the nonnegative orthant of R^n and its interior, respectively. Moreover, let $I = [a, b]$ be a real interval and let $P = \{1, 2, \ldots, p\}$, $J = \{1, 2, \ldots, l\}$ and $K = \{1, \ldots, s\}$.

In this paper, we shall assume that $x(t)$ is an n-dimensional piecewise smooth function of t, and $\dot{x}(t)$ is the derivative of $x(t)$ with respect to t in $[a, b]$. Further, we denote by X the space of piecewise smooth functions $x : I \to R^n$ with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$z = Dx \Longleftrightarrow x(t) = x(a) + \int_a^t z(s)ds,$$

where $x(a)$ is a given boundary value. Therefore, $\frac{d}{dt} \equiv D$ except at discontinuities. Further, we denote by U the space of piecewise smooth control functions $u : I \to R^m$, with norm $\|u\|_\infty$.

The multiobjective fractional variational control problem is to choose, under given conditions, a control $u(t)$ such that the state vector $x(t)$ is brought from the specified initial state $x(a) = \alpha$ to some specified final state $x(b) = \beta$ in such a way to minimize a given vector-valued fractional functional. A more precise mathematical formulation is given in the following optimization problem (MFP):

$$\operatorname*{Minimize}_{x,u} \left(\frac{\int_a^b \left\{ f^1(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) + \sqrt{x(t)^T A_1(t) x(t)} + \sqrt{u(t)^T B_1(t) u(t)} \right\} dt}{\int_a^b \left\{ q^1(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - \sqrt{x(t)^T C_1(t) x(t)} - \sqrt{u(t)^T E_1(t) u(t)} \right\} dt}, \right.$$
$$\ldots,$$
$$\left. \frac{\int_a^b \left\{ f^p(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) + \sqrt{x(t)^T A_p(t) x(t)} + \sqrt{u(t)^T B_p(t) u(t)} \right\} dt}{\int_a^b \left\{ q^p(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - \sqrt{x(t)^T C_p(t) x(t)} - \sqrt{u(t)^T E_p(t) u(t)} \right\} dt} \right)$$

subject to
$$g(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \leqq 0, \quad t \in I,$$
$$h(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) = 0, \quad t \in I,$$
$$x(a) = \alpha, \quad x(b) = \beta,$$

where

$$f=\left(f^1,\ldots,f^p\right):I\times R^n\times R^n\times R^m\times R^m\to R^p,$$
$$q=(q^1,\ldots,q^p):I\times R^n\times R^n\times R^m\times R^m\to R^p$$

are p-dimensional functions and each their component is a continuously differentiable real scalar function,

$$g=(g^1,\ldots,g^l):I\times R^n\times R^n\times R^m\times R^m\to R^l$$

and

$$h=\left(h^1,\ldots,h^s\right):I\times R^n\times R^n\times R^m\times R^m\to R^s$$

are assumed to be continuously differentiable l-dimensional and s-dimensional functions, respectively. For each $t\in I$, let $A_i(t)$, $C_i(t)$, $1,\ldots,p$, be piecewise positive semidefinite $n\times n$ matrices with $A_i(\cdot)$ and $C_i(\cdot)$ continuous on I and, moreover, $B_i(t)$, $E_i(t)$, $i=1,\ldots,p$, be piecewise positive semidefinite $m\times m$ matrices with $B_i(\cdot)$ and $E_i(\cdot)$ continuous on I. Further, we assume that

$$f^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t))+\sqrt{x(t)^TA_i(t)x(t)}+\sqrt{u(t)^TB_i(t)u(t)}\geqq 0,\quad i\in P,$$
$$q^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t))-\sqrt{x(t)^TC_i(t)x(t)}-\sqrt{u(t)^TE_i(t)u(t)}>0,\quad i\in P.$$

For notational simplicity, we write $x(t)$ and $\dot{x}(t)$ as x and $\dot{x}$, respectively. We denote the partial derivatives of f^i, $i=1,\ldots,p$, with respect to t, x and $\dot{x}$, respectively, by f^i_t, f^i_x, $f^i_{\dot{x}}$ such that $f^i_x=(\frac{\partial f^i}{\partial x_1},\ldots\frac{\partial f^i}{\partial x_n})$ and $f^i_{\dot{x}}=(\frac{\partial f^i}{\partial \dot{x}_1},\ldots,\frac{\partial f^i}{\partial \dot{x}_n})$. In the similar manner, we formulate f^i_u, $f^i_{\dot{u}}$, $i=1,\ldots,p$. Similarly, the partial derivatives of the vector-valued function g and the vector-valued function h can be written, using matrices with l rows and s rows instead of one, respectively.

Let S denote the set of all feasible solutions in (MFP), i.e.

$$S=\{(x,u):x\in X,u\in U\text{ verifying the constraints of (MFP)}\}.$$

Definition 2.1. $(\bar{x},\bar{u})\in S$ is said to be a weakly efficient solution for (MFP) if there is no other $(x,u)\in S$ such that

$$\frac{\int_a^b\left\{f^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t))+\sqrt{x(t)^TA_i(t)x(t)}+\sqrt{u(t)^TB_i(t)u(t)}\right\}dt}{\int_a^b\left\{q^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t))-\sqrt{x(t)^TC_i(t)x(t)}-\sqrt{u(t)^TE_i(t)u(t)}\right\}dt}$$
$$<\frac{\int_a^b\left\{f^i\left(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t),\dot{\bar{u}}(t)\right)+\sqrt{\bar{x}(t)^TA_i(t)\bar{x}(t)}+\sqrt{\bar{u}(t)^TB_i(t)\bar{u}(t)}\right\}dt}{\int_a^b\left\{q^i\left(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t),\dot{\bar{u}}(t)\right)-\sqrt{\bar{x}(t)^TC_i(t)\bar{x}(t)}-\sqrt{\bar{u}(t)^TE_i(t)\bar{u}(t)}\right\}dt},\quad i\in P.$$

Definition 2.2. $(\bar{x}, \bar{u}) \in S$ is said to be an efficient solution for (MFP) if there is no other $(x, u) \in S$ such that

$$\frac{\int_a^b \left\{ f^i(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) + \sqrt{x(t)^T A_1(t) x(t)} + \sqrt{u(t)^T B_1(t) u(t)} \right\} dt}{\int_a^b \left\{ q^i(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - \sqrt{x(t)^T C_1(t) x(t)} - \sqrt{u(t)^T E_1(t) u(t)} \right\} dt}$$
$$\leqq \frac{\int_a^b \left\{ f^i\left(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{u}(t), \dot{\bar{u}}(t)\right) + \sqrt{\bar{x}(t)^T A_i(t) \bar{x}(t)} + \sqrt{\bar{u}(t)^T B_i(t) \bar{u}(t)} \right\} dt}{\int_a^b \left\{ q^i\left(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{u}(t), \dot{\bar{u}}(t)\right) - \sqrt{\bar{x}(t)^T C_i(t) \bar{x}(t)} - \sqrt{\bar{u}(t)^T E_i(t) \bar{u}(t)} \right\} dt}, \quad i \in P$$

with at least one strict inequality for some $i \in P$.

In order to prove the results in this paper, we use the inequality which follows directly from the generalized Schwarz inequality (see Liu [21, Lemma 3.1]).

Now, let $\Gamma(t)$ be an $n \times n$ positive semidefinite symmetric matrix for each $t \in I$, with $\Gamma(\cdot)$ continuous on I. The following generalized Schwarz inequality is required in the sequel:

$$x^T \Gamma w \leqq \sqrt{x^T \Gamma x} \sqrt{w^T \Gamma w} \quad \text{for all} \quad x, w \in R^n, \tag{2.1}$$

where the equality holds when $\Gamma x = \beta \Gamma w$ for some $\beta \geqq 0$.

Hence, if $w^T \Gamma w \leqq 1$, then we have

$$x^T \Gamma w \leqq \sqrt{x^T \Gamma x}. \tag{2.2}$$

In [8], Bector *et al.* introduced the definition of a univex function as a generalization of invexity defined by Hanson [13]. In this section, we generalize the aforesaid definitions of univexity and generalized univexity notions to the continuous vectorial case.

Following the notational convenience, we use

$$\varphi(t, x, \dot{x}, u, \dot{u}) \quad \text{for} \quad \varphi(t, x(t), \dot{x}(t), u(t), \dot{u}(t)).$$

Let $\Psi : X \times U \to R^p$ be defined by

$$\Psi(x, u) = \int_a^b \varphi(t, x, \dot{x}, u, \dot{u}) dt,$$

where $\varphi = (\varphi^1, \dots, \varphi^p) : I \times X \times X \times U \times U \to R^p$ and $(\bar{x}, \bar{u}) \in X \times U$ be given.

Definition 2.3. If there exist

$$b = (b_1, \dots, b_p) : X \times U \times X \times U \to R^p_+,$$
$$\Phi = (\Phi_1, \dots, \Phi_p) : R^p \to R^p,$$
$$\eta : I \times R^n \times R^n \times R^m \times R^m \times R^n \times R^n \times R^m \times R^m \to R^n$$

and

$$\theta : I \times R^n \times R^n \times R^m \times R^m \times R^n \times R^n \times R^m \times R^m \to R^m$$

such that the inequalities

$$\begin{aligned} & b_i(x,u,\bar{x},\bar{u})\Phi_i(\Psi_i(x,u)-\Psi_i(\bar{x},\bar{u})) \\ & \geqq \int_a^b \Big\{ [\eta(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]^T \left[\varphi_x^i\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right) - \frac{d}{dt}\varphi_{\dot{x}}^i\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right) \right] \\ & \qquad + \left[\theta\left(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\right]^T \left[\varphi_u^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \frac{d}{dt}\varphi_{\dot{u}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \right] \Big\} dt, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1,\ldots,p \end{aligned} \tag{2.3}$$

hold for every $(x,u) \in X \times U$, then Ψ is said to be a (vector-valued) univex functional at $(\bar{x},\bar{u}) \in X \times U$ on $X \times U$ (with respect to b, Φ, η, θ). If inequalities (2.3) are satisfied for each $(\bar{x},\bar{u}) \in X \times U$, then Ψ is said to be a (vector-valued) univex functional on $X \times U$ (with respect to b, Φ, η, θ). If inequalities (2.3) are satisfied for every $(x,u),(\bar{x},\bar{u}) \in A$, where A is a nonempty subset of $X \times U$, then Ψ is said to be a (vector-valued) univex functional on A (with respect to b, Φ, η, θ).

Now, we give the example of a univex functional.

Example 2.4. Let

$$A = \left\{(x,u) \in R \times R : u(t) > 0 \text{ for all } t \in [a,b]\right\}$$

and $\Psi : A \to R$ be defined by

$$\Psi(x,u) = \int_a^b \frac{x^2(t)}{u(t)} dt.$$

Let us define $b(x,u,\bar{x},\bar{u}) = 1$ for each $(x,u),(\bar{x},\bar{u}) \in A$, $\Phi(a) = e^a - 1$,

$$\eta(t,x,u,\bar{x},\bar{u}) = \begin{bmatrix} \frac{\bar{u}}{u}(x-\bar{x}) \\ \frac{\bar{u}}{u}(u-\bar{u}) \end{bmatrix}.$$

Then, it can be shown by Definition 2.3 that the functional Ψ is univex on the set A with respect to b, Φ and η given above.

Remark 2.5. For some properties of a class of univex functions, the readers are advised to consult [8].

Definition 2.6. If there exist

$$\begin{aligned} & b = (b_1,\ldots,b_p) : X \times U \times X \times U \to R_+^p, \\ & \Phi = (\Phi_1,\ldots,\Phi_p) : R^p \to R^p, \\ & \eta : I \times R^n \times R^n \times R^m \times R^m \times R^n \times R^n \times R^m \times R^m \to R^n \end{aligned}$$

and

$$\theta : I \times R^n \times R^n \times R^m \times R^m \times R^n \times R^n \times R^m \times R^m \to R^m$$

such that the relation

$$\sum_{i=1}^{p}\int_{a}^{b}\Big\{[\eta(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]^T\Big[\varphi_x^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\frac{d}{dt}\varphi_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\Big]$$
$$+[\theta(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]^T\Big[\varphi_u^i\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)-\frac{d}{dt}\varphi_{\dot{u}}^i\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\Big]\Big\}dt\geqq 0 \tag{2.4}$$

$$\Longrightarrow \sum_{i=1}^{p} b_i(x,u,\bar{x},\bar{u})\Phi_i\left(\Psi_i(x,u)-\Psi_i(\bar{x},\bar{u})\right)\geqq 0,$$

holds for every $(x,u)\in X\times U$, then the functional Ψ is said to be pseudo-univex at $(\bar{x},\bar{u})\in X\times U$ on $X\times U$ (with respect to b, Φ, η, θ). If (2.4) is satisfied for each $(\bar{x},\bar{u})\in X\times U$, then the functional Ψ is said to be pseudo-univex on $X\times U$ (with respect to b, Φ, η, θ). If inequalities (2.4) are satisfied for every $(x,u),(\bar{x},\bar{u})\in A$, where A is a nonempty subset of $X\times U$, then Ψ is said to be a (vector-valued) pseudo-univex functional on A (with respect to b, Φ, η, θ).

Definition 2.7. If there exist

$$b=(b_1,\ldots,b_p):X\times U\times X\times U\to R_+^p,$$
$$\Phi=(\Phi_1,\ldots,\Phi_p):R^p\to R^p,$$
$$\eta: I\times R^n\times R^n\times R^m\times R^m\times R^n\times R^n\times R^m\times R^m\to R^n$$

and $\theta: I\times R^n\times R^n\times R^m\times R^m\times R^n\times R^n\times R^m\times R^m\to R^m$ such that the relation

$$\sum_{i=1}^{p}\int_{a}^{b}\Big\{[\eta(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]^T\left[\varphi_x^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\frac{d}{dt}\varphi_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right]$$
$$+[\theta(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]^T\left[\varphi_u^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\frac{d}{dt}\varphi_{\dot{u}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right]\Big\}dt\geqq 0 \tag{2.5}$$

$$\Longrightarrow \sum_{i=1}^{p} b_i(x,u,\bar{x},\bar{u})\Phi_i\left(\Psi_i(x,u)-\Psi_i(\bar{x},\bar{u})\right)>0,$$

holds for every $(x,u)\in X\times U$, $(x,u)\neq(\bar{x},\bar{u})$, then the functional Ψ is said to be strictly pseudo-univex at $(\bar{x},\bar{u})\in X\times U$ on $X\times U$ (with respect to b, Φ, η, θ). If (2.5) is satisfied for each $(\bar{x},\bar{u})\in X\times U$, then the functional Ψ is said to be strictly pseudo-univex on $X\times U$ (with respect to b, Φ, η, θ). If inequalities (2.5) are satisfied for every $(x,u),(\bar{x},\bar{u})\in A$, $(x,u)\neq(\bar{x},\bar{u})$, where A is a nonempty subset of $X\times U$, then Ψ is said to be a (vector-valued) strictly pseudo-univex functional on A (with respect to b, Φ, η, θ).

Definition 2.8. If there exist

$$b=(b_1,\ldots,b_p):X\times U\times X\times U\to R^p_+,$$
$$\Phi=(\Phi_1,\ldots,\Phi_p):R^p\to R^p,$$
$$\eta:I\times R^n\times R^n\times R^m\times R^m\times R^n\times R^n\times R^m\times R^m\to R^n$$

and

$$\theta:I\times R^n\times R^n\times R^m\times R^m\times R^n\times R^n\times R^m\times R^m\to R^m$$

such that the relation

$$\sum_{i=1}^{p}b_i(x,u,\bar{x},\bar{u})\Phi_i\left[\Psi_i(x,u)-\Psi_i\left(\bar{x},\bar{u}\right)\right]\leqq 0\Longrightarrow \tag{2.6}$$
$$\sum_{i=1}^{p}\int_a^b\Big\{[\eta(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]^T\Big[\varphi^i_x(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\frac{d}{dt}\varphi^i_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\Big]$$
$$+[\theta(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]^T\Big[\varphi^i_u\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)-\frac{d}{dt}\varphi^i_{\dot{u}}\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\Big]\Big\}dt\leqq 0$$

holds for every $(x,u)\in X\times U$, then the functional Ψ is said to be quasi-univex at $(\bar{x},\bar{u})\in X\times U$ on $X\times U$ (with respect to b, Φ, η, θ). If (2.6) is satisfied for each $(\bar{x},\bar{u})\in X\times U$, then the functional Ψ is said to be quasi-univex on $X\times U$ (with respect to b, Φ, η, θ). If inequalities (2.6) are satisfied for every $(x,u),(\bar{x},\bar{u})\in A$, where A is a nonempty subset of $X\times U$, then Ψ is said to be a (vector-valued) quasi-univex functional on A (with respect to b, Φ, η, θ).

3. OPTIMALITY CONDITIONS

First, we write the necessary optimality conditions for the multiobjective fractional variational control problem (MFP), using the relationship between a weakly efficient solution (an efficient solution) of the problem (MFP) and a weakly efficient solution (an efficient solution) of the associated auxiliary vector nonfractional variational control problem.

Now, we consider, therefore, the auxiliary vector nonfractional variational control problem (MFP)(z) for the considered multiobjective fractional variational control problem (MFP) as follows:

$$\begin{aligned}
\underset{x,u}{\text{Minimize}} \Big(\int_a^b \Big\{ & f^1(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\
& + \sqrt{x(t)^T A_1(t) x(t)} + \sqrt{u(t)^T B_1(t) u(t)} \\
& - z_1 \Big[q^1(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\
& \quad - \sqrt{x(t)^T C_1(t) x(t)} - \sqrt{u(t)^T E_1(t) u(t)} \Big] \Big\} dt, \\
\dots, \\
\int_a^b \Big\{ & f^p(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\
& + \sqrt{x(t)^T A_p(t) x(t)} + \sqrt{u(t)^T B_p(t) u(t)} \\
& - z_p \Big[q^p(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\
& \quad - \sqrt{x(t)^T C_p(t) x(t)} - \sqrt{u(t)^T E_p(t) u(t)} \Big] \Big\} dt \Big) \\
\text{subject to} \quad & g(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \leqq 0, \quad t \in I, \\
& h(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) = 0, \quad t \in I, \\
& x(a) = \alpha, \quad x(b) = \beta,
\end{aligned}$$

where $z = (z_1, \dots, z_p) \in R^p_+$ is a parameter. We denote by $S(z)$ the set of all feasible solutions of (MFP)(z).

Definition 3.1. $(\bar{x}, \bar{u}) \in S(\bar{z})$ is said to be a weakly efficient solution of (MFP)($\bar{z}$) iff there is no another feasible solution (x, u) of (MFP)($\bar{z}$) such that the inequalities

$$\begin{aligned}
& \int_a^b \Big\{ f^i(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) + \sqrt{x(t)^T A_i x(t)} + \sqrt{u(t)^T B_i u(t)} \\
& \quad - \bar{z}_i \left[q^i(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) - \sqrt{x(t) C_i(t) x(t)} - \sqrt{u(t)^T E_i(t) u(t)} \right] \Big\} dt \\
& < \int_a^b \Big\{ f^i(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{u}(t), \dot{\bar{u}}(t)) + \sqrt{\bar{x}(t)^T A_i \bar{x}(t)} + \sqrt{\bar{u}(t)^T B_i \bar{u}(t)} \\
& \quad - \bar{z}_i \left[q^i(t, \bar{x}(t), \dot{\bar{x}}(t), \bar{u}(t), \dot{\bar{u}}(t)) - \sqrt{\bar{x}(t)^T C_i \bar{x}(t)} + \sqrt{\bar{u}(t)^T E_i \bar{u}(t)} \right] \Big\} dt, \ \forall i \in P
\end{aligned}$$

hold.

Definition 3.2. $(\bar{x},\bar{u}) \in S(\bar{z})$ is said to be an efficient solution of (MFP)$(\bar{z})$ iff there is no another feasible solution (x,u) of (MFP)$(\bar{z})$ such that the following inequalities hold:

$$\begin{aligned}
&\int_a^b \Big\{ f^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) + \sqrt{x(t)^T A_i x(t)} + \sqrt{u(t)^T B_i u(t)} \\
&\quad - \bar{z}_i \left[q^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) - \sqrt{x(t) C_i(t) x(t)} - \sqrt{u(t)^T E_i(t) u(t)} \right] \Big\} dt \\
&\leqq \int_a^b \Big\{ f^i(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t),\dot{\bar{u}}(t)) + \sqrt{\bar{x}(t)^T A_i \bar{x}(t)} + \sqrt{\bar{u}(t)^T B_i \bar{u}(t)} \\
&\quad - \bar{z}_i \left[q^i(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t),\dot{\bar{u}}(t)) - \sqrt{\bar{x}(t)^T C_i \bar{x}(t)} + \sqrt{\bar{u}(t)^T E_i \bar{u}(t)} \right] \Big\} dt, \ \forall i \in P,
\end{aligned}$$

$$\begin{aligned}
&\int_a^b \Big\{ f^{i^*}(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) + \sqrt{x(t)^T A_{i^*} x(t)} + \sqrt{u(t)^T B_{i^*} u(t)} \\
&\quad - \bar{z}_{i^*} \left[q^{i^*}(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) - \sqrt{x(t) C_{i^*}(t) x(t)} - \sqrt{u(t)^T E_{i^*}(t) u(t)} \right] \Big\} dt \\
&< \int_a^b \Big\{ f^{i^*}(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t),\dot{\bar{u}}(t)) + \sqrt{\bar{x}(t)^T A_{i^*} \bar{x}(t)} + \sqrt{\bar{u}(t)^T B_{i^*} \bar{u}(t)} \\
&\quad - \bar{z}_{i^*} \left[q^{i^*}\big(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t),\dot{\bar{u}}(t)\big) - \sqrt{\bar{x}(t)^T C_{i^*} \bar{x}(t)} + \sqrt{\bar{u}(t)^T E_{i^*} \bar{u}(t)} \right] \Big\} dt \\
&\hspace{20em} \text{for some } i^* \in P.
\end{aligned}$$

Now, we give the result which connects a weakly efficient solution (an efficient solution) of (MFP) with a weakly efficient solution (an efficient solution) of its associated vector control problem (MFP)(z).

Lemma 3.3. $(\bar{x},\bar{u}) \in S$ *is a weakly efficient solution* (*an efficient solution*) *of the considered multiobjective fractional control problem* (MFP) *if and only if there exists* $\bar{z} \in R^p_+$ *such that* $(\bar{x},\bar{u})$ *is a weakly efficient solution* (*an efficient solution*) *of its associated vector control problem* (MFP)$(\bar{z})$ *and, moreover,*

$$\bar{z}_i = \frac{\int_a^b \Big\{ f^i\big(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t),\dot{\bar{u}}(t)\big) + \sqrt{\bar{x}(t)^T A_i(t)\bar{x}(t)} + \sqrt{\bar{u}(t)^T B_i(t)\bar{u}(t)} \Big\} dt}{\int_a^b \Big\{ q^i\big(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t),\dot{\bar{u}}(t)\big) - \sqrt{\bar{x}(t)^T C_i(t)\bar{x}(t)} - \sqrt{\bar{u}(t)^T E_i(t)\bar{u}(t)} \Big\} dt}, \ \forall i \in P. \tag{3.1}$$

Proof. Let $(\bar{x}, \bar{u}) \in S$ be a weakly efficient solution of (MFP) and $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_p) \in R^p$ be defined by (3.1). We proceed by contradiction. Suppose, contrary to the results, that there exists $(\tilde{x}, \tilde{u}) \in S$ such that

$$\begin{aligned}
&\int_a^b \Big\{ f^i(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}) + \sqrt{\tilde{x}^T A_i \tilde{x}} + \sqrt{\tilde{u}^T B_i \tilde{u}} \\
&\qquad - \bar{z}_i \Big[q^i(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}) - \sqrt{\tilde{x}^T C_i \tilde{x}} - \sqrt{\tilde{u}^T E_i \tilde{u}} \Big] \Big\} dt \\
&< \int_a^b \Big\{ f^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sqrt{\bar{x}^T A_i \bar{x}} + \sqrt{\bar{u}^T B_i \bar{u}} \\
&\qquad - \bar{z}_i \Big[q^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \sqrt{\bar{x}^T C_i \bar{x}} + \sqrt{\bar{u}^T E_i \bar{u}} \Big] \Big\} dt, \quad \forall i \in P.
\end{aligned}$$

Hence, by (3.1), we have

$$\begin{aligned}
&\int_a^b \Big\{ f^i(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}) + \sqrt{\tilde{x}^T A_i \tilde{x}} + \sqrt{\tilde{u}^T B_i \tilde{u} dt} \Big\} dt \\
&< \frac{\int_a^b \Big\{ f^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sqrt{\bar{x}^T A_i \bar{x}} + \sqrt{\bar{u}^T B_i \bar{u}} \Big\} dt}{\int_a^b \Big\{ q^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \sqrt{\bar{x}^T C_i \bar{x}} - \sqrt{\bar{u}^T E_i \bar{u}} \Big\} dt} \\
&\quad \times \int_a^b \Big\{ q^i(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}) - \sqrt{\tilde{x}^T C_i \tilde{x}} - \sqrt{\tilde{u}^T E_i \tilde{u}} \Big\} dt, \quad \forall i \in P.
\end{aligned}$$

Thus, the inequalities

$$\begin{aligned}
&\frac{\int_a^b \Big\{ f^i\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}\right) + \sqrt{\tilde{x}^T A_i \tilde{x}} + \sqrt{\tilde{u}^T B_i \tilde{u} dt} \Big\} dt}{\int_a^b \Big\{ q^i\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}\right) - \sqrt{\tilde{x}^T C_i \tilde{x}} - \sqrt{\tilde{u}^T E_i \tilde{u}} \Big\} dt} \\
&< \frac{\int_a^b \Big\{ f^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sqrt{\bar{x}^T A_i \bar{x}} + \sqrt{\bar{u}^T B_i \bar{u}} \Big\} dt}{\int_a^b \Big\{ q^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \sqrt{\bar{x}^T C_i \bar{x}} - \sqrt{\bar{u}^T E_i \bar{u}} \Big\} dt}, \quad \forall i \in P
\end{aligned} \tag{3.2}$$

hold, contradicting the assumption that $(\bar{x}, \bar{u}) \in S$ is a weakly efficient solution of (MFP).

Conversely, suppose that $(\bar{x}, \bar{u}) \in S$ is not a weakly efficient solution of (MFP). This means that there exists $(\tilde{x}, \tilde{u}) \in S$ such that

$$\frac{\int_a^b \left\{ f^i(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}) + \sqrt{\tilde{x}^T A_i \tilde{x}} + \sqrt{\tilde{u}^T B_i \tilde{u} dt} \right\} dt}{\int_a^b \left\{ q^i\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}\right) - \sqrt{\tilde{x}^T C_i \tilde{x}} - \sqrt{\tilde{u}^T E_i \tilde{u}} \right\} dt} < \frac{\int_a^b \left\{ f^i\left(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right) + \sqrt{\bar{x}^T A_i \bar{x}} + \sqrt{\bar{u}^T B_i \bar{u}} \right\} dt}{\int_a^b \left\{ q^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \sqrt{\bar{x}^T C_i \bar{x}} - \sqrt{\bar{u}^T E_i \bar{u}} \right\} dt} = \bar{z}_i, \quad \forall i \in P.$$

Hence, the following inequalities

$$\begin{aligned} &\int_a^b \Big\{ f^i(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}) + \sqrt{\tilde{x}^T A_i \tilde{x}} + \sqrt{\tilde{u}^T B_i \tilde{u}} \\ &\qquad - \bar{z}_i \left[q^i(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}) - \sqrt{\tilde{x}^T C_i \tilde{x}} - \sqrt{\tilde{u}^T E_i \tilde{u}} \right] \Big\} dt \\ &< 0 = \int_a^b \Big\{ f^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sqrt{\bar{x}^T A_i \bar{x}} + \sqrt{\bar{u}^T B_i \bar{u}} \\ &\qquad\qquad - \bar{z}_i \left[q^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \sqrt{\bar{x}^T C_i \bar{x}} + \sqrt{\bar{u}^T E_i \bar{u}} \right] \Big\} dt, \quad \forall i \in P \end{aligned}$$

hold, contradicting the assumption that $(\bar{x}, \bar{u})$ is a weakly efficient solution (an efficient solution) of its associated vector control problem (MFP)$(\bar{z})$. This completes the proof of this lemma. □

Now, we present the theorem which is the continuous version of the result given in [22].

Theorem 3.4. *A solution $(\bar{x}, \bar{u}) \in S(\bar{z})$ is an efficient solution in the vector control problem* (MFP)$(\bar{z})$ *if and only if $(\bar{x}, \bar{u})$ solves each scalar control problem* $(\mathrm{FPP})_{i^*}(\bar{x}, \bar{u})$:

$$\begin{aligned} \textit{Minimize} \int_a^b \Big\{ & f^{i^*}(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) + \sqrt{x(t)^T A_{i^*}(t) x(t)} + \sqrt{u(t)^T B_{i^*}(t) u(t)} \\ & - \bar{z}_{i^*} \Big[q^{i^*}(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\ & \qquad - \sqrt{x(t) C_{i^*}(t) x(t)} - \sqrt{u(t) E_{i^*}(t) u(t)} \Big] \Big\} dt \\ \textit{subject to} \quad & g(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \leqq 0, \quad t \in I, \\ & h(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) = 0, \quad t \in I, \\ & x(a) = \alpha, \quad x(b) = \beta, \end{aligned}$$

$$\int_a^b \Big\{ f^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) + \sqrt{x(t)^T A_i(t) x(t)} + \sqrt{u(t)^T B_i(t) u(t)}$$
$$- \bar{z}_i \Big[q^i(t,x(t),\dot{x}(t),u(t),\dot{u}(t)) - \sqrt{x(t)^T C_i(t) x(t)} - \sqrt{u(t)^T E_i(t) u(t)} \Big] \Big\} dt$$
$$\leqq \int_a^b \Big\{ f^i(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t),\dot{\bar{u}}(t)) + \sqrt{\bar{x}(t)^T A_i(t) \bar{x}(t)} + \sqrt{\bar{u}(t)^T B_i(t) \bar{u}(t)}$$
$$- \bar{z}_i \Big[q^i(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t),\dot{\bar{u}}(t)) - \sqrt{\bar{x}(t)^T C_i(t) \bar{x}(t)} + \sqrt{\bar{u}(t)^T E_i(t) \bar{u}(t)} \Big] \Big\} dt,$$
$$\forall i \neq i^*.$$

In order to prove sufficient optimality conditions for the considered multiobjective variational programming problem (MFP), we recall the Karush–Kuhn–Tucker necessary optimality conditions for this vector optimization problem. This theorem is the continuous version of Theorem 3.1 of [38] in fractional multiobjective programming and also Theorem 2.1 of [17].

Theorem 3.5. *Let* $(\bar{x}, \bar{u})$ *be a normal weakly efficient solution of the vector optimization problem* (MFP). *Then there exist* $\bar{\lambda} \in R^p$, $\bar{z} \in R^p$ *and piecewise smooth functions* $\bar{\xi}(\cdot) : I \to R^l$, $\bar{\zeta}(\cdot) : I \to R^s$, $\bar{r}(\cdot) : I \to R^n$, $\bar{w}(\cdot) : I \to R^n$, $\bar{\delta}(\cdot) : I \to R^m$, $\bar{\vartheta}(\cdot) : I \to R^m$ *such that*

$$\begin{aligned}
&\sum_{i=1}^p \bar{\lambda}_i \Big\{ \big[f_x^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + A_i(t)\bar{r}(t) \big] \\
&\qquad - \bar{z}_i \big[q_x^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - C_i(t)\bar{w}(t) \big] \Big\} \\
&+ \bar{\xi}(t)^T g_x(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \bar{\zeta}(t)^T h_x(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \\
&= \frac{d}{dt}\Big(\sum_{i=1}^p \bar{\lambda}_i \big\{ f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{z}_i q_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \big\} \\
&\qquad + \bar{\xi}(t)^T g_{\dot{x}}\big(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\big) + \bar{\zeta}(t)^T h_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \Big), \quad t \in I,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\sum_{i=1}^p \bar{\lambda}_i \Big\{ \big[f_u^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + B_i(t)\bar{\delta}(t) \big] \\
&\qquad - \bar{z}_i \big[q_u^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - E_i(t)\bar{\vartheta}(t) \big] \Big\} \\
&+ \bar{\xi}(t)^T g_u(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \bar{\zeta}(t)^T h_u(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \\
&= \frac{d}{dt}\Big(\sum_{i=1}^p \bar{\lambda}_i \big\{ f_{\dot{u}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{z}_i q_{\dot{u}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \big\} \\
&\qquad + \bar{\xi}(t)^T g_{\dot{u}}\big(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\big) + \bar{\zeta}(t)^T h_{\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \Big), \quad t \in I,
\end{aligned} \tag{3.4}$$

$$\bar{\xi}(t)^T g(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) = 0, \tag{3.5}$$

$$\int_a^b \left\{ f^i(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t),\dot{\bar{u}}(t)) + \sqrt{\bar{x}(t)^T A_i(t)\bar{x}(t)} + \sqrt{\bar{u}(t)^T B_i(t)\bar{u}(t)} \right\} dt$$
$$- \bar{z}_i \left[\int_a^b \left\{ q^i\left(t,\bar{x}(t),\dot{\bar{x}}(t),\bar{u}(t),\dot{\bar{u}}(t)\right) - \sqrt{\bar{x}(t)^T C_i(t)\bar{x}(t)} - \sqrt{\bar{u}(t)^T E_i(t)\bar{u}(t)} \right\} dt \right]$$
$$= 0, \quad i \in P, \tag{3.6}$$

$$\bar{x}^T(t)A_i(t)\bar{r}(t) = \sqrt{\bar{x}(t)^T A_i(t)\bar{x}(t)}, \quad \bar{u}(t)^T B_i(t)\bar{\delta}(t) = \sqrt{\bar{u}(t)^T B_i \bar{u}(t)}, \tag{3.7}$$

$$\begin{aligned} \bar{x}(t)^T C_i(t)\bar{w}(t) &= \sqrt{\bar{x}(t)^T C_i(t)\bar{x}(t)}, \\ \bar{u}(t)^T E_i(t)\bar{\vartheta}(t) &= \sqrt{\bar{u}(t)^T E_i(t)\bar{u}(t)}, \quad t \in I,\ i = 1,\ldots,p, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \bar{r}(t)^T A_i(t)\bar{r}(t) \leqq 1, \quad \bar{\delta}(t)^T B_i(t)\bar{\delta}(t) \leqq 1, \\ \bar{w}(t)^T C_i(t)\bar{w}(t) \leqq 1, \quad \bar{\vartheta}(t)^T E_i(t)\bar{\vartheta}(t) \leqq 1, \quad t \in I,\ i = 1,\ldots,p, \end{aligned} \tag{3.9}$$

$$\bar{\lambda} \geq 0, \quad \bar{\lambda}^T e = 1, \quad \bar{\xi}(t) \geqq 0. \tag{3.10}$$

Now, we prove the sufficiency of the above necessary optimality conditions under appropriate univexity and generalized univexity hypotheses. For notational convenience, we use $\bar{\xi}$ for $\bar{\xi}(t)$ and $\bar{\zeta}$ for $\bar{\zeta}(t)$.

Theorem 3.6. *Let* $(\bar{x},\bar{u})$ *be a feasible solution in the considered multiobjective variational programming problem* (MFP) *and the necessary optimality conditions* (3.3)–(3.10) *be satisfied at* $(\bar{x},\bar{u})$ *with* $\bar{\lambda} \in R^p$ *and piecewise smooth functions* $\bar{\xi}(\cdot) : I \to R^l$ *and* $\bar{\zeta}(\cdot) : I \to R^s$, $\bar{r}(\cdot) : I \to R^n$, $\bar{w}(\cdot) : I \to R^n$, $\bar{\delta}(\cdot) : I \to R^m$, $\bar{\vartheta}(\cdot) : I \to R^m$. *Further, assume that the following hypotheses are fulfilled:*

(a)

$$\Bigg(\int_a^b \Big\{ f^1(t,\cdot,\cdot,\cdot,\cdot) + \cdot^T A_1(t)\bar{r}(t) + \cdot^T B_1(t)\bar{\delta}(t)$$
$$- \bar{z}_1 \left[q^1(t,\cdot,\cdot,\cdot,\cdot) - \cdot^T C_1(t)\bar{w}(t) - \cdot^T E_1(t)\bar{\vartheta}(t) \right] \Big\} dt,$$
$$\ldots,$$
$$\int_a^b \Big\{ f^p(t,\cdot,\cdot,\cdot,\cdot) + \cdot^T A_p(t)\bar{r}(t) + \cdot^T B_p(t)\bar{\delta}(t)$$
$$- \bar{z}_p \left[q^p(t,\cdot,\cdot,\cdot,\cdot) - \cdot^T C_p(t)\bar{w}(t) - \cdot^T E_p(t)\bar{\vartheta}(t) \right] \Big\} dt \Bigg)$$

is a strictly univex function (*a univex function*) *at* $(\bar{x},\bar{u})$ *on* S *with respect to* $b = (b_1,\ldots,b_p)$, $\Phi = (\Phi_1,\ldots,\Phi_p)$, η, θ,

(b)

$$\left(\int_a^b \bar{\xi}_1(t)g^1(t,\cdot,\cdot,\cdot,\cdot)dt,\dots,\int_a^b \bar{\xi}_l(t)g^l(t,\cdot,\cdot,\cdot,\cdot)dt\right)$$

is univex at $(\bar{x},\bar{u})$ *on* S *with respect to* $b_g=(b_{g_1},\dots,b_{g_l})$, $\Phi_g=(\Phi_{g_1},\dots,\Phi_{g_l})$, η, θ,

(c)

$$\left(\int_a^b \bar{\zeta}_1(t)h^1(t,\cdot,\cdot,\cdot,\cdot)dt,\dots,\int_a^b \bar{\zeta}_s(t)h^s(t,\cdot,\cdot,\cdot,\cdot)dt\right)$$

is univex at $(\bar{x},\bar{u})$ *on* S *with respect to* $b_h=(b_{h_1},\dots,b_{h_s})$, $\Phi_h=(\Phi_{h_1},\dots,\Phi_{h_s})$, η, θ,

(d) $b_i(x,u,\bar{x},\bar{u})>0$, $i=1,\dots,p$, *for every* $(x,u)\in X\times U$,

(e) Φ_i, $i=1,\dots,p$, *are strictly increasing functionals satisfying* $a<0\Longrightarrow\Phi_i(a)<0$, $\Phi_i(0)\leqq 0$,

(f) Φ_{g_j}, $j\in J$, *are increasing functionals satisfying* $a\leqq 0\Longrightarrow\Phi_{g_j}(a)\leqq 0$,

(g) Φ_{h_k}, $k\in K$, *are increasing functionals satisfying* $a\leqq 0\Longrightarrow\Phi_{h_k}(a)\leqq 0$.

Then $(\bar{x},\bar{u})$ *is an efficient solution* (*a weakly efficient solution*) *of* (MFP).

Proof. Suppose, contrary to the result, that $(\bar{x},\bar{u})$ is not an efficient solution of (MFP). Then, there exists $(\tilde{x},\tilde{u})\in S$ such that

$$\begin{aligned}&\int_a^b \Big\{f^i(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}})+\sqrt{\tilde{x}^TA_i\tilde{x}}+\sqrt{\tilde{u}^TB_i\tilde{u}}\\&\qquad-\bar{z}_i\Big[q^i(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}})-\sqrt{\tilde{x}^TC_i\tilde{x}}-\sqrt{\tilde{u}^TE_i\tilde{u}}\Big]\Big\}dt\\&\leqq\int_a^b \Big\{f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sqrt{\bar{x}^TA_i\bar{x}}+\sqrt{\bar{u}^TB_i\bar{u}}\\&\qquad-\bar{z}_i\Big[q^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\sqrt{\bar{x}^TC_i\bar{x}}-\sqrt{\bar{u}^TE_i\bar{u}}\Big]\Big\}dt,\quad\forall i\in P,\end{aligned}\tag{3.11}$$

$$\begin{aligned}&\int_a^b \Big\{f^{i^*}(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}})+\sqrt{\tilde{x}^TA_{i^*}\tilde{x}}+\sqrt{\tilde{u}^TB_{i^*}\tilde{u}}\\&\qquad-\bar{z}_{i^*}\Big[q^{i^*}(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}})-\sqrt{\tilde{x}^TC_{i^*}\tilde{x}}-\sqrt{\tilde{u}^TE_{i^*}\tilde{u}}\Big]\Big\}dt\\&<\int_a^b \Big\{f^{i^*}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\sqrt{\bar{x}^TA_{i^*}\bar{x}}+\sqrt{\bar{u}^TB_{i^*}\bar{u}}\\&\qquad-\bar{z}_{i^*}\Big[q^{i^*}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\sqrt{\bar{x}^TC_{i^*}\bar{x}}-\sqrt{\bar{u}^TE_{i^*}\bar{u}}\Big]\Big\}dt\quad\text{for some } i^*\in P.\end{aligned}\tag{3.12}$$

By the necessary optimality condition (3.7) and the generalized Schwarz inequality (see (2.2)), the inequalities (3.11) and (3.12) yield

$$\begin{aligned}&\int_a^b \Big\{ f^i(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}) + \tilde{x}^T A_i \bar{r} + \tilde{u}^T B_i \bar{\delta} \\ &\qquad - \bar{z}_i \left[q^i(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}) - \tilde{x}^T C_i \bar{w} - \tilde{u}^T E_i \bar{\vartheta} \right] \Big\} dt \\ &\leqq \int_a^b \Big\{ f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \bar{x}^T A_i \bar{r} + \bar{u}^T B_i \bar{\delta} \\ &\qquad - \bar{z}_i \left[q^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{x}^T C_i \bar{w} - \bar{u}^T E_i \bar{\vartheta} \right] \Big\} dt, \quad \forall i \in P,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&\int_a^b \Big\{ f^{i^*}(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}) + \tilde{x}^T A_{i^*} \bar{r} + \tilde{u}^T B_{i^*} \bar{\delta} \\ &\qquad - \bar{z}_{i^*} \left[q^{i^*}(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}) - \tilde{x}^T C_{i^*} \bar{w} - \tilde{u}^T E_{i^*} \bar{\vartheta} \right] \Big\} dt \\ &< \int_a^b \Big\{ f^{i^*}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \bar{x}^T A_{i^*} \bar{r} + \bar{u}^T B_{i^*} \bar{\delta} \\ &\qquad - \bar{z}_{i^*} \left[q^{i^*}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{x}^T C_{i^*} \bar{w} - \bar{u}^T E_{i^*} \bar{\vartheta} \right] \Big\} dt \quad \text{for some } i^* \in P.\end{aligned} \tag{3.14}$$

By hypotheses (d) and (e), it follows that $b_i(\tilde{x},\tilde{u},\bar{x},\bar{u}) > 0$, $i = 1,\ldots,k$, and Φ_i, $i = 1,\ldots,k$, are strictly increasing functionals with $\Phi_i(0) \leqq 0$. Thus, (3.13) and (3.14) give

$$\begin{aligned}&b_i(\tilde{x},\tilde{u},\bar{x},\bar{u}) \Phi_i \Bigg(\int_a^b \Big\{ f^i\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}\right) + \tilde{x}^T A_i \bar{r} + \tilde{u}^T B_i \bar{\delta} \\ &\qquad - \bar{z}_i \left[q^i(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}) - \tilde{x}^T C_i \bar{w} - \tilde{u}^T E_i \bar{\vartheta} \right] \Big\} dt \\ &\qquad - \int_a^b \Big\{ f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \bar{x}^T A_i \bar{r} + \bar{u}^T B_i \bar{\delta} \\ &\qquad - \bar{z}_i \left[q^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{x}^T C_i \bar{w} - \bar{u}^T E_i \bar{\vartheta} \right] \Big\} dt \Bigg) \\ &\leqq b_i(\tilde{x},\tilde{u},\bar{x},\bar{u}) \Phi_i(0) \leqq 0, \quad \forall i \in P,\end{aligned} \tag{3.15}$$

$$
\begin{aligned}
b_{i^*}(\tilde{x},\tilde{u},\bar{x},\bar{u})\Phi_{i^*}\Bigg(\int_a^b \Big\{ & f^{i^*}\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}\right)+\tilde{x}^T A_{i^*}\bar{r}+\tilde{u}^T B_{i^*}\bar{\delta} \\
& -\bar{z}_{i^*}[q^{i^*}(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}})-\tilde{x}^T C_{i^*}\bar{w}-\tilde{u}^T E_{i^*}\bar{\vartheta}]\Big\}dt \\
-\int_a^b \Big\{ & f^{i^*}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\bar{x}^T A_{i^*}\bar{r}+\bar{u}^T B_{i^*}\bar{\delta} \\
& -\bar{z}_{i^*}[q^{i^*}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\bar{x}^T C_{i^*}\bar{w}-\bar{u}^T E_{i^*}\bar{\vartheta}]\Big\}dt\Bigg) \\
< b_{i^*}(\tilde{x},\tilde{u},\bar{x},\bar{u})\Phi_{i^*}(0) \leqq 0 \text{ for some } i^* \in P. &
\end{aligned} \tag{3.16}
$$

Since the hypotheses (a)–(c) are fulfilled, by Definition 2.3, the inequalities

$$
\begin{aligned}
b_i(\tilde{x},\tilde{u},\bar{x},\bar{u})\Phi_i\Bigg(\int_a^b \Big\{ & f^i\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}\right)+\tilde{x}^T A_i\bar{r}+\tilde{u}^T B_i\bar{\delta} \\
& -\bar{z}_i\left[q^i(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}})-\tilde{x}^T C_i\bar{w}-\tilde{u}^T E_i\bar{\vartheta}\right]\Big\}dt \\
-\int_a^b \Big\{ & f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\bar{x}^T A_i\bar{r}+\bar{u}^T B_i\bar{\delta} \\
& -\bar{z}_i\left[q^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\bar{x}^T C_i\bar{w}-\bar{u}^T E_i\bar{\vartheta}\right]\Big\}dt\Bigg) \\
> \int_a^b \Big\{ & \left[\eta(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right]^T\Big(f_x^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+A_i(t)\bar{r}(t) \\
& -\bar{z}_i[q_x^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-C_i(t)\bar{w}(t)] \\
& -\frac{d}{dt}\left[f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\bar{z}_i q_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right]\Big) \\
& +\left[\theta(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right]^T\Big(f_u^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+B_i(t)\bar{\delta}(t) \\
& -\bar{z}_i[q_u^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-E(t)\bar{\vartheta}(t)] \\
& -\frac{d}{dt}[f_{\dot{u}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\bar{z}_i q_{\dot{u}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]\Big)\Big\}dt, \\
& i \in P,
\end{aligned} \tag{3.17}
$$

$$b_{g_j}(\tilde{x},\tilde{u},\bar{x},\bar{u})\Phi_{g_j}\left(\int_a^b \bar{\xi}_j g^j\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}\right)dt-\int_a^b \bar{\xi}_j(t)g^j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})dt\right)$$
$$\geqq \int_a^b \Big\{ \left[\eta\left(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\right]^T\left[\bar{\xi}_j g^j_x(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\frac{d}{dt}\left(\bar{\xi}_j g^j_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\right]$$
$$+\Big(\left[\theta\left(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\right]^T\Big[\bar{\xi}_j g^j_u(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$
$$-\frac{d}{dt}\left(\bar{\xi}_j g^j_{\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Big]\Big)\Big\}dt, \quad j\in J(t), \tag{3.18}$$

$$b_{h_k}(\tilde{x},\tilde{u},\bar{x},\bar{u})\Phi_{h_k}\left(\int_a^b \bar{\zeta}_k h^k\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}\right)dt-\int_a^b \bar{\zeta}_k h^k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})dt\right)$$
$$\geqq \int_a^b \Big\{[\eta\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)]^T\Big[\bar{\zeta}_k h^k_x(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\frac{d}{dt}\left(\bar{\zeta}_k h^k_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Big]$$
$$+\Big([\theta\left(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)]^T\Big[\bar{\zeta}_k h^k_u\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)$$
$$-\frac{d}{dt}\left(\bar{\zeta}_k h^k_{\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)\Big]\Big)\Big\}dt, \quad k\in K \tag{3.19}$$

hold. Combining (3.15), (3.16) and (3.17), we get

$$\begin{aligned}\int_a^b \Big\{&\left[\eta\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\right]^T\Big(f^i_x(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+A_i(t)\bar{r}(t)\\
&-\bar{z}_i\left[q^i_x(t,\bar{x},\dot{\bar{x}},\bar{u}(t),\dot{\bar{u}})-C_i(t)\bar{w}(t)\right]\\
&+f^i_u(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+B_i(t)\bar{\delta}(t)\\
&-\bar{z}_i\left[q^i_u(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-E_i(t)\bar{\vartheta}(t)\right]\Big)\\
-\frac{d}{dt}\Big(&\left[\eta\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\right]^T\left[f^i_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\bar{z}_i q^i_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right.\\
&\left.+f^i_{\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\bar{z}_i q^i_{\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right]\Big)\Big\}dt<0, \quad \forall i\in P.\end{aligned} \tag{3.20}$$

By the necessary optimality condition (3.10), we have $\bar{\lambda} \geq 0$, $\bar{\lambda}^T e = 1$. Thus, (3.20) gives

$$\begin{aligned}\sum_{i=1}^{p} \bar{\lambda}_i \int_a^b \Big\{ & [\eta(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})]^T \Big[f_x^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + A_i(t)\bar{r}(t) \\ & - \bar{z}_i [q_x^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - C_i(t)\bar{w}(t)] \\ & - \frac{d}{dt}\Big(f_{\dot{x}}^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \bar{z}_i q_{\dot{x}}^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \Big) \Big] \\ & + [\theta(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})]^T \Big[f_u^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + B_i(t)\bar{\delta}(t) \\ & - \bar{z}_i [q_u^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - E_i(t)\bar{\vartheta}(t)] \\ & - \frac{d}{dt}\big(f_{\dot{u}}^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \\ & - \bar{z}_i q_{\dot{u}}^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \big) \Big] \Big\} dt < 0. \end{aligned} \tag{3.21}$$

Using the feasibility of $(\tilde{x}, \tilde{u})$ in the problem (MFP) together with the necessary optimality condition (3.5), we get

$$\int_a^b \bar{\xi}_j g^j\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}\right) dt - \int_a^b \bar{\xi}_j g^j\left(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right) dt \leqq 0, \quad j \in J(t). \tag{3.22}$$

By hypothesis (g), each Φ_{g_j}, $j \in J(t)$, is an increasing functional. Since $b_{g_j}(\tilde{x}, \tilde{u}, \bar{x}, \bar{u}) \geqq 0$, $j \in J$, therefore, (3.22) gives

$$\begin{aligned} & \sum_{j=1}^{l} b_{g_j}(\tilde{x}, \tilde{u}, \bar{x}, \bar{u}) \Phi_{g_j} \left(\int_a^b \bar{\xi}_j g^j\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}\right) dt - \int_a^b \bar{\xi}_j g^j\left(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right) dt \right) \\ & \leqq \sum_{j=1}^{l} b_{g_j}(\tilde{x}, \tilde{u}, \bar{x}, \bar{u}) \Phi_{g_j}(0) \leqq 0. \end{aligned} \tag{3.23}$$

Combining (3.18) and (3.23), we obtain

$$\begin{aligned} \sum_{j=1}^{l} \int_a^b \Bigg\{ & \left[\eta\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right)\right]^T \left[\bar{\xi}_j g_x^j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \frac{d}{dt}\left(\bar{\xi}_j g_{\dot{x}}^j\left(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right)\right) \right] \\ & + \left[\theta\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right)\right]^T \left[\bar{\xi}_j g_u^j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \frac{d}{dt} \bar{\xi}_j g_{\dot{u}}^j\left(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right) \right] \Bigg\} dt \\ \leqq 0, \quad j \in J(t). \end{aligned} \tag{3.24}$$

By the feasibility of $(\tilde{x}, \tilde{u})$ and $(\bar{x}, \bar{u})$ in (MFP) together with

$$b_{h_k}(\tilde{x}, \tilde{u}, \bar{x}, \bar{u}) \geqq 0, \quad k \in K,$$

we have

$$\begin{aligned} & b_{h_k}(\tilde{x}, \tilde{u}, \bar{x}, \bar{u}) \Phi_{h_k} \left(\int_a^b \bar{\zeta}_k h^k \left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}\right) dt - \int_a^b \bar{\zeta}_k h^k \left(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right) dt \right) \\ & = b_{h_k}(\tilde{x}, \tilde{u}, \bar{x}, \bar{u}) \Phi_{h_k}(0) \leqq 0, \quad k \in K. \end{aligned} \tag{3.25}$$

Combining (3.19) and (3.25), we obtain

$$\begin{aligned} \int_a^b \Big\{ & \left[\eta\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right)\right]^T \left[\bar{\zeta}_k h_x^k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \frac{d}{dt}\left(\bar{\zeta}_k h_{\dot{x}}^k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right)\right] \\ & + \left[\theta\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right)\right]^T \Big[\bar{\zeta}_k h_u^k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \\ & \qquad - \frac{d}{dt}\left(\bar{\zeta}_k h_{\dot{u}}^k\left(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right)\right)\Big]\Big\} dt \leqq 0, \quad k \in K. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{k=1}^{s} \int_a^b \Big\{ & [\eta(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})]^T \left[\bar{\zeta}_k h_x^k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \frac{d}{dt}\left(\bar{\zeta}_k h_{\dot{x}}^k\left(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right)\right)\right] \\ & + [\theta(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})]^T \Big[\bar{\zeta}_k h_u^k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \\ & \qquad - \frac{d}{dt}\left(\bar{\zeta}_k h_{\dot{u}}^k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right)\Big]\Big\} dt \leqq 0. \end{aligned} \tag{3.26}$$

Adding both sides of (3.21), (3.24) and (3.26), we get that the following inequality

$$\int_a^b \Big\{ [\eta(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]^T \Big[\sum_{i=1}^{p} \bar{\lambda}_i (f_x^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + A_i(t)\bar{r}(t)$$

$$- \bar{z}_i [q_x^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - C_i(t)\bar{w}(t)])$$

$$+ \sum_{j=1}^{l} \bar{\xi}_j g_x^j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \bar{\zeta}_k h_x^k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$- \frac{d}{dt}\Big(\sum_{i=1}^{p} \bar{\lambda}_i [f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{z}_i q_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]$$

$$+ \sum_{j=1}^{l} \bar{\xi}_j g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k=1}^{s} \bar{\zeta}_k h_{\dot{x}}^k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \Big) \Big]$$

$$+ [\theta(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]^T \Big[\sum_{i=1}^{p} \bar{\lambda}_i (f_u^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + B_i(t)\bar{\delta}(t)$$

$$- \bar{z}_i [q_u^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - E(t)\bar{\vartheta}(t)])$$

$$+ \sum_{j=1}^{l} \bar{\xi}_j g_u^j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k=1}^{s} \bar{\zeta}_k h_u^k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$$

$$- \frac{d}{dt}\Big(\sum_{i=1}^{p} \bar{\lambda}_i [f_{\dot{u}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{z}_i q_{\dot{u}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]$$

$$+ \sum_{j=1}^{l} \bar{\xi}_j g_{\dot{u}}^j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k=1}^{s} \bar{\zeta}_k h_{\dot{u}}^k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \Big) \Big] \Big\} dt < 0$$

holds, contradicting the necessary optimality conditions (3.3) and (3.4). □

In order to illustrate the sufficient optimality conditions established in Theorem 3.6, we give an example of a multiobjective fractional variational control problem involving the nondifferentiable terms in the numerators and in the denominators of each objective function.

Example 3.7. Consider the following multiobjective fractional variational programming problem defined (MFP1) by

$$\underset{x}{\text{Minimize}} \left(\frac{\int_0^1 \left(x^2 - x + t + 1 + \sqrt{xA_1x}\right) dt}{\int_0^1 \left(x + t + 1 - \sqrt{xC_1x}\right) dt}, \frac{\int_0^1 \left(x^2 - x + 2t + 2 + \sqrt{xA_2x}\right) dt}{\int_0^1 \left(x + 2t + 2 - \sqrt{xC_2x}\right) dt} \right),$$

$$g_1(t,x) = x^2 - x \leqq 0,$$

$$x(0) = x(1) = 0,$$

where $A_1 = A_2 = 1$, $C_1 = C_2 = 1$. Note that $S = \left\{x \in R : x^2 - x \leqq 0\right\}$ and $\bar{x}(t) = 0$, $t \in [0,1]$, is a feasible solution in (MFP1). We now show that the Karush–Kuhn–Tucker necessary optimality conditions (3.3)–(3.10) are satisfied at $\bar{x}(t)$ for (MFP1). In fact, by (3.3) and (3.5), we have that, respectively,

$$\begin{aligned}&\bar{\lambda}_1\left(2\bar{x}-1+A_1\bar{r}-\bar{z}_1\left(1-C_1\bar{w}\right)\right)+\bar{\lambda}_2\left(2\bar{x}-1+A_2\bar{r}-\bar{z}_2\left(1-C_2\bar{w}\right)\right)\\&+\bar{\xi}_1\left(2\bar{x}-1\right)=0,\end{aligned}\tag{3.27}$$

$$\bar{\xi}_1\left(x^2-x\right)=0.\tag{3.28}$$

Further, by the Karush–Kuhn–Tucker necessary optimality condition (3.6), it follows that

$$\bar{z}_1=\frac{\int_0^1\left(\bar{x}^2-\bar{x}+t+1+\bar{x}\right)dt}{\int_0^1\left(\bar{x}+t+1-\bar{x}\right)dt}=\frac{\int_0^1(t+1)dt}{\int_0^1(t+1)dt}=1,\tag{3.29}$$

$$\bar{z}_2=\frac{\int_0^1\left(\bar{x}^2-\bar{x}+2t+2+\bar{x}\right)dt}{\int_0^1\left(\bar{x}+2t+2-\bar{x}\right)dt}=\frac{\int_0^1(2t+2)\,dt}{\int_0^1(2t+2)dt}=1.\tag{3.30}$$

If we set $\bar{r}(t)=1$ and $\bar{w}(t)=1$, $t\in[0,1]$ in (3.27), then also the Karush–Kuhn–Tucker necessary optimality conditions (3.7)–(3.9) are fulfilled. Further, by $A_1=A_2=1$, $C_1=C_2=1$ and the Karush–Kuhn–Tucker necessary optimality condition (3.10), (3.27) yields

$$4\bar{x}+\bar{\xi}_1(2\bar{x}-1)=0.\tag{3.31}$$

Consider two cases.

(i) $\bar{\xi}_1\neq 0$. Then (3.31) gives $\bar{x}=\frac{\bar{\xi}_1}{4+2\bar{\xi}_1}$. But this solution doesn't satisfy (3.27).
(ii) $\bar{\xi}_1=0$. Then, $\bar{x}(t)=0$, $t\in[0,1]$, satisfies the the Karush–Kuhn–Tucker necessary optimality conditions (3.3)–(3.10) for (MFP1).

Now, we show that the sufficient optimality conditions established in Theorem 3.6 are also satisfied at $\bar{x}$ for (MFP1). Let us set $\Phi_1(a)=e^a-1$, $\Phi_2(a)=e^a-1$, $\Phi_{g_1}(a)=a$, $b_1(x,\bar{x})=b_2(x,\bar{x})=b_{g_1}(x,\bar{x})=1$ for any $x\in S$, $\eta(x,\bar{x})=\frac{1}{2}(2x-x^2)$. Then, by Definition 2.3, it can be shown that the appropriate univexity hypotheses are fulfilled at $\bar{x}$ on S with respect to the functions given above. Then, by Theorem 3.6, $\bar{x}$ is an efficient solution in (MFP1).

Now, we prove the sufficient conditions for optimality of $(\bar{x},\bar{u})\in S$ for the considered multiobjective fractional variational programming problem (MFP) under generalized univexity assumptions.

Theorem 3.8. *Let* $(\bar{x}, \bar{u})$ *be a feasible solution in the considered multiobjective variational programming problem* (MFP) *and the necessary optimality conditions* (3.3)–(3.10) *be satisfied at* $(\bar{x}, \bar{u})$ *with* $\bar{\lambda} \in R^p$ *and piecewise smooth functions* $\bar{\xi}(\cdot) : I \to R^l$ *and* $\bar{\zeta}(\cdot) : I \to R^s$, $\bar{r}(\cdot) : I \to R^n$, $\bar{w}(\cdot) : I \to R^n$, $\bar{\delta}(\cdot) : I \to R^m$, $\bar{\vartheta}(\cdot) : I \to R^m$. *Further, assume that the following hypotheses are fulfilled:*

(a)

$$\Bigg(\int_a^b \bar{\lambda}_i \Big\{ f^1(t,\cdot,\cdot,\cdot,\cdot) + \cdot^T A_1(t)\bar{r}(t) + \cdot^T B_1(t)\bar{\delta}(t) - \bar{z}_1 \left[q^1(t,\cdot,\cdot,\cdot,\cdot) - \cdot^T C_1(t)\bar{w}(t) - \cdot^T E_1(t)\bar{\vartheta}(t) \right] \Big\} dt,$$

$$\ldots,$$

$$\int_a^b \bar{\lambda}_i \Big\{ f^p(t,\cdot,\cdot,\cdot,\cdot) + \cdot^T A_p(t)\bar{r}(t) + \cdot^T B_p(t)\bar{\delta}(t) - \bar{z}_p \left[q^p(t,\cdot,\cdot,\cdot,\cdot) - \cdot^T C_p(t)\bar{w}(t) - \cdot^T E_p(t)\bar{\vartheta}(t) \right] \Big\} dt \Bigg)$$

is a strictly pseudo-univex function (*a pseudo-univex function*) *at* $(\bar{x}, \bar{u})$ *on* S *with respect to* $b = (b_1, \ldots, b_p)$, $\Phi = (\Phi_1, \ldots, \Phi_p)$, η, θ,

(b)

$$\left(\int_a^b \bar{\xi}_1(t) g^1(t,\cdot,\cdot,\cdot,\cdot) dt, \ldots, \int_a^b \bar{\xi}_l(t) g^l(t,\cdot,\cdot,\cdot,\cdot) dt \right)$$

is a quasi-univex function at $(\bar{x}, \bar{u})$ *on* S *with respect to* $b_g = (b_{g_1}, \ldots, b_{g_l})$, $\Phi_g = (\Phi_{g_1}, \ldots, \Phi_{g_l})$, η, θ,

(c)

$$\left(\int_a^b \bar{\zeta}_1(t) h^1(t,\cdot,\cdot,\cdot,\cdot) dt, \ldots, \int_a^b \bar{\zeta}_s(t) h^s(t,\cdot,\cdot,\cdot,\cdot) dt \right)$$

is a quasi-univex function at $(\bar{x}, \bar{u})$ *on* S *with respect to* $b_h = (b_{h_1}, \ldots, b_{h_s})$, $\Phi_h = (\Phi_{h_1}, \ldots, \Phi_{h_s})$, η, θ,

(d) $b_i(x, u, \bar{x}, \bar{u}) > 0$, $i = 1, \ldots, p$, *for every* $(x, u) \in X \times U$,

(e) Φ_i, $i = 1, \ldots, p$, *are strictly increasing functionals satisfying* $a < 0 \Longrightarrow \Phi_i(a) < 0$, $\Phi_i(0) \leqq 0$,

(f) Φ_{g_j}, $j \in J(t)$, *are increasing functionals satisfying* $a \leqq 0 \Longrightarrow \Phi_{g_j}(a) \leqq 0$,

(g) Φ_{h_k}, $k \in K$, *are increasing functionals satisfying* $a \leqq 0 \Longrightarrow \Phi_{h_k}(a) \leqq 0$.

Then $(\bar{x}, \bar{u})$ *is an efficient solution* (*a weakly efficient solution*) *of* (MFP).

Proof. We proceed by contradiction. Suppose, contrary to the result, that $(\bar{x}, \bar{u})$ is not an efficient solution of (MFP). Then there exists $(\tilde{x}, \tilde{u}) \in S$ such that

$$\begin{aligned}
&\int_a^b \Big\{ f^i(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}) + \sqrt{\tilde{x}^T A_i \tilde{x}} + \sqrt{\tilde{u}^T B_i \tilde{u}} \\
&\qquad - \bar{z}_i \Big[q^i(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}) - \sqrt{\tilde{x}^T C_i \tilde{x}} - \sqrt{\tilde{u}^T E_i \tilde{u}} \Big] \Big\} dt \\
&\leqq \int_a^b \Big\{ f^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sqrt{\bar{x}^T A_i \bar{x}} + \sqrt{\bar{u}^T B_i \bar{u}} \\
&\qquad - \bar{z}_i \Big[q^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \sqrt{\bar{x}^T C_i \bar{x}} - \sqrt{\bar{u}^T E_i \bar{u}} \Big] \Big\} dt, \quad \forall i \in P,
\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}
&\int_a^b \Big\{ f^{i^*}\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}\right) + \sqrt{\tilde{x}^T A_{i^*} \tilde{x}} + \sqrt{\tilde{u}^T B_{i^*} \tilde{u}} \\
&\qquad - \bar{z}_{i^*} \Big[q^{i^*}\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}\right) - \sqrt{\tilde{x}^T C_{i^*} \tilde{x}} - \sqrt{\tilde{u}^T E_{i^*} \tilde{u}} \Big] \Big\} dt \\
&< \int_a^b \Big\{ f^{i^*}\left(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right) + \sqrt{\bar{x}^T A_{i^*} \bar{x}} + \sqrt{\bar{u}^T B_{i^*} \bar{u}} \\
&\qquad - \bar{z}_{i^*} \Big[q^{i^*}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \sqrt{\bar{x}^T C_{i^*} \bar{x}} - \sqrt{\bar{u}^T E_{i^*} \bar{u}} \Big] \Big\} dt \quad \text{for some } i^* \in P.
\end{aligned} \tag{3.33}$$

By the necessary optimality conditions (3.7) and (3.8) and the generalized Schwarz inequality (see (2.1)), (3.32) and (3.33) yield

$$\begin{aligned}
&\int_a^b \bar{\lambda}_i \Big\{ f^i(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}) + \tilde{x}^T A_i \bar{r} + \tilde{u}^T B_i \bar{\delta} \\
&\qquad - \bar{z}_i \left[q^i\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}\right) - \tilde{x}^T C_i \bar{w} - \tilde{u}^T E_i \bar{\vartheta} \right] \Big\} dt \\
&\leqq \int_a^b \bar{\lambda}_i \Big\{ f^i\left(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right) + \bar{x}^T A_i \bar{r} + \bar{u}^T B_i \bar{\delta} \\
&\qquad - \bar{z}_i \left[q^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \bar{x}^T C_i \bar{w} - \bar{u}^T E_i \bar{\vartheta} \right] \Big\} dt, \quad \forall i \in P,
\end{aligned} \tag{3.34}$$

$$\int_a^b \bar{\lambda}_{i^*}\Big\{f^{i^*}(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}) + \tilde{x}^T A_{i^*}\bar{r} + \tilde{u}^T B_{i^*}\bar{\delta} - \bar{z}_{i^*}\Big[q^{i^*}(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}) - \tilde{x}^T C_{i^*}\bar{w} - \tilde{u}^T E_{i^*}\bar{\vartheta}\Big]\Big\}dt$$
$$< \int_a^b \bar{\lambda}_{i^*}\Big\{f^{i^*}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \bar{x}^T A_{i^*}\bar{r} + \bar{u}^T B_{i^*}\bar{\delta} - \bar{z}_{i^*}\Big[q^{i^*}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{x}^T C_{i^*}\bar{w} - \bar{u}^T E_{i^*}\bar{\vartheta}\Big]\Big\}dt \quad \text{for some } i^* \in P. \tag{3.35}$$

By hypotheses (d) and (e), $b_i(\tilde{x},\tilde{u},\bar{x},\bar{u}) > 0$, $i = 1,\ldots,k$, and Φ_i, $i = 1,\ldots,k$, are strictly increasing functionals with $\Phi_i(0) \leqq 0$. Thus, (3.13) and (3.14) imply

$$b_i(\tilde{x},\tilde{u},\bar{x},\bar{u})\Phi_i\Bigg(\int_a^b \bar{\lambda}_i\{f^i(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}) + \tilde{x}^T A_i\bar{r} + \tilde{u}^T B_i\bar{\delta} - \bar{z}_i[q^i(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}) - \tilde{x}^T C_i\bar{w} - \tilde{u}^T E_i\bar{\vartheta}]\}dt$$
$$- \int_a^b \bar{\lambda}_i\{f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \bar{x}^T A_i\bar{r} + \bar{u}^T B_i\bar{\delta} - \bar{z}_i\left[q^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{x}^T C_i\bar{w} - \bar{u}^T E_i\bar{\vartheta}\right]\}dt\Bigg)$$
$$\leqq b_i(\tilde{x},\tilde{u},\bar{x},\bar{u})\,\Phi_i(0) \leqq 0, \quad \forall i \in P, \tag{3.36}$$

$$b_{i^*}(\tilde{x},\tilde{u},\bar{x},\bar{u})\,\Phi_{i^*}\Bigg(\int_a^b \bar{\lambda}_{i^*}\Big\{f^{i^*}(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}) + \tilde{x}^T A_{i^*}\bar{r} + \tilde{u}^T B_{i^*}\bar{\delta} - \bar{z}_{i^*}\Big[q^{i^*}\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}\right) - \tilde{x}^T C_{i^*}\bar{w} - \tilde{u}^T E_{i^*}\bar{\vartheta}\Big]\Big\}dt$$
$$- \int_a^b \bar{\lambda}_{i^*}\Big\{f^{i^*}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \bar{x}^T A_{i^*}\bar{r} + \bar{u}^T B_{i^*}\bar{\delta} - \bar{z}_{i^*}\Big[q^{i^*}\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right) - \bar{x}^T C_{i^*}\bar{w} - \bar{u}^T E_{i^*}\bar{\vartheta}\Big]\Big\}dt\Bigg)$$
$$< b_{i^*}(\tilde{x},\tilde{u},\bar{x},\bar{u})\,\Phi_{i^*}(0) \leqq 0 \text{ for some } i^* \in P. \tag{3.37}$$

Hence, the inequalities (3.36) and (3.37) yield

$$\sum_{i=1}^{p} b_i(\tilde{x},\tilde{u},\bar{x},\bar{u})\Phi_i\Bigg(\int_a^b \bar{\lambda}_i\Big\{f^i(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}})+\tilde{x}^T A_i\bar{r}+\tilde{u}^T B_i\bar{\delta} - \bar{z}_i\left[q^i(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}})-\tilde{x}^T C_i\bar{w}-\tilde{u}^T E_i\bar{\vartheta}\right]\Big\}dt - \int_a^b \bar{\lambda}_i\Big\{f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\bar{x}^T A_i\bar{r}+\bar{u}^T B_i\bar{\delta} - \bar{z}_i\left[q^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-\bar{x}^T C_i\bar{w}-\bar{u}^T E_i\bar{\vartheta}\right]\Big\}dt\Bigg)<0.$$

By hypothesis (a) and Definition 2.7, the above inequality implies

$$\begin{aligned}\sum_{i=1}^{p}\bar{\lambda}_i\int_a^b\Big\{&\left[\eta\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\right]^T\Big[f_x^i\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)+A_i(t)\bar{r}(t)\\&-\bar{z}_i\left[q_x^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-C_i(t)\bar{w}(t)\right]\\&-\frac{d}{dt}\left(f_{\dot{x}}^i\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)-\bar{z}_i q_{\dot{x}}^i\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\right)\Big]\\&+\left[\theta\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\right]^T\Big[f_u^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\\&+B_i(t)\bar{\delta}(t)\\&-\bar{z}_i\left[q_u^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})-E_i(t)\bar{\vartheta}(t)\right]\\&-\frac{d}{dt}\big(f_{\dot{u}}^i\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\\&-\bar{z}_i q_{\dot{u}}^i\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\big)\Big]\Big\}dt<0.\end{aligned}\tag{3.38}$$

From the feasibility of $(\tilde{x},\tilde{u})$ in (MFP) and by the necessary optimality condition (3.5), it follows that

$$\int_a^b \bar{\xi}_j g^j\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}\right)dt-\int_a^b \bar{\xi}_j g^j\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)dt\leqq 0,\ j\in J(t).\tag{3.39}$$

By hypothesis (h), we have that Φ_{g_j}, $j\in J(t)$, is an increasing functional and $\Phi_{g_j}(0)\leqq 0$, $j\in J(t)$. Since $b_{g_j}(\tilde{x},\tilde{u},\bar{x},\bar{u})\geqq 0$, $j\in J$, therefore, (3.23) gives

$$\begin{aligned}&\sum_{j=1}^{l} b_{g_j}(\tilde{x},\tilde{u},\bar{x},\bar{u})\Phi_{g_j}\left(\int_a^b \bar{\xi}_j g^j\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}}\right)dt-\int_a^b \bar{\xi}_j g^j\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)dt\right)\\&\leqq\sum_{j=1}^{l} b_{g_j}(\tilde{x},\tilde{u},\bar{x},\bar{u})\Phi_{g_j}(0)\leqq 0.\end{aligned}\tag{3.40}$$

Hence, by Definition 2.8, (3.40) implies

$$\int_a^b \sum_{j=1}^{l} \Bigg\{ \left[\eta\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right)\right]^T \left[\bar{\xi}_j g_x^j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \frac{d}{dt}\left(\bar{\xi}_j g_{\dot{x}}^j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right)\right] + \Bigg(\left[\theta\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right)\right]^T \Bigg[\bar{\xi}_j g_u^j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \frac{d}{dt}\left(\bar{\xi}_j g_{\dot{u}}^j\left(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right)\right)\Bigg]\Bigg)\Bigg\} dt \leqq 0, \ j \in J(t). \tag{3.41}$$

By the feasibility of $(\tilde{x}, \tilde{u})$ and $(\bar{x}, \bar{u})$ in (MFP) together with $b_{h_k}(\tilde{x}, \tilde{u}, \bar{x}, \bar{u}) \geqq 0$, $k \in K$, it follows that

$$\begin{aligned} &\sum_{k=1}^{s} b_{h_k}(\tilde{x}, \tilde{u}, \bar{x}, \bar{u}) \Phi_{h_k} \left(\int_a^b \bar{\zeta}_k h^k\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}\right) dt - \int_a^b \bar{\zeta}_k h^k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) dt \right) \\ &= \sum_{k=1}^{s} b_{h_k}(\tilde{x}, \tilde{u}, \bar{x}, \bar{u}) \Phi_{h_k}(0) \leqq 0. \end{aligned} \tag{3.42}$$

Thus, by Definition 2.8, (3.42) implies

$$\int_a^b \sum_{k=1}^{s} \Bigg\{ \left[\eta\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right)\right]^T \left[\bar{\zeta}_k h_x^k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \frac{d}{dt}\left(\bar{\zeta}_k h_{\dot{x}}^k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})\right)\right] + \Bigg(\left[\theta\left(t, \tilde{x}, \dot{\tilde{x}}, \tilde{u}, \dot{\tilde{u}}, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right)\right]^T \Bigg[\bar{\zeta}_k h_u^k\left(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right) - \frac{d}{dt}\left(\bar{\zeta}_k h_{\dot{u}}^k\left(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}\right)\right)\Bigg]\Bigg)\Bigg\} dt \leqq 0. \tag{3.43}$$

Adding both sides of (3.38), (3.41) and (3.43), we get that the inequality

$$\int_a^b \Bigg\{ \left[\eta\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\right]^T \Bigg[\sum_{i=1}^{p}\bar{\lambda}_i\Big(f_x^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + A_i(t)\bar{r}(t) - \bar{z}_i\left[q_x^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - C_i(t)\bar{w}(t)\right]\Big) + \sum_{j=1}^{l}\bar{\xi}_j g_x^j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k=1}^{s}\bar{\zeta}_k h_x^k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \frac{d}{dt}\Big(\sum_{i=1}^{p}\bar{\lambda}_i\left(f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{z}_i q_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right) + \xi^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \zeta^T h_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\Big)\Bigg]$$

$$+ \left[\theta\left(t,\tilde{x},\dot{\tilde{x}},\tilde{u},\dot{\tilde{u}},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\right]^T \Bigg[\sum_{i=1}^{p}\bar{\lambda}_i\Big(f_u^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + B_i(t)\bar{\delta}(t) - \bar{z}_i\left[q_u^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - E_i(t)\bar{\vartheta}(t)\right]\Big) + \sum_{j=1}^{l}\bar{\xi}_j g_u^j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sum_{k=1}^{s}\bar{\zeta}_k h_u^k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \frac{d}{dt}\Big(\sum_{i=1}^{p}\bar{\lambda}_i\left(f_{\dot{u}}\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right) - \bar{z}_i q_{\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right) + \xi^T g_{\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \zeta^T h_{\dot{u}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\Big)\Bigg]\Bigg\} dt < 0$$

holds, contradicting the necessary optimality conditions (3.3) and (3.4). □

4. MOND–WEIR TYPE DUALITY

In this section, we prove duality results between the considered multiobjective fractional variational control problem (MFP) and its parametric Mond–Weir type multiobjective variational dual problem (VMWD) defined as follows:

$$\underset{y,v}{\text{Maximize}}$$

$$\Bigg(\int_a^b \Big\{ f^1(t,y(t),\dot y(t),v(t),\dot v(t)) + \sqrt{y(t)^T A_1(t) y(t)} + \sqrt{v(t)^T B_1(t) v(t)}$$

$$- z_1 \left[q^1(t,y(t),\dot y(t),v(t),\dot v(t)) - \sqrt{y(t)^T C_1(t) y(t)} - \sqrt{v(t)^T E_1(t) v(t)} \right] \Big\} dt,$$

$$\ldots,$$

$$\int_a^b \Big\{ f^p\left(t,y(t),\dot y(t),v(t),\dot v(t)\right) + \sqrt{y(t)^T A_p(t) y(t)} + \sqrt{v(t)^T B_p(t) v(t)}$$

$$- z_p \left[q^p\left(t,y(t),\dot y(t),v(t),\dot v(t)\right) - \sqrt{y(t)^T C_p(t) y(t)} - \sqrt{v(t)^T E_p(t) v(t)} \right] \Big\} dt \Bigg)$$

subject to $x\left(a\right) = \alpha,\ x(b) = \beta,$

$$\sum_{i=1}^{p} \lambda_i \big[f_y^i(t,y(t),\dot y(t),v(t),\dot v(t)) + A_i(t) r(t)$$

$$- z_i \left[q_y^i(t,y(t),\dot y(t),v(t),\dot v(t)) - C_i(t) w(t) \right] \big]$$

$$+ \xi(t)^T g_y\left(t,y(t),\dot y(t),v(t),\dot v(t)\right) + \zeta(t)^T h_y(t,y(t),\dot y(t),v(t),\dot v(t))$$

$$= \frac{d}{dt} \Bigg[\sum_{i=1}^{p} \lambda_i \left(f_{\dot y}^i(t,y(t),\dot y(t),v(t),\dot v(t)) - z_i q_{\dot y}^i\left(t,y(t),\dot y(t),v(t),\dot v(t)\right)\right)$$

$$+ \xi(t)^T g_{\dot y}(t,y(t),\dot y(t),v(t),\dot v(t)) + \zeta(t)^T h_{\dot y}\left(t,y(t),\dot y(t),v(t),\dot v(t)\right) \Bigg], \quad t \in I,$$

$$\sum_{i=1}^{p} \lambda_i \big[f_v^i(t,y(t),\dot y(t),v(t),\dot v(t)) + B_i(t) \delta(t)$$

$$- z_i \left[q_v^i\left(t,y(t),\dot y(t),v(t),\dot v(t)\right) - E_i(t) \vartheta(t) \right] \big]$$

$$+ \xi(t)^T g_v(t,y(t),\dot y(t),v(t),\dot v(t)) + \zeta(t)^T h_v\left(t,y(t),\dot y(t),v(t),\dot v(t)\right) \big]$$

$$= \frac{d}{dt} \Bigg[\sum_{i=1}^{p} \lambda_i \left(f_{\dot v}^i(t,y(t),\dot y(t),v(t),\dot v(t)) - q_{\dot v}^i(t,y(t),\dot y(t),v(t),\dot v(t))\right)$$

$$+ \xi(t)^T g_{\dot v}(t,y(t),\dot y(t),v(t),\dot v(t)) + \zeta(t)^T h_{\dot y}\left(t,y(t),\dot y(t),v(t),\dot v(t)\right) \Bigg], \quad t \in I,$$

$$\int_a^b \xi_j(t) g^j\left(t,y(t),\dot y(t),v(t),\dot v(t)\right) dt \geqq 0, \quad j \in J,$$

$$\int_a^b \zeta_k(t) h^k\left(t,y(t),\dot y(t),v(t),\dot v(t)\right) dt = 0, \quad k \in K,$$

$$r(t) \in R^n, \quad \delta(t) \in R^m, \quad r(t)^T A_i(t) r(t) \leqq 1, \quad \delta(t)^T B_i(t)\delta(t) \leqq 1,$$
$$w(t) \in R^n, \quad \vartheta(t) \in R^m, \quad w(t)^T C_i(t) w(t) \leqq 1, \quad \vartheta(t)^T E_i(t)\vartheta(t) \leqq 1,$$

$$y^T(t)A_i(t)r(t) = \sqrt{y(t)^T A_i(t)y(t)},$$
$$v(t)^T B_i(t)\delta(t) = \sqrt{v(t)^T B_i v(t)},$$
$$y(t)^T C_i(t)w(t) = \sqrt{y(t)^T C_i(t)y(t)},$$
$$v(t)^T E_i(t)\vartheta(t) = \sqrt{v(t)^T E_i(t)v(t)},$$

for $t \in I$, $i = 1, \ldots, p$,

$$\lambda \geq 0, \quad \lambda^T e = 1, \quad \xi(t) \geqq 0,$$

where $e = (1, \ldots, 1) \in R^p$ is a p-dimensional vector. It may be noted here that the above dual constraints are written using the Karush–Kuhn–Tucker necessary optimality conditions for the problem (MFP).

Let Ω_{MW} be the set of all feasible solutions $(y, v, \lambda, \xi, \zeta, r, \delta, w, \vartheta)$ in Mond–Weir dual problem (VMWD). We denote by Y the set

$$Y = \{(y, v) \in X \times U : (y, v, z, \lambda, \xi, \zeta, r, \delta, w, \vartheta) \in \Omega_{MW}\}.$$

Now, under univexity and generalized univexity, we prove several duality results between (MFP) and (VMWD).

Theorem 4.1 (Weak duality)**.** *Let* (x, u) *and* $(y, v, z, \lambda, \xi, \zeta, r, \delta, w, \vartheta)$ *be any feasible solutions in* (MFP) *and* (VMWD)*. Further, assume that at least one of the following hypotheses are fulfilled:*

(A)

(a)

$$\Bigg(\int_a^b \Big\{f^1(t,\cdot,\cdot,\cdot,\cdot) + \cdot^T A_1(t)\bar{r}(t) + \cdot^T B_1(t)\delta(t) - z_1[q^1(t,\cdot,\cdot,\cdot,\cdot) - \cdot^T C_1(t)w(t) - \cdot^T E_1(t)\vartheta(t)]\Big\}dt,$$
$$\ldots,$$
$$\int_a^b \Big\{f^p(t,\cdot,\cdot,\cdot,\cdot) + \cdot^T A_p(t)r(t) + \cdot^T B_p(t)\delta(t) - z_p[q^p(t,\cdot,\cdot,\cdot,\cdot) - \cdot^T C_p(t)w(t) - \cdot^T E_p(t)\vartheta(t)]\Big\}dt\Bigg)$$

is univex at (y, v) *on* $S \cup Y$ *with respect to* $b = (b_1, \ldots, b_p)$*,* $\Phi = (\Phi_1, \ldots, \Phi_p)$*,* η*,* θ*,*

(b)

$$\left(\int_a^b \xi_1(t)g^1(t,\cdot,\cdot,\cdot,\cdot)\,dt,\ldots,\int_a^b \xi_l(t)g^l(t,\cdot,\cdot,\cdot,\cdot)dt\right)$$

is univex at (y,v) *on* $S\cup Y$ *with respect to* $b_g=(b_{g_1},\ldots,b_{g_l})$, $\Phi_g=(\Phi_{g_1},\ldots,\Phi_{g_l})$, η, θ,

(c)

$$\left(\int_a^b \zeta_1(t)h^1(t,\cdot,\cdot,\cdot,\cdot)\,dt,\ldots,\int_a^b \zeta_s(t)h^s(t,\cdot,\cdot,\cdot,\cdot)dt\right)$$

is univex at (y,v) *on* $S\cup Y$ *with respect to* $b_h=(b_{h_1},\ldots,b_{h_s})$, $\Phi_h=(\Phi_{h_1},\ldots,\Phi_{h_s})$, η, θ,

(d) $b_i(x,u,y,v)>0$, $i=1,\ldots,p$,

(e) Φ_i, $i=1,\ldots,k$, *are strictly increasing functionals satisfying* $a<0\Longrightarrow\Phi_i(a)<0$, $\Phi_i(0)\leqq 0$, *and moreover,* Φ_{g_j}, $j\in J$, Φ_{h_k}, $k\in K$, *are increasing functionals satisfying* $a\leqq 0\Longrightarrow\Phi_{g_j}(a)\leqq 0$, $a\leqq 0\Longrightarrow\Phi_{h_k}(a)\leqq 0$, *respectively.*

(B)

(a)

$$\Bigg(\int_a^b \lambda_1\Big\{f^1(t,\cdot,\cdot,\cdot,\cdot)+\cdot^T A_1(t)r(t)+\cdot^T B_1(t)\delta(t) - z_1[q^1(t,\cdot,\cdot,\cdot,\cdot)-\cdot^T C_1(t)w(t)-\cdot^T E_1(t)\vartheta(t)]\Big\}dt,$$
$$\ldots,$$
$$\int_a^b \lambda_p\Big\{f^p(t,\cdot,\cdot,\cdot,\cdot)+\cdot^T A_p(t)r(t)+\cdot^T B_p(t)\delta(t) - z_p[q^p(t,\cdot,\cdot,\cdot,\cdot)-\cdot^T C_p(t)w(t)-\cdot^T E_p(t)\vartheta(t)]\Big\}dt\Bigg)$$

is pseudo-univex at (y,v) *on* $S\cup Y$ *with respect to* $b=(b_1,\ldots,b_p)$, $\Phi=(\Phi_1,\ldots,\Phi_p)$, η, θ,

(b)

$$\left(\int_a^b \xi_1(t)g^1(t,\cdot,\cdot,\cdot,\cdot)\,dt,\ldots,\int_a^b \xi_l(t)g^l(t,\cdot,\cdot,\cdot,\cdot)dt\right)$$

is univex at (y,v) *on* $S\cup Y$ *with respect to* $b_g=(b_{g_1},\ldots,b_{g_l})$, $\Phi_g=(\Phi_{g_1},\ldots,\Phi_{g_l})$, η, θ,

(c)

$$\left(\int_a^b \zeta_1(t)h^1(t,\cdot,\cdot,\cdot,\cdot)\,dt,\ldots,\int_a^b \zeta_s(t)h^s(t,\cdot,\cdot,\cdot,\cdot)dt\right)$$

is univex at (y,v) *on* $S \cup Y$ *with respect to* $b_h = (b_{h_1},\ldots,b_{h_s})$, $\Phi_h = (\Phi_{h_1},\ldots,\Phi_{h_s})$, η, θ,

(d) $b_i(x,u,y,v) > 0$, $i = 1,\ldots,p$,

(e) Φ_i, $i = 1,\ldots,p$, *are strictly increasing functionals satisfying* $a < 0 \Longrightarrow \Phi_i(a) < 0$, $\Phi_i(0) \leqq 0$, *and, moreover,* Φ_{g_j}, $j = 1,\ldots,l$, Φ_{h_1}, $k = 1,\ldots,s$, *are increasing functionals satisfying* $a \leqq 0 \Longrightarrow \Phi_{g_j}(a) \leqq 0$, $a \leqq 0 \Longrightarrow \Phi_{h_k}(a) \leqq 0$.

Then, the following cannot hold

$$\begin{aligned}
&\int_a^b \Big\{f^i(t,x,\dot{x},u,\dot{u}) + \sqrt{x(t)^T A_i(t)x(t)} + \sqrt{u(t)^T B_i(t)u(t)} \\
&\quad - z_i\left(q^i\left(t,x(t),\dot{x}(t),u(t),\dot{u}(t)\right) - \sqrt{x(t)C_i(t)x(t)} - \sqrt{u(t)E_i(t)u(t)}\right)\Big\}dt \\
&< \int_a^b \Big\{f^i\left(t,y(t),\dot{y}(t),v(t),\dot{v}(t)\right) + \sqrt{y(t)^T A_i(t)y(t)} + \sqrt{v(t)^T B_i(t)v(t)} \\
&\quad - z_i\left(q^i(t,y(t),\dot{y}(t),v(t),\dot{v}(t)) - \sqrt{y(t)C_i(t)y(t)} - \sqrt{v(t)E_i(t)v(t)}\right)\Big\}dt, \\
&\hfill \forall i \in P.
\end{aligned} \tag{4.1}$$

Proof. Let (x,u) and $(y,v,z,\lambda,\xi,\zeta,r,\delta,w,\vartheta)$ be any feasible solutions in problems (MFP) and (VMWD), respectively. We proceed by contradiction. Suppose, contrary to the result, that the inequalities (4.1) are fulfilled. By $(y,v,z,\lambda,\xi,\zeta,r,\delta,w,\vartheta) \in \Omega_{MW}$ and also the generalized Schwarz, (4.1) yields

$$\begin{aligned}
&\int_a^b \left\{f^i(t,x,\dot{x},u,\dot{u}) + x^T A_i r + u^T B_i \delta - z_i\left[q^i\left(t,x,\dot{x},u,\dot{u}\right) - x^T C_i w - u^T E_i \vartheta\right]\right\}dt \\
&< \int_a^b \left\{f^i(t,y,\dot{y},v,\dot{v}) + y^T A_i \bar{r} + v^T B_i \delta - z_i\left[q^i(t,y,\dot{y},v,\dot{v}) - y^T C_i w - v^T E_i \vartheta\right]\right\}dt, \\
&\hfill \forall i \in P.
\end{aligned} \tag{4.2}$$

Now, we prove this theorem under hypothesis (A).

By hypotheses (d) and (e), $b_i(x,u,y,v) > 0$, $i = 1,\dots,k$, and Φ_i, $i = 1,\dots,k$, are strictly increasing functionals with $\Phi_i(0) \leqq 0$. Thus, (4.2) gives

$$
\begin{aligned}
b_i(x,u,y,v)\Phi_i\Bigg(\int_a^b &\{f^i(t,x,\dot{x},u,\dot{u}) + x^T A_i r + u^T B_i \delta \\
&\quad - z_i\left[q^i(t,x,\dot{x},u,\dot{u}) - x^T C_i w - u^T E_i \vartheta\right]\}dt \\
&- \int_a^b \{f^i(t,y,\dot{y},v,\dot{v}) + y^T A_i r + v^T B_i \delta \\
&\quad - z_i\left[q^i(t,y,\dot{y},v,\dot{v}) - y^T C_i w - v^T E_i \vartheta\right]\}dt\Bigg) \\
< \Phi_i(0) = 0, \quad \forall i \in P.
\end{aligned}
\tag{4.3}
$$

Using hypothesis (a), by Definition 2.3, (4.3) yields

$$
\begin{aligned}
\int_a^b \Big\{&[\eta(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v})]^T \Big[f_y^i(t,y,\dot{y},v,\dot{v}) + A_i(t)r(t) \\
&\quad - z_i\left(q_y^i(t,y,\dot{y},v,\dot{v}) - C_i(t)w(t)\right) \\
&\quad - \frac{d}{dt}\left(f_{\dot{y}}^i(t,y,\dot{y},v,\dot{v}) - z_i q_{\dot{y}}^i(t,y,\dot{y},v,\dot{v})\right)\Big] \\
&+ [\theta(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v})]^T \Big[f_v^i(t,y,\dot{y},v,\dot{v}) + B_i(t)\delta(t) \\
&\quad - z_i\left(q_v^i(t,y,\dot{y},v,\dot{v}) - E(t)\vartheta(t)\right) \\
&\quad - \frac{d}{dt}\left(f_{\dot{v}}^i(t,y,\dot{y},v,\dot{v}) - z_i q_{\dot{v}}^i(t,y,\dot{y},v,\dot{v})\right)\Big]\Big\}dt < 0, \\
&\qquad \forall i \in P.
\end{aligned}
$$

By the last constraint of (VMWD), the above inequality implies

$$
\begin{aligned}
\int_a^b \sum_{i=1}^p \lambda_i \Big\{&[\eta(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v})]^T \Big[f_y^i(t,y,\dot{y},v,\dot{v}) + A_i(t)r(t) \\
&\quad - z_i\left(q_y^i(t,y,\dot{y},v,\dot{v}) - C_i(t)w(t)\right) \\
&\quad - \frac{d}{dt}\left(f_{\dot{y}}^i(t,y,\dot{y},v,\dot{v}) - z_i q_{\dot{y}}^i(t,y,\dot{y},v,\dot{v})\right)\Big] \\
&+ [\theta(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v})]^T \Big[f_v^i(t,y,\dot{y},v,\dot{v}) + B_i(t)\delta(t) \\
&\quad - z_i\left(q_v^i(t,y,\dot{y},v,\dot{v}) - E(t)\vartheta(t)\right) \\
&\quad - \frac{d}{dt}\left(f_{\dot{v}}^i(t,y,\dot{y},v,\dot{v}) - z_i q_{\dot{v}}^i(t,y,\dot{y},v,\dot{v})\right)\Big]\Big\}dt < 0.
\end{aligned}
\tag{4.4}
$$

From the feasibility of (x, u) in (MFP) and the feasibility of $(y, v, z, \lambda, \xi, \zeta, r, \delta, w, \vartheta)$ in (VMWD), we get

$$\int_a^b \xi_j g^j(t, x, \dot{x}, u, \dot{u})dt - \int_a^b \xi_j g^j(t, y, \dot{y}, v, \dot{v})dt \leqq 0, \quad j \in J(t). \tag{4.5}$$

By hypothesis (d), Φ_{g_j}, $j \in J$, is an increasing functional. Since

$$b_{g_j}(x, u, y, v) \geqq 0, \quad j \in J,$$

(4.5) gives

$$\begin{aligned} &b_{g_j}(x, u, y, v)\Phi_{g_j}\left(\int_a^b \xi_j g^j(t, x, \dot{x}, u, \dot{u})dt - \int_a^b \xi_j g^j(t, y, \dot{y}, v, \dot{v})dt\right) \\ &\leqq b_{g_j}(x, u, y, v)\Phi_{g_j}(0) \leqq 0, \quad j \in J(t). \end{aligned} \tag{4.6}$$

By assumption, each function $\xi^j g^j\,(t, \cdot, \cdot, \cdot, \cdot)$, $j \in J$, is univex at (y, v) on $S \cup Y$ with respect to b_{g_j}, Φ_{g_j}, η. Hence, by (4.5), Definition 2.3 yields

$$\begin{aligned} \int_a^b \sum_{j=1}^{l} \Big\{ &[\eta(t, x, \dot{x}, u, \dot{u}, y, \dot{y}, v, \dot{v})]^T \left[\xi_j g^j_y(t, y, \dot{y}, v, \dot{v}) - \frac{d}{dt}\left(\xi_j g^j_{\dot{y}}\,(t, y, \dot{y}, v, \dot{v})\right)\right] \\ &+ [\theta(t, x, \dot{x}, u, \dot{u}, y, \dot{y}, v, \dot{v})]^T \left[\xi_j g^j_v(t, y, \dot{y}, v, \dot{v}) - \frac{d}{dt}\left(\xi_j g^j_{\dot{v}}(t, y, \dot{y}, v, \dot{v})\right)\right]\Big\} dt \leqq 0. \end{aligned} \tag{4.7}$$

By the feasibility of (x, u) in (MFP) and (y, v) in (VMWD) together with $b_{h_k}(x, u, y, v) \geqq 0$, $k \in K$, we have

$$\begin{aligned} &b_{h_k}(x, u, y, v)\Phi_{h_k}\left(\int_a^b \zeta_k h^k(t, y, \dot{y}, v, \dot{v})dt - \int_a^b \zeta_k h^k(t, y, \dot{y}, v, \dot{v})dt\right) \\ &= b_{h_k}(x, u, y, v)\Phi_{h_k}(0) \leqq 0, \quad k \in K. \end{aligned} \tag{4.8}$$

By assumption, each function $\zeta^k h^k\,(t, \cdot, \cdot, \cdot, \cdot)$, $k \in K$, is univex at (y, v) on $S \cup Y$ with respect to b_{h_k}, Φ_{h_k}, η. Hence, by (4.8), Definition 2.3 yields

$$\begin{aligned} \sum_{k=1}^{s} \int_a^b \Big\{ &[\eta(t, x, \dot{x}, u, \dot{u}, y, \dot{y}, v, \dot{v})]^T \left[\zeta_k h^k_y(t, y, \dot{y}, v, \dot{v}) - \frac{d}{dt}\left(\zeta_k h^k_{\dot{y}}\,(t, y, \dot{y}, v, \dot{v})\right)\right] \\ &+ [\theta(t, x, \dot{x}, u, \dot{u}, y, \dot{y}, v, \dot{v})]^T \left[\zeta_k h^k_v(t, y, \dot{y}, v, \dot{v}) - \frac{d}{dt}\zeta_k h^k_{\dot{v}}(t, y, \dot{y}, v, \dot{v})\right]\Big\} dt \leqq 0. \end{aligned} \tag{4.9}$$

Combining (4.4), (4.7) and (4.9), we get that the inequality

$$\begin{aligned}
\int_a^b \Big\{ & \left[\eta\left(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)\right]^T \Bigg[\sum_{i=1}^{p} \lambda_i \Big(f_y^i(t,y,\dot{y},v,\dot{v}) + A_i(t)r(t) \\
& \qquad - z_i[q_y^i(t,y,\dot{y},v,\dot{v}) - C_i(t)w(t)]\Big) \\
& \qquad + \sum_{j=1}^{l} \xi_j g_y^j(t,y,\dot{y},v,\dot{v}) + \sum_{k=1}^{s} \zeta_k h_y^k(t,y,\dot{y},v,\dot{v}) \\
& \qquad - \frac{d}{dt}\Big(\sum_{i=1}^{p} \lambda_i \left(f_{\dot{y}}^i(t,y,\dot{y},v,\dot{v}) - z_i q_{\dot{y}}(t,y,\dot{y},v,\dot{v})\right) \\
& \qquad + \xi^T g_{\dot{y}}(t,y,\dot{y},v,\dot{v}) \\
& \qquad + \zeta^T h_{\dot{y}}(t,y,\dot{y},v,\dot{v})\Big)\Bigg] \\
& + [\theta\left(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v}\right)]^T \Bigg[\sum_{i=1}^{p} \lambda_i \Big(f_v^i\left(t,y,\dot{y},v,\dot{v}\right) + B_i(t)\delta(t) \\
& \qquad - z_i \left[q_v^i(t,y,\dot{y},v,\dot{v}) - E(t)\vartheta(t)\right]\Big) \\
& \qquad + \sum_{j=1}^{l} \xi_j g_v^j(t,y,\dot{y},v,\dot{v}) + \sum_{k=1}^{s} \zeta_k h_v^k(t,y,\dot{y},v,\dot{v}) \\
& \qquad - \frac{d}{dt}\Big(\sum_{i=1}^{p} \lambda_i \left(f_{\dot{v}}(t,y,\dot{y},v,\dot{v}) - z_i q_{\dot{v}}(t,y,\dot{y},v,\dot{v})\right) \\
& \qquad + \xi^T g_{\dot{v}}(t,y,\dot{y},v,\dot{v}) \\
& \qquad + \zeta^T h_{\dot{v}}(t,y,\dot{y},v,\dot{v})\Big)\Bigg]\Big\} dt < 0
\end{aligned} \tag{4.10}$$

holds, contradicting the first constraint of (VMWD).

Now, we prove this theorem under hypotheses (B).

We proceed by contradiction. Suppose, contrary to the result of the theorem, that the inequalities (4.1) are satisfied. In the similar way as in the proof of this theorem under hypothesis (A), we obtain inequalities (4.2). Multiplying each inequality (4.2) by λ_i, $i = 1, \ldots, p$, using hypothesis (b) and then summing the resulting inequalities, we get

$$\sum_{i=1}^{p} b_i(x,u,y,v)\Phi_i\Bigg(\int_a^b \lambda_i\big\{f^i(t,x,\dot{x},u,\dot{u}) + x^T A_i r + u^T B_i\delta - z_i(q^i(t,x,\dot{x},u,\dot{u}) - x^T C_i w - u^T E_i\vartheta)\big\}dt\Bigg)$$
$$< \sum_{i=1}^{p} b_i(x,u,y,v)\Phi_i\Bigg(\int_a^b \lambda_i\big\{f^i(t,y,\dot{y},v,\dot{v}) + y^T A_i r + v^T B_i\delta - z_i(q^i(t,y,\dot{y},v,\dot{v}) - y^T C_i w - v^T E_i\vartheta)\big\}dt\Bigg).$$

Hence, using hypothesis (a), by Definition 2.6, the above inequality gives

$$\int_a^b \sum_{i=1}^{p} \lambda_i\Big\{[\eta(t,x,\dot{x},u,\dot{u},\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})]^T\Big[f^i_y(t,y,\dot{y},v,\dot{v}) + A_i(t)r(t) - z_i[q^i_y(t,y,\dot{y},v,\dot{v}) - C_i(t)w(t)] - \frac{d}{dt}\left(f^i_{\dot{y}}(t,y,\dot{y},v,\dot{v}) - z_i q^i_{\dot{y}}(t,y,\dot{y},v,\dot{v})\right)\Big]$$
$$+[\theta(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v})]^T\Big[f^i_v(t,y,\dot{y},v,\dot{v}) + B_i(t)\delta(t) - z_i[q^i_v(t,y,\dot{y},v,\dot{v}) - E(t)\vartheta(t)] - \frac{d}{dt}(f_{\dot{v}}(t,y,\dot{y},v,\dot{v}) - z_i q^i_{\dot{v}}(t,y,\dot{y},v,\dot{v}))\Big]\Big\}dt < 0. \tag{4.11}$$

From the feasibility of (x,u) in (MFP) and the feasibility of $(y,v,z,\lambda,\xi,\zeta,r,\delta,w,\vartheta)$ in (VMWD) together with hypothesis (e), we get

$$\sum_{j=1}^{l} b_{g_j}(x,u,y,v)\Phi_{g_j}\left(\int_a^b \xi_j g^j(t,x,\dot{x},u,\dot{u})dt - \int_a^b \xi_j g^j(t,y,\dot{y},v,\dot{v})dt\right)$$
$$\leqq \sum_{j=1}^{l} b_{g_j}(x,u,y,v)\Phi_{g_j}(0) \leqq 0. \tag{4.12}$$

Using hypothesis (c), by Definition 2.8, (4.12) yields

$$\int_a^b \sum_{j=1}^{l} \Big\{ [\eta(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v})]^T \Big[\xi_j g_y^j(t,y,\dot{y},v,\dot{v}) - \frac{d}{dt}\Big(\xi_j g_{\dot{y}}^j(t,y,\dot{y},v,\dot{v})\Big) \Big] + [\theta(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v})]^T \Big[\xi_j g_v^j(t,y,\dot{y},v,\dot{v}) - \frac{d}{dt}\Big(\xi_j g_{\dot{v}}^j(t,y,\dot{y},v,\dot{v})\Big) \Big] \Big\} dt \leqq 0. \tag{4.13}$$

Again from the feasibility of (x,u) in (MFP) and the feasibility of $(y,v,z,\lambda,\xi,\zeta,r,\delta,w,\vartheta)$ in (VMWD) together with hypothesis (e), we get

$$\sum_{k=1}^{s} b_{h_k}(x,u,y,v)\Phi_{h_k}\left(\int_a^b \zeta_k h^k(t,x,\dot{x},u,\dot{u})dt - \int_a^b \zeta_k h^k(t,y,\dot{y},v,\dot{v})dt \right) = \sum_{k=1}^{s} b_{h_k}(x,u,y,v)\Phi_{h_k}(0) \leqq 0. \tag{4.14}$$

Using hypothesis (d), by Definition 2.8, (4.14) yields

$$\int_a^b \sum_{k=1}^{s} \Big\{ [\eta(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v})]^T \Big[\zeta_k h_y^k(t,y,\dot{y},v,\dot{v}) - \frac{d}{dt}\big(\zeta_k h_{\dot{y}}^k(t,y,\dot{y},v,\dot{v})\big) \Big] + [\theta(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v})]^T \Big[\zeta_k h_v^k(t,y,\dot{y},v,\dot{v}) - \frac{d}{dt}\big(\zeta_k h_{\dot{v}}^k(t,y,\dot{y},v,\dot{v})\big) \Big] \Big\} dt \leqq 0. \tag{4.15}$$

Combining (4.10), (4.13) and (4.15), we get that the inequality (4.10) holds, contradicting the first constraint of (VMWD).
This completes the proof of this theorem. □

If some stronger univexity hypotheses are assumed, then the following result can be proved.

Theorem 4.2 (Weak duality)**.** *Let* (x,u) *and* $(y,v,z,\lambda,\xi,\zeta,r,\delta,w,\vartheta)$ *be any feasible solutions in problems* (MFP) *and* (VMWD)*. Further, assume that the following hypotheses are fulfilled:*

(A)

(a)

$$\Bigg(\int_a^b \Big\{f^1(t,\cdot,\cdot,\cdot,\cdot)+\cdot^T A_1(t)\bar{r}(t)+\cdot^T B_1(t)\delta(t) - z_1[q^1(t,\cdot,\cdot,\cdot,\cdot)-\cdot^T C_1(t)w(t)-\cdot^T E_1(t)\vartheta(t)]\Big\}dt, \dots, \int_a^b \Big\{f^p(t,\cdot,\cdot,\cdot,\cdot)+\cdot^T A_p(t)r(t)+\cdot^T B_p(t)\delta(t) - \bar{z}_p[q^p(t,\cdot,\cdot,\cdot,\cdot)-\cdot^T C_p(t)w(t)-\cdot^T E_p(t)\vartheta(t)]\Big\}dt\Bigg)$$

is strictly univex at (y,v) *on* $S\cup Y$ *with respect to* b_f, Φ_f, η, θ, *where* $b_f(x,u,y,v)>0$,

(b)

$$\left(\int_a^b \xi_1(t)g^1(t,\cdot,\cdot,\cdot,\cdot)\,dt,\dots,\int_a^b \xi_l(t)g^l(t,\cdot,\cdot,\cdot,\cdot)dt\right)$$

is univex at (y,v) *on* $S\cup Y$ *with respect to* $b_g=(b_{g_1},\dots,b_{g_l})$, $\Phi_g=(\Phi_{g_1},\dots,\Phi_{g_l})$, η, θ,

(c)

$$\left(\int_a^b \zeta_1(t)h^1(t,\cdot,\cdot,\cdot,\cdot)\,dt,\dots,\int_a^b \zeta_s(t)h^s(t,\cdot,\cdot,\cdot,\cdot)dt\right)$$

is univex at (y,v) *on* $S\cup Y$ *with respect to* $b_h=(b_{h_1},\dots,b_{h_s})$, $\Phi_h=(\Phi_{h_1},\dots,\Phi_{h_s})$, η, θ,

(d) $b_i(x,u,y,v)>0$, $i=1,\dots,p$,

(e) Φ_{f_i}, $i=1,\dots,k$, Φ_{g_j}, $j=1,\dots,l$, Φ_{h_1}, $k=1,\dots,s$, *are increasing functionals satisfying* $a\leqq 0\Longrightarrow\Phi_{f_i}(a)<0$, $\Phi_{f_i}(0)\leqq 0$, $a\leqq 0\Longrightarrow\Phi_{g_j}(a)\leqq 0$, $a\leqq 0\Longrightarrow\Phi_{h_k}(a)\leqq 0$.

(B)

(a)

$$\left(\int_a^b \lambda_1\Big\{f^1(t,\cdot,\cdot,\cdot,\cdot) + \cdot^T A_1(t)\bar{r}(t) + \cdot^T B_1(t)\delta(t) - z_1[q^1(t,\cdot,\cdot,\cdot,\cdot) - \cdot^T C_1(t)w(t) - \cdot^T E_1(t)\vartheta(t)]\Big\}dt, \ldots, \int_a^b \lambda_1\Big\{f^p(t,\cdot,\cdot,\cdot,\cdot) + \cdot^T A_p(t)r(t) + \cdot^T B_p(t)\delta(t) - \bar{z}_p[q^p(t,\cdot,\cdot,\cdot,\cdot) - \cdot^T C_p(t)w(t) - \cdot^T E_p(t)\vartheta(t)]\Big\}dt\right)$$

is strictly pseudo-univex at (y,v) *on* $S \cup Y$ *with respect to* $b = (b_1, \ldots, b_p)$, $\Phi = (\Phi_1, \ldots, \Phi_p)$, η, θ,

(b) $b_i(x,u,y,v) > 0$, $i = 1, \ldots, p$,

(c)

$$\left(\int_a^b \xi_1(t)g^1(t,\cdot,\cdot,\cdot,\cdot)\,dt, \ldots, \int_a^b \xi_l(t)g^l(t,\cdot,\cdot,\cdot,\cdot)dt\right)$$

is univex at (y,v) *on* $S \cup Y$ *with respect to* $b_g = (b_{g_1}, \ldots, b_{g_l})$, $\Phi_g = (\Phi_{g_1}, \ldots, \Phi_{g_l})$, η, θ,

(d)

$$\left(\int_a^b \zeta_1(t)h^1(t,\cdot,\cdot,\cdot,\cdot)\,dt, \ldots, \int_a^b \zeta_s(t)h^s(t,\cdot,\cdot,\cdot,\cdot)dt\right)$$

is univex at (y,v) *on* $S \cup Y$ *with respect to* $b_h = (b_{h_1}, \ldots, b_{h_s})$, $\Phi_h = (\Phi_{h_1}, \ldots, \Phi_{h_s})$, η, θ,

(e) Φ_i, $i = 1, \ldots, p$, *are increasing functionals,* Φ_{g_j}, $j = 1, \ldots, l$, Φ_{h_1}, $k = 1, \ldots, s$, *are increasing functionals satisfying* $a \leqq 0 \Longrightarrow \Phi_{g_j}(a) \leqq 0$, $a \leqq 0 \Longrightarrow \Phi_{h_k}(a) \leqq 0$.

Then, the following cannot hold

$$\int_a^b \Big\{ f^i(t,x,\dot{x},u,\dot{u}) + \sqrt{x^T A_i(t) x} + \sqrt{u^T B_i(t) u} - z_i \Big(q^i(t,x,\dot{x},u,\dot{u}) - \sqrt{x^T C_i(t) x} - \sqrt{u^T E_i(t) u} \Big) \Big\} dt$$
$$\leqq \int_a^b \Big\{ f^i(t,y,\dot{y},v,\dot{v}) + \sqrt{y^T A_i(t) y} + \sqrt{v^T B_i(t) v} - z_i \Big(q^i(t,y,\dot{y},v,\dot{v}) - \sqrt{y^T C_i(t) y} - \sqrt{v^T E_i(t) v} \Big) \Big\} dt, \quad i \in P,$$

and

$$\int_a^b \Big\{ f^{i^*}(t,x,\dot{x},u,\dot{u}) + \sqrt{x^T A_{i^*}(t) x} + \sqrt{u^T B_{i^*}(t) u} - z_{i^*} \Big(q^{i^*}(t,x,\dot{x},u,\dot{u}) + \sqrt{x^T C_{i^*}(t) x} + \sqrt{u^T E_{i^*}(t) u} \Big) \Big\} dt$$
$$< \int_a^b \Big\{ f^{i^*}(t,y,\dot{y},v,\dot{v}) + \sqrt{y^T A_{i^*}(t) y} + \sqrt{v^T B_{i^*}(t) v} - z_{i^*} \Big(q^{i^*}(t,y,\dot{y},v,\dot{v}) + \sqrt{y^T C_{i^*}(t) y} + \sqrt{v^T E_{i^*}(t) v} \Big) \Big\} dt$$

for at least one $i^* \in P$.

Theorem 4.3 (Strong duality). *Let* $(\bar{x}(t), \bar{u}(t))$ *be a weakly efficient solution* (*an efficient solution*) *of the considered multiobjective variational programming problem* (MFP). *Further, assume that the Kuhn–Tucker constraint qualification is satisfied for* (MFP). *Then, there exists* $\bar{\lambda} \in R^p$, $\bar{z} \in R^p$ *and piecewise smooth functions* $\bar{\xi}(\cdot) : I \to R^l$ *and* $\bar{\zeta}(\cdot) : I \to R^s$, $\bar{r}(\cdot) : I \to R^n$, $\bar{w}(\cdot) : I \to R^n$, $\bar{\delta}(\cdot) : I \to R^m$, $\bar{\vartheta}(\cdot) : I \to R^m$ *such that* $\big(\bar{x}(t), \bar{u}(t), \bar{z}, \bar{\lambda}, \bar{\xi}(t), \bar{\zeta}(t), \bar{r}(t), \bar{\delta}(t), \bar{w}(t), \bar{\vartheta}(t)\big)$ *is a feasible solution for the Mond–Weir type multiobjective variational dual problem* (VMWD). *If also the weak duality theorem holds between* (MFP) *and* (VMWD), *then* $\big(\bar{x}, \bar{u}, \bar{z}, \bar{\lambda}, \bar{\xi}(t), \bar{\zeta}(t), \bar{r}(t), \bar{\delta}(t), \bar{w}(t), \bar{\vartheta}(t)\big)$ *is a weakly efficient solution* (*an efficient solution*) *of a maximum type of* (VMWD) *and the objective function values are equal.*

Proof. By assumption, $(\bar{x}(t), \bar{u}(t))$ is an efficient solution in the considered multiobjective variational programming problem (MFP). Hence, by Theorem 3.5, there exists $\bar{\lambda} \in R^p$, $\bar{z} \in R^p$ and piecewise smooth functions $\bar{\xi}(\cdot) : I \to R^l$ and $\bar{\zeta}(\cdot) : I \to R^s$, $\bar{r}(\cdot) : I \to R^n$, $\bar{w}(\cdot) : I \to R^n$, $\bar{\delta}(\cdot) : I \to R^m$, $\bar{\vartheta}(\cdot) : I \to R^m$ such that the Karush–Kuhn–Tucker optimality conditions (3.3)–(3.5) are satisfied. Thus, by the Karush–Kuhn–Tucker optimality conditions (3.3)–(3.5), it follows that

$(\bar{x}(t), \bar{u}(t), \bar{z}, \bar{\lambda}, \bar{\xi}(t), \bar{\zeta}(t), \bar{r}(t), \bar{\delta}(t), \bar{w}(t), \bar{\vartheta}(t))$ is a feasible solution the Mond–Weir type multiobjective variational dual problem (VMWD) and the two objective functionals have same values. Efficiency of $(\bar{x}(t), \bar{u}(t), \bar{z}, \bar{\lambda}, \bar{\xi}(t), \bar{\zeta}(t), \bar{r}(t), \bar{\delta}(t), \bar{w}(t), \bar{\vartheta}(t))$ in the problem (VMWD) follows directly from weak duality (Theorem 4.2). The proof in the case when $(\bar{x}, \bar{u})$ is a weakly efficient is similar and it follows from Theorem 4.1. □

Theorem 4.4 (Strict converse duality). *Let* $(\bar{x}, \bar{u})$ *and* $\left(\bar{y}, \bar{v}, \bar{z}, \bar{\lambda}, \bar{\xi}, \bar{\zeta}, \bar{r}, \bar{\delta}, \bar{w}, \bar{\vartheta}\right)$ *be weakly efficient solutions of* (MFP) *and* (VMWD)*, respectively, the Kuhn–Tucker constraint qualification is satisfied for* (MFP)*. Further, assume that the following hypotheses are fulfilled:*

(a)

$$\Bigg(\int_a^b \Big\{ f^1(t,\cdot,\cdot,\cdot,\cdot) + \cdot^T A_1(t)\bar{\bar{r}}(t) + \cdot^T B_1(t)\bar{\delta}(t) - \bar{\bar{z}}_1[q^1(t,\cdot,\cdot,\cdot,\cdot) - \cdot^T C_1(t)\bar{w}(t) - \cdot^T E_1(t)\bar{\vartheta}(t)]\Big\} dt, \ldots, \int_a^b \Big\{ f^p(t,\cdot,\cdot,\cdot,\cdot) + \cdot^T A_p(t)\bar{\bar{r}}(t) + \cdot^T B_p(t)\bar{\delta}(t) - \bar{\bar{z}}_p[q^p(t,\cdot,\cdot,\cdot,\cdot) - \cdot^T C_p(t)\bar{w}(t) - \cdot^T E_p(t)\bar{\vartheta}(t)]\Big\} dt\Bigg)$$

is strictly univex at $(\bar{y}, \bar{v})$ *on* $S \cup Y$ *with respect to* $b = (b_1, \ldots, b_p)$, $\Phi = (\Phi_1, \ldots, \Phi_p)$, η, θ,

(b)

$$\left(\int_a^b \bar{\xi}_1(t) g^1(t,\cdot,\cdot,\cdot,\cdot)dt, \ldots, \int_a^b \bar{\xi}_l(t) g^l(t,\cdot,\cdot,\cdot,\cdot)dt\right)$$

is univex at $(\bar{y}, \bar{v})$ *on* $S \cup Y$ *with respect to* $b_g = (b_{g_1}, \ldots, b_{g_l})$, $\Phi_g = (\Phi_{g_1}, \ldots, \Phi_{g_l})$, η, θ,

(c)

$$\left(\int_a^b \bar{\zeta}_1(t) h^1(t,\cdot,\cdot,\cdot,\cdot)dt, \ldots, \int_a^b \bar{\zeta}_s(t) h^s(t,\cdot,\cdot,\cdot,\cdot)dt\right)$$

is univex at $(\bar{y}, \bar{v})$ *on* $S \cup Y$ *with respect to* $b_h = (b_{h_1}, \ldots, b_{h_s})$, $\Phi_h = (\Phi_{h_1}, \ldots, \Phi_{h_s})$, η, θ,

(d) $b_i(\bar{x}, \bar{u}, \bar{y}, \bar{v}) > 0$, $i = 1, \ldots, p$,

(e) Φ_i, $i = 1, \ldots, p$, *are strictly increasing functionals satisfying* $a < 0 \Longrightarrow \Phi_i(a) < 0$, $\Phi_i(0) \leqq 0$, Φ_{g_j}, $j \in J$, Φ_{h_k}, $k \in K$, *are increasing functionals satisfying* $a \leqq 0 \Longrightarrow \Phi_{g_j}(a) \leqq 0$, $a \leqq 0 \Longrightarrow \Phi_{h_k}(a) \leqq 0$, $k \in K$.

Then $(\bar{x}, \bar{u}) = (\bar{y}, \bar{v})$.

Proof. Suppose, contrary to the result, that $(\bar{x},\bar{u}) \neq (\bar{y},\bar{v})$. Since $(\bar{x},\bar{u})$ and $\left(\bar{y},\bar{v},\bar{z},\bar{\lambda},\bar{\xi},\bar{\zeta},\bar{r},\bar{\delta},\bar{w},\bar{\vartheta}\right)$ are weakly efficient solutions of (MFP) and (VMWD), respectively, by the strong duality theorem (Theorem 4.3), we have

$$\begin{aligned}
&\int_a^b \Bigg\{ f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sqrt{\bar{x}^T A_i(t)\bar{x}} + \sqrt{\bar{u}^T B_i(t)\bar{u}} \\
&\qquad - \bar{z}_i \left(q^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \sqrt{\bar{x}^T C_i(t)\bar{x}} - \sqrt{\bar{u}^T E_i(t)\bar{u}} \right) \Bigg\} dt \\
&= \int_a^b \Bigg\{ f^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) + \sqrt{\bar{y}^T A_i(t)\bar{y}} + \sqrt{\bar{v}^T B_i(t)\bar{v}} \\
&\qquad - \bar{z}_i \left(q^i\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right) - \sqrt{\bar{y}^T C_i(t)\bar{y}} - \sqrt{\bar{v}^T E_i(t)\bar{v}} \right) \Bigg\} dt, \quad i \in P.
\end{aligned}$$

By $\left(\bar{y},\bar{v},\bar{z},\bar{\lambda},\bar{\xi},\bar{\zeta},\bar{r},\bar{\delta},\bar{w},\bar{\vartheta}\right) \in \Omega_{MW}$ and the generalized Schwarz inequality (see (2.1)), (4.1) yields

$$\begin{aligned}
&\int_a^b \left\{ f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \bar{x}^T A_i \bar{r} + \bar{u}^T B_i \bar{\delta} - \bar{z}_i \left[q^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{x}^T C_i \bar{w} - \bar{u}^T E_i \bar{\vartheta} \right] \right\} dt \\
&= \int_a^b \left\{ f^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) + \bar{y}^T A_i \bar{r} + \bar{v}^T B_i \bar{\delta} - \bar{z}_i \left[q^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) - \bar{y}^T C_i \bar{w} - \bar{v}^T E_i \bar{\vartheta} \right] \right\} dt,
\end{aligned}$$

$i \in P$.

$$\tag{4.16}$$

Hence, by assumption (e), (4.16) yields

$$\begin{aligned}
&\sum_{i=1}^p b_i(\bar{x},\bar{u},\bar{y},\bar{v})\Phi_i\Bigg(\int_a^b \Big\{ f^i\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right) + \bar{x}^T A_i \bar{r} + \bar{u}^T B_i \bar{\delta} \\
&\qquad\qquad - \bar{z}_i \left[q^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{x}^T C_i \bar{w} - \bar{u}^T E_i \bar{\vartheta} \right] \Big\} dt \\
&\qquad - \int_a^b \Big\{ f^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) + \bar{y}^T A_i \bar{r} + \bar{v}^T B_i \bar{\delta} \\
&\qquad\qquad - \bar{z}_i [q^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) - \bar{y}^T C_i \bar{w} - \bar{v}^T E_i \bar{\vartheta}] \Big\} dt \Bigg) \\
&= \sum_{i=1}^p b_i(\bar{x},\bar{u},\bar{y},\bar{v})\Phi_i(0) \leqq 0.
\end{aligned} \tag{4.17}$$

Using assumption (a), by Definition 2.3, (4.17) gives

$$\int_a^b \Big\{ \left[\eta\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}},\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)\right]^T \Big[f_y^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) + A_i(t)\bar{r}(t) - \bar{z}_i\left[q_y^i\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right) - C_i(t)\bar{w}(t)\right] - \frac{d}{dt}\left(f_{\dot{y}}^i\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right) - \bar{z}_i q_{\dot{y}}^i\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)\right)\Big] + \left[\theta\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}},\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)\right]^T \Big[f_v^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) + B_i(t)\bar{\delta}(t) - \bar{z}_i\left[q_v^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) - E(t)\bar{\vartheta}(t)\right] - \frac{d}{dt}\big(f_{\dot{v}}^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) - \bar{z}_i q_{\dot{v}}^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}})\big)\Big]\Big\} dt < 0.$$

Thus, $\left(\bar{y},\bar{v},\bar{z},\bar{\lambda},\bar{\xi}(t),\bar{\zeta}(t),\bar{r}(t),\bar{\delta}(t),\bar{w}(t),\bar{\vartheta}(t)\right) \in \Omega_{MW}$ yields

$$\int_a^b \sum_{i=1}^p \bar{\lambda}_i \Big\{ [\eta(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}},\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}})]^T \Big[f_y^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) + A_i(t)\bar{r}(t) - \bar{z}_i\left[q_y^i\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right) - C_i(t)\bar{w}(t)\right] - \frac{d}{dt}\left(f_{\dot{y}}^i\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right) - \bar{z}_i q_{\dot{y}}^i\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)\right)\Big] + \left[\theta\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}},\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)\right]^T \Big[f_v^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) + B_i(t)\bar{\delta}(t) - \bar{z}_i\left[q_v^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) - E(t)\bar{\vartheta}(t)\right] - \frac{d}{dt}\big(f_{\dot{v}}^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) - \bar{z}_i q_{\dot{v}}^i(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}})\big)\Big]\Big\} dt < 0. \tag{4.18}$$

Using $(\bar{x},\bar{u}) \in S$, $\left(\bar{y},\bar{v},\bar{z},\bar{\lambda},\bar{\xi},\bar{\zeta},\bar{r},\bar{\delta},\bar{w},\bar{\vartheta}\right) \in \Omega_{MW}$ together with hypothesis (e), we get

$$b_{g_j}(\bar{x},\bar{u},\bar{y},\bar{v})\Phi_{g_j}\left(\int_a^b \xi_j g^j\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}\right)dt - \int_a^b \xi_j g^j\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)dt\right) \leqq b_{g_j}(\bar{x},\bar{u},\bar{y},\bar{v})\Phi_{g_j}(0) \leqq 0, \quad j \in J(t). \tag{4.19}$$

From hypothesis (b), by Definition 2.3, (4.19) yields

$$\begin{aligned}\int_a^b \sum_{j=1}^{l} \Big\{ & \left[\eta\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}},\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)\right]^T \left[\bar{\xi}_j g_y^j\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)+\bar{\xi}_j g_v^j\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)\right] \\ & -\frac{d}{dt}\Big(\left[\eta\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}},\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)\right]^T \Big[\bar{\xi}_j g_{\dot{y}}^j\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right) \\ & \qquad + \bar{\xi}_j g_{\dot{v}}^j\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)\Big]\Big)\Big\} dt \leqq 0. \end{aligned} \tag{4.20}$$

By $(\bar{x},\bar{u}) \in S$, $\left(\bar{y},\bar{v},\bar{z},\bar{\lambda},\bar{\xi},\bar{\zeta},\bar{r},\bar{\delta},\bar{w},\bar{\vartheta}\right) \in \Omega_{MW}$ and $b_{h_k}\left(\bar{x},\bar{u},\bar{y},\bar{v}\right) \geqq 0$, $k \in K$, we obtain

$$\begin{aligned} & \sum_{k=1}^{s} b_{h_k}(\bar{x},\bar{u},\bar{y},\bar{v})\Phi_{h_k}\left(\int_a^b \bar{\zeta}_k h^k(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})dt - \int_a^b \bar{\zeta}_k h^k\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)dt\right) \\ & = \sum_{k=1}^{s} b_{h_k}\left(x,u,\bar{y},\bar{v}\right)\Phi_{h_k}(0) \leqq 0. \end{aligned} \tag{4.21}$$

Using assumptions (c) and (e), by Definition 2.3, (4.21) yields

$$\begin{aligned}\int_a^b \sum_{k=1}^{s} \Big\{ & \left[\eta\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}},\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)\right]^T \left[\bar{\zeta}_k h_y^k\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right) - \frac{d}{dt}\left(\bar{\zeta}_k h_{\dot{y}}^k(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}})\right)\right] \\ & + \left[\theta\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}},\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)\right]^T \Big[\bar{\zeta}_k h_v^k(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}) \\ & \qquad - \frac{d}{dt}\left(\bar{\zeta}_k h_{\dot{v}}^k\left(t,\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)\right)\Big]\Big\} dt \leqq 0. \end{aligned} \tag{4.22}$$

Combining (4.18), (4.20) and (4.22), we get that the inequality

$$\int_a^b \Bigg\{ \left[\eta\left(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}},\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}}\right)\right]^T \Big[\sum_{i=1}^p \bar{\lambda}_i (f_y^i(t,y,\dot{y},v,\dot{v}) + A_i(t)r(t)$$

$$- \bar{z}_i[q_y^i(t,y,\dot{y},v,\dot{v}) - C_i(t)w(t)])$$

$$+ \sum_{j=1}^l \xi_j g_y^j\left(t,y,\dot{y},v,\dot{v}\right) + \sum_{k=1}^s \zeta_k h_y^k(t,y,\dot{y},v,\dot{v})$$

$$-\frac{d}{dt}\Big(\sum_{i=1}^p \lambda_i \big(f_{\dot{y}}^i(t,y,\dot{y},v,\dot{v})$$

$$- \bar{z}_i q_{\dot{y}}(t,y,\dot{y},v,\dot{v})\big)\Big)\Big]$$

$$+[\theta(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}},\bar{y},\dot{\bar{y}},\bar{v},\dot{\bar{v}})]^T \Big[\sum_{i=1}^p \bar{\lambda}_i (f_v^i(t,y,\dot{y},v,\dot{v}) + B_i(t)\delta(t)$$

$$- \bar{z}_i\left[q_v^i(t,y,\dot{y},v,\dot{v}) - E(t)\vartheta(t)\right])$$

$$+ \sum_{j=1}^l \xi_j g_v^j(t,y,\dot{y},v,\dot{v}) + \sum_{k=1}^s \bar{\zeta}_k h_v^k(t,y,\dot{y},v,\dot{v})$$

$$-\frac{d}{dt}\Big(\sum_{i=1}^p \lambda_i \big(f_{\dot{v}}(t,y,\dot{y},v,\dot{v})$$

$$- z_i q_{\dot{v}}(t,y,\dot{y},v,\dot{v})\big)\Big)$$

$$+ \xi^T g_{\dot{v}}(t,y,\dot{y},v,\dot{v}) + \zeta^T h_{\dot{v}}(t,y,\dot{y},v,\dot{v})\Big]\Bigg\} dt < 0$$

holds, contradicting the first constraint of (VMWD). This completes the proof of this theorem. □

5. WOLFE TYPE DUALITY

In this section, we prove duality results between the considered multiobjective fractional variational control problem (MFP) and its Wolfe type multiobjective variational dual problem (VWD).

The problem (VWD) is defined as follows:

$$\underset{y,v}{\text{Maximize}} \left(\int_a^b \Big\{ f^1(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) + \sqrt{y(t)^T A_1(t) y(t)} + \sqrt{v(t)^T B_1(t) v(t)} \right.$$
$$- z_1 \left[q^1(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) - \sqrt{y(t)^T C_1(t) y(t)} - \sqrt{v(t)^T E_1(t) v(t)} \right]$$
$$+ \xi(t)^T g\,(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) + \zeta(t)^T h(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) \Big\}\, dt,$$
$$\ldots,$$
$$\int_a^b \Big\{ f^p\,(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) + \sqrt{y(t)^T A_p(t) y(t)} + \sqrt{v(t)^T B_p(t) v(t)}$$
$$- z_p \left[q^p(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) - \sqrt{y(t)^T C_p(t) y(t)} - \sqrt{v(t)^T E_p(t) v(t)} \right]$$
$$\left. + \xi(t)^T g(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) + \zeta(t)^T h\,(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) \Big\} dt \right)$$

subject to $x(a) = \alpha,\ x(b) = \beta$,

$$\sum_{i=1}^p \lambda_i \big[f_y^i(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) + A_i(t) r(t)$$
$$- z_i \left[q_y^i(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) - C_i(t) w(t) \right] \big]$$
$$+ \xi(t)^T g_y\,(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) + \zeta(t)^T h_y(t, y(t), \dot{y}(t), v(t), \dot{v}(t))$$
$$= \frac{d}{dt} \Big[\sum_{i=1}^p \lambda_i \left(f_{\dot{y}}^i(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) - z_i q_{\dot{y}}^i\,(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) \right)$$
$$+ \xi(t)^T g_{\dot{y}}(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) + \zeta(t)^T h_{\dot{y}}\,(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) \Big], \quad t \in I,$$

$$\sum_{i=1}^p \lambda_i \big[f_v^i(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) + B_i(t) \delta(t)$$
$$- z_i [q_v^i\,(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) - E_i(t) \vartheta(t)]]$$
$$+ \xi(t)^T g_v(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) + \zeta(t)^T h_v\,(t, y(t), \dot{y}(t), v(t), \dot{v}(t))]$$
$$= \frac{d}{dt} \Big[\sum_{i=1}^p \lambda_i \left(f_{\dot{v}}^i(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) - q_{\dot{v}}^i(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) \right)$$
$$+ \xi(t)^T g_{\dot{v}}(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) + \zeta(t)^T h_{\dot{y}}\,(t, y(t), \dot{y}(t), v(t), \dot{v}(t)) \Big], \quad t \in I,$$

$$r(t) \in R^n, \quad \delta(t) \in R^m, \quad r(t)^T A_i(t) r(t) \leqq 1, \quad \delta(t)^T B_i(t)\delta(t) \leqq 1,$$
$$w(t) \in R^n, \quad \vartheta(t) \in R^m, \quad w(t)^T C_i(t) w(t) \leqq 1, \quad \vartheta(t)^T E_i(t)\vartheta(t) \leqq 1,$$

$$\begin{aligned} y^T(t)A_i(t)r(t) &= \sqrt{y(t)^T A_i(t) y(t)}, \\ v(t)^T B_i(t)\delta(t) &= \sqrt{v(t)^T B_i v(t)}, \\ y(t)^T C_i(t) w(t) &= \sqrt{y(t)^T C_i(t) y(t)}, \\ v(t)^T E_i(t)\vartheta(t) &= \sqrt{v(t)^T E_i(t) v(t)}, \end{aligned}$$

for $t \in I$, $i = 1, \dots, p$,

$$\lambda \geq 0, \quad \lambda^T e = 1, \quad \xi(t) \geqq 0,$$

where $e = (1, \dots, 1) \in R^p$ is a p-dimensional vector. It may be noted here that the above dual constraints are written using the Karush–Kuhn–Tucker necessary optimality conditions for the problem (MFP).

Let Ω_W be the set of all feasible solutions $(y, v, z, \lambda, \xi, \zeta, r, \delta, w, \vartheta)$ in Wolfe type multiobjective variational dual problem (VWD). We denote by Y_W the set

$$Y_W = \{(y, v) \in X \times U : (y, v, z, \lambda, \xi, \zeta, r, \delta, w, \vartheta) \in \Omega_W\}.$$

Further, we define the vector-valued function Γ for (MFP(z)) as follows:

$$\Gamma(y, v, z, \xi, \zeta, r, \delta, w, \vartheta) = (\Gamma_1(y, v, z, \xi, \zeta, r, \delta, w, \vartheta), \dots, \Gamma_p(y, v, z, \xi, \zeta, r, \delta, w, \vartheta)),$$

where

$$\begin{aligned} &\Gamma_i(y, v, z, \xi, \zeta, r, \delta, w, \vartheta) \\ &= \int_a^b \Big\{ f^i(t, y, \dot{y}, v, \dot{v}) + y^T A_i(t) r(t) + v^T B_i(t)\delta(t) \\ &\qquad - z_i \left(q^i(t, y, \dot{y}, v, \dot{v}) - y^T C_i(t) w(t) - v^T E_i(t)\vartheta(t) \right) \\ &\qquad + \xi(t)^T g(t, y, \dot{y}, v, \dot{v}) + \zeta(t)^T h(t, y, \dot{y}, v, \dot{v}) \Big\} dt, \quad i \in P. \end{aligned}$$

Theorem 5.1 (Weak duality). *Let* (x, u) *and* $(y, v, z, \lambda, \xi, \zeta, r, \delta, w, \vartheta)$ *be any feasible solutions in problems* (MFP) *and* (VWD). *Further, assume that the following hypotheses are fulfilled:*

(A)
- (a) *the function* $\Gamma(\cdot, \cdot, z, \xi, \zeta, r, \delta, w, \vartheta)$ *is univex at* (y, v) *on* $S \cup Y_W$ *with respect to* $b = (b_1, \dots, b_p)$, $\Phi = (\Phi_1, \dots, \Phi_p)$, η, θ,
- (b) $b_i(x, u, y, v) > 0$, $i = 1, \dots, p$,
- (c) Φ_i, $i = 1, \dots, p$, *are strictly increasing functionals satisfying* $a < 0 \Longrightarrow \Phi_i(a) < 0$, $\Phi_i(0) \leqq 0$.

(B)
(a) *the function* $\Gamma(\cdot,\cdot,z,\xi,\zeta,r,\delta,w,\vartheta)$ *is pseudo-univex at* (y,v) *on* $S\cup Y$ *with respect to* $b=(b_1,\ldots,b_p)$, $\Phi=(\Phi_1,\ldots,\Phi_p)$, η, θ,
(b) $b_i(x,u,y,v)>0$, $i=1,\ldots,p$,
(c) Φ_i, $i=1,\ldots,p$, *are strictly increasing functionals satisfying* $a<0 \Longrightarrow \Phi_i(a)<0$, $\Phi_i(0)\leqq 0$.

Then, the following cannot hold

$$\begin{aligned}&\int_a^b \Big\{ f^i(t,x,\dot{x},u,\dot{u})+\sqrt{x^TA_i(t)x}+\sqrt{u^TB_i(t)u}\\&\qquad -z_i\Big(q^i(t,x,\dot{x},u,\dot{u})-\sqrt{x^TC_i(t)x}-\sqrt{u^TE_i(t)u}\Big)\Big\}dt\\&<\int_a^b \Big\{ f^i(t,y,\dot{y},v,\dot{v})+\sqrt{y^TA_i(t)y}+\sqrt{v^TB_i(t)v}\\&\qquad -z_i(q^i(t,y,\dot{y},v,\dot{v})-\sqrt{y^TC_i(t)y}-\sqrt{v^TE_i(t)v})\\&\qquad +\xi(t)^Tg(t,y,\dot{y},v,\dot{v})+\zeta(t)^Th(t,y,\dot{y},v,\dot{v})\Big\}dt,\quad i\in P.\end{aligned}\tag{5.1}$$

Proof. Suppose, contrary to the result of the theorem, that the inequalities (5.1) are satisfied. From $(x,u)\in S$ and $(y,v,z,\lambda,\xi,\zeta,r,\delta,w,\vartheta)\in\Omega_W$, we have

$$\begin{aligned}&\int_a^b \Big\{ f^i(t,x,\dot{x},u,\dot{u})+\sqrt{x^TA_i(t)x}+\sqrt{u^TB_i(t)u}\\&\qquad -z_i\left(q^i(t,x,\dot{x},u,\dot{u})-\sqrt{x^TC_i(t)x}-\sqrt{u^TE_i(t)u}\right)\\&\qquad +\xi(t)^Tg(t,x,\dot{x},u,\dot{u})+\zeta(t)^Th(t,x,\dot{x},u,\dot{u})\Big\}dt\\&<\int_a^b \Big\{ f^i(t,y,\dot{y},v,\dot{v})+\sqrt{y^TA_i(t)y}+\sqrt{v^TB_i(t)v}\\&\qquad -z_i\left(q^i\left(t,y,\dot{y},v,\dot{v}\right)-\sqrt{y^TC_i(t)y}-\sqrt{v^TE_i(t)v}\right)\\&\qquad +\xi(t)^Tg(t,y,\dot{y},v,\dot{v})+\zeta(t)^Th\left(t,y,\dot{y},v,\dot{v}\right)\}dt,\quad i\in P.\end{aligned}\tag{5.2}$$

By $(y, v, z, \lambda, \xi, \zeta, r, \delta, w, \vartheta) \in \Omega_{MW}$ and the generalized Schwarz inequality (see Lemma 2.1), (5.2) yields

$$\begin{aligned}
&\int_a^b \Big\{ f^i(t, x, \dot{x}, u, \dot{u}) + x^T A_i r + u^T B_i \delta \\
&\quad - z_i \left(q^i\left(t, x, \dot{x}, u, \dot{u}\right) - x^T C_i w - u^T E_i \vartheta \right) \\
&\quad + \xi(t)^T g(t, x, \dot{x}, u, \dot{u}) \\
&\quad + \zeta(t)^T h(t, x, \dot{x}, u, \dot{u}) \Big\} dt \\
&< \int_a^b \Big\{ f^i(t, y, \dot{y}, v, \dot{v}) + y^T A_i r + v^T B_i \delta \\
&\quad - z_i \left(q^i(t, y, \dot{y}, v, \dot{v}) - y^T C_i w - v^T E_i \vartheta \right) \\
&\quad + \xi(t)^T g(t, y, \dot{y}, v, \dot{v}) \\
&\quad + \zeta(t)^T h\left(t, y, \dot{y}, v, \dot{v}\right) \Big\} dt, \quad i = 1, \ldots, p.
\end{aligned} \tag{5.3}$$

Now, we prove this theorem under hypothesis (A).
Using hypotheses (b) and (c), (5.3) gives

$$\begin{aligned}
&b_i(x, u, y, v) \Phi_i \Bigg(\int_a^b \Big\{ f^i(t, x, \dot{x}, u, \dot{u}) + x^T A_i r + u^T B_i \delta \\
&\quad - z_i \left(q^i\left(t, x, \dot{x}, u, \dot{u}\right) - x^T C_i w - u^T E_i \vartheta \right) \\
&\quad + \xi(t)^T g(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \\
&\quad + \zeta(t)^T h(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) \Big\} dt \\
&\quad - \int_a^b \Big\{ f^i(t, y, \dot{y}, v, \dot{v}) + y^T A_i r + v^T B_i \delta \\
&\quad - z_i \left(q^i(t, y, \dot{y}, v, \dot{v}) - y^T C_i w - v^T E_i \vartheta \right) \\
&\quad + \xi(t)^T g\left(t, y, \dot{y}, v, \dot{v}\right) + \zeta(t)^T h(t, y, \dot{y}, v, \dot{v}) \Big\} dt \Bigg) \\
&< b_i(x, u, y, v) \Phi_i(0) \leqq 0, \quad i = 1, \ldots, p.
\end{aligned} \tag{5.4}$$

Using hypothesis (a), by Definition 2.3, the above inequalities yield

$$\int_a^b \Big\{ [\eta(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v})]^T \Big[f_y^i(t,y,\dot{y},v,\dot{v}) + A_i(t)r(t) - z_i[q_y^i(t,y,\dot{y},v,\dot{v}) - C_i(t)w(t)] + \xi^T g_y(t,y,\dot{y},v,\dot{v}) + \zeta^T h_y(t,y,\dot{y},v,\dot{v}) - \frac{d}{dt}\big(f_{\dot{y}}^i(t,y,\dot{y},v,\dot{v}) - z_i q_{\dot{y}}(t,y,\dot{y},v,\dot{v}) + \xi^T g_{\dot{y}}(t,y,\dot{y},v,\dot{v}) + \zeta^T h_{\dot{y}}(t,y,\dot{y},v,\dot{v})\big)\Big] + [\theta(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v})]^T \Big[f_v^i(t,y,\dot{y},v,\dot{v}) + B_i(t)\delta(t) - z_i[q_v^i(t,y,\dot{y},v,\dot{v}) - E_i(t)\vartheta(t)] + \xi^T g_v(t,y,\dot{y},v,\dot{v}) + \zeta^T h_v(t,y,\dot{y},v,\dot{v}) - \frac{d}{dt}\big(f_{\dot{v}}(t,y,\dot{y},v,\dot{v}) - z_i q_{\dot{v}}(t,y,\dot{y},v,\dot{v}) + \xi^T g_{\dot{v}}(t,y,\dot{y},v,\dot{v}) + \zeta^T h_{\dot{v}}(t,y,\dot{y},v,\dot{v})\big)\Big]\Big\} dt < 0. \tag{5.5}$$

From the last constraint of (VWD), it follows that $\lambda \geq 0$, $\lambda^T e = 1$. Then, (5.5) implies that the inequality

$$\int_a^b \Big\{ [\eta(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v})]^T \Big[\sum_{i=1}^p \lambda_i \Big(f_y^i(t,y,\dot{y},v,\dot{v}) + A_i(t)r(t) - z_i[q_y^i(t,y,\dot{y},v,\dot{v}) - C_i(t)w(t)]\Big) + \xi^T g_y(t,y,\dot{y},v,\dot{v}) + \zeta^T h_y(t,y,\dot{y},v,\dot{v}) - \frac{d}{dt}\Big(\sum_{i=1}^p \lambda_i (f_{\dot{y}}^i(t,y,\dot{y},v,\dot{v}) - z_i q_{\dot{y}}(t,y,\dot{y},v,\dot{v})) + \xi^T g_{\dot{y}}(t,y,\dot{y},v,\dot{v}) + \zeta^T h_{\dot{y}}(t,y,\dot{y},v,\dot{v})\Big)\Big] + [\theta(t,x,\dot{x},u,\dot{u},y,\dot{y},v,\dot{v})]^T \Big[\sum_{i=1}^p \lambda_i \Big(f_v^i(t,y,\dot{y},v,\dot{v}) + B_i(t)\delta(t) - z_i[q_v^i(t,y,\dot{y},v,\dot{v}) - E_i(t)\vartheta(t)]\Big) + \xi^T g_v(t,y,\dot{y},v,\dot{v}) + \zeta^T h_v(t,y,\dot{y},v,\dot{v}) - \frac{d}{dt}\Big(\sum_{i=1}^p \lambda_i (f_{\dot{v}}(t,y,\dot{y},v,\dot{v}) - z_i q_{\dot{v}}(t,y,\dot{y},v,\dot{v})) + \xi^T g_{\dot{v}}(t,y,\dot{y},v,\dot{v}) + \zeta^T h_{\dot{v}}(t,y,\dot{y},v,\dot{v})\Big)\Big]\Big\} dt < 0.$$

holds, contradicting the constraints of (VWD).

Now, we prove this theorem under hypothesis (B). In the similar way as in the proof under hypothesis (A), we obtain inequalities (5.4). By hypothesis a), it follows that $\Gamma(\cdot,\cdot,z,\xi,\zeta,r,\delta,w,\vartheta)$ is univex at (y,v) on $S \cup Y_W$ with respect to $b = (b_1,\ldots,b_p)$, $\Phi = (\Phi_1,\ldots,\Phi_p)$, η, θ. Hence, by Definition 2.6, (5.4) yields (5.5). The last part of this proof is the same as in the proof of this theorem under hypothesis (A).

This completes the proof of this theorem. □

Theorem 5.2 (Strong duality)**.** *Let* $(\bar{x},\bar{u})$ *be a weakly efficient solution* (*an efficient solution*) *of the considered multiobjective fractional variational control problem* (MFP)*. Further, assume that the Kuhn–Tucker constraint qualification is satisfied for* (MFP)*. Then there exists* $\bar{\lambda} \in R^p$ *and piecewise smooth functions* $\bar{\xi}(\cdot) : I \to R^l$ *and* $\bar{\zeta}(\cdot) : I \to R^s$, $\bar{r}(\cdot) : I \to R^n$, $\bar{w}(\cdot) : I \to R^n$, $\bar{\delta}(\cdot) : I \to R^m$, $\bar{\vartheta}(\cdot) : I \to R^m$ *such that* $\left(\bar{x},\bar{u},\bar{z},\bar{\lambda},\bar{\xi},\bar{\zeta},\bar{r},\bar{\delta},\bar{w},\bar{\vartheta}\right)$ *is a feasible solution for the Wolfe type multiobjective variational dual problem* (VWD)*. If also the weak duality theorem holds between* (MFP) *and* (VWD)*, then* $\left(\bar{x},\bar{u},\bar{z},\bar{\lambda},\bar{\xi},\bar{\zeta},\bar{r},\bar{\delta},\bar{w},\bar{\vartheta}\right)$ *is a weakly efficient solution* (*an efficient solution*) *of a maximum type of* (VWD) *and the objective function values are equal.*

Proof. By assumption, $(\bar{x},\bar{u})$ is a weakly efficient solution in the considered multiobjective variational programming problem (MFP). Hence, by Lemma 3.3, $(\bar{x},\bar{u})$ is also a weakly efficient solution of the multiobjective nonfractional variational control problem (MFP($\bar{z}$)). Then, by Theorem 3.5, there there exist $\bar{\lambda} \in R^p$ and piecewise smooth functions $\bar{\xi}(\cdot) : I \to R^l$ and $\bar{\zeta}(\cdot) : I \to R^s$, $\bar{r}(\cdot) : I \to R^n$, $\bar{w}(\cdot) : I \to R^n$, $\bar{\delta}(\cdot) : I \to R^m$, $\bar{\vartheta}(\cdot) : I \to R^m$ such that the Karush–Kuhn–Tucker optimality conditions (3.3)–(3.4) are satisfied. Thus, $\left(\bar{x},\bar{u},\bar{z},\bar{\lambda},\bar{\xi},\bar{\zeta},\bar{r},\bar{\delta},\bar{w},\bar{\vartheta}\right)$ is a feasible solution in Wolfe dual problem (VWD) and the two objective functionals have same values.

Now, we show that $\left(\bar{x},\bar{u},\bar{z},\bar{\lambda},\bar{\xi},\bar{\zeta},\bar{r},\bar{\delta},\bar{w},\bar{\vartheta}\right)$ is a weakly efficient solution in Wolfe type dual problem (VWD) for $\bar{z}$. We proceed by contradiction. Suppose that $\left(\bar{x},\bar{u},\bar{z},\bar{\lambda},\bar{\xi},\bar{\zeta},\bar{r},\bar{\delta},\bar{w},\bar{\vartheta}\right)$ is not weakly efficient in (VWD) for the given $\bar{z}$. Then, there exists $(\widehat{y},\widehat{v},\bar{z},\widehat{\lambda},\widehat{\xi},\widehat{\zeta},\widehat{r},\widehat{\delta},\widehat{w},\widehat{\vartheta}) \in \Omega_W$ such that

$$\begin{aligned}
&\int_a^b \Big\{ f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) + \sqrt{\bar{x}^T A_i(t)\bar{x}} + \sqrt{\bar{u}^T B_i(t)\bar{u}} \\
&\qquad - \bar{z}_i \left(q^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \sqrt{\bar{x}^T C_i(t)\bar{x}} - \sqrt{\bar{u}^T E_i(t)\bar{u}} \right) + \bar{\xi}(t)^T g(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \\
&\qquad + \bar{\zeta}(t)^T h(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) \Big\} dt \\
&< \int_a^b \Big\{ f^i(t,\widehat{y},\dot{\widehat{y}},\widehat{v},\dot{\widehat{v}}) + \sqrt{\widehat{y}^T A_i(t)\widehat{y}} + \sqrt{\widehat{v}^T B_i(t)\widehat{v}} \\
&\qquad - \bar{z}_i \left(q^i(t,\widehat{y},\dot{\widehat{y}},\widehat{v},\dot{\widehat{v}}) - \sqrt{\widehat{y}^T C_i(t)\widehat{y}} - \sqrt{\widehat{v}^T E_i(t)\widehat{v}} \right) + \widehat{\xi}(t)^T g(t,\widehat{y},\dot{\widehat{y}},\widehat{v},\dot{\widehat{v}}) \\
&\qquad + \widehat{\zeta}(t)^T h(tt,\widehat{y},\dot{\widehat{y}},\widehat{v},\dot{\widehat{v}}) \Big\} dt, \quad i \in P.
\end{aligned}$$

Using the feasibility of $(\bar{x}, \bar{u})$ in (MFP) together with the Karush–Kuhn–Tucker necessary optimality condition (3.5), we get that the inequality

$$\begin{aligned}
&\int_a^b \Big\{ f^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sqrt{\bar{x}^T A_i(t)\bar{x}} + \sqrt{\bar{u}^T B_i(t)\bar{u}} \\
&\qquad - \bar{z}_i \Big(q^i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) - \sqrt{\bar{x}^T C_i(t)\bar{x}} - \sqrt{\bar{u}^T E_i(t)\bar{u}} \Big) \Big\} dt \\
&< \int_a^b \Big\{ f^i(t, \widehat{y}, \dot{\widehat{y}}, \widehat{v}, \dot{\widehat{v}}) + \sqrt{\widehat{y}^T A_i(t)\widehat{y}} + \sqrt{\widehat{v}^T B_i(t)\widehat{v}} \\
&\qquad - \widehat{z}_i \Big(q^i \Big(t, \widehat{y}, \dot{\widehat{y}}, \widehat{v}, \dot{\widehat{v}} \Big) - \sqrt{\widehat{y}^T C_i(t)\widehat{y}} - \sqrt{\widehat{v}^T E_i(t)\widehat{v}} \Big) \\
&\qquad + \widehat{\xi}(t)^T g(t, \widehat{y}, \dot{\widehat{y}}, \widehat{v}, \dot{\widehat{v}}) + \widehat{\zeta}(t)^T h(t, \widehat{y}, \dot{\widehat{y}}, \widehat{v}, \dot{\widehat{v}}) \Big\} dt, \quad i \in P.
\end{aligned}$$

holds, contradicting the weak duality theorem (Theorem 5.1).

Thus, $(\bar{x}, \bar{u}, \bar{z}, \bar{\lambda}, \bar{\xi}, \bar{\zeta}, \bar{r}, \bar{\delta}, \bar{w}, \bar{\vartheta})$ is a weakly efficient solution of a maximum type in (VWD). □

Theorem 5.3 (Converse duality). *Let* $\left(\bar{x}, \bar{u}, \bar{z}, \bar{\lambda}, \bar{\xi}, \bar{\zeta}, \bar{r}, \bar{\delta}, \bar{w}, \bar{\vartheta}\right) \in \Omega_W$ *be a weakly efficient solution* (*an efficient solution*) *of a maximum type in Wolfe type dual problem* (VWD) *and* $(\bar{x}, \bar{u}) \in S$. *Further, we assume that at least one of the following sets of hypotheses is fulfilled:*

(A)
- (a) *the function* $\Gamma\left(\cdot, \cdot, \bar{z}, \bar{\lambda}, \bar{\xi}, \bar{\zeta}, \bar{r}, \bar{\delta}, \bar{w}, \bar{\vartheta}\right)$ *is univex at* (y, v) *on* $S \cup Y_W$ *with respect to* $b = (b_1, \ldots, b_p)$, $\Phi = (\Phi_1, \ldots, \Phi_p)$, η, θ,
- (b) $b_i(\bar{x}, \bar{u}, \bar{y}, \bar{v}) > 0$, $i = 1, \ldots, p$,
- (c) Φ_i, $i = 1, \ldots, p$, *are strictly increasing functionals satisfying* $a < 0 \Longrightarrow \Phi_i(a) < 0$, $\Phi_i(0) \leqq 0$.

(B)
- (a) *the function* $\Gamma\left(\cdot, \cdot, \bar{z}, \bar{\lambda}, \bar{\xi}, \bar{\zeta}, \bar{r}, \bar{\delta}, \bar{w}, \bar{\vartheta}\right)$ *is pseudo-univex at* (y, v) *on* $S \cup Y$ *with respect to* $b = (b_1, \ldots, b_p)$, $\Phi = (\Phi_1, \ldots, \Phi_p)$, η, θ,
- (b) $b_i(\bar{x}, \bar{u}, \bar{y}, \bar{v}) > 0$, $i = 1, \ldots, p$,
- (c) Φ_i, $i = 1, \ldots, p$, *are strictly increasing functionals satisfying* $a < 0 \Longrightarrow \Phi_i(a) < 0$, $\Phi_i(0) \leqq 0$.

Then $(\bar{x}, \bar{u})$ *is a weakly efficient* (*an efficient solution*) *of the considered multiobjective variational programming problem* (MFP).

The proof of the above result is similar to the proof of Theorem 5.1.

Theorem 5.4 (Strict converse duality). *Let* $(\bar{x}, \bar{u})$ *and* $\left(\bar{y}, \bar{v}, \bar{z}, \bar{\lambda}, \bar{\xi}, \bar{\zeta}, \bar{r}, \bar{\delta}, \bar{w}, \bar{\vartheta}\right)$ *be weakly efficient solutions of* (MFP) *and* (VWD), *respectively, and, moreover, the Kuhn–Tucker constraint qualification be satisfied for* (MFP). *Further, assume that the following hypotheses are fulfilled:*

(A)

(a) *the function* $\Gamma\left(\cdot,\cdot,\bar{z},\bar{\lambda},\bar{\xi},\bar{\zeta},\bar{r},\bar{\delta},\bar{w},\bar{\vartheta}\right)$ *is strictly univex at* $(\bar{y},\bar{v})$ *on* $S \cup Y_W$ *with respect to* $b = (b_1,\ldots,b_p)$, $\Phi = (\Phi_1,\ldots,\Phi_p)$, η, θ,

(b) $b_i(\bar{x},\bar{u},\bar{y},\bar{v}) > 0$, $i = 1,\ldots,p$,

(c) Φ_i, $i = 1,\ldots,p$, *are strictly increasing functionals satisfying* $a < 0 \Longrightarrow \Phi_i(a) < 0$, $\Phi_i(0) \leqq 0$.

(B)

(a) *the function* $\Gamma(\cdot,\cdot,\bar{z},\bar{\lambda},\bar{\xi},\bar{\zeta},\bar{r},\bar{\delta},\bar{w},\bar{\vartheta})$ *is strictly pseudo-univex at* $(\bar{y},\bar{v})$ *on* $S \cup Y$ *with respect to* $b = (b_1,\ldots,b_p)$, $\Phi = (\Phi_1,\ldots,\Phi_p)$, η, θ,

(b) $b_i(\bar{x},\bar{u},\bar{y},\bar{v}) > 0$, $i = 1,\ldots,p$,

(c) Φ_i, $i = 1,\ldots,p$, *are strictly increasing functionals satisfying* $a < 0 \Longrightarrow \Phi_i(a) < 0$, $\Phi_i(0) \leqq 0$.

Then $(\bar{x},\bar{u}) = (\bar{y},\bar{v})$.

6. CONCLUSION

In this paper, a nondifferentiable multiobjective fractional variational control problem involving the nondifferentiable terms in the numerators and in the denominators has been considered. By using Dinkelbach approach, parametric optimality conditions of Karush–Kuhn–Tucker type have been derived for such nondifferentiable vector optimization problems. Under univexity and generalized univexity hypotheses, the sufficiency of these necessary optimality conditions have also been established. Further, parametric Mond–Weir vector optimization dual problem and parametric Wolfe dual problem have been constructed for the considered nondifferentiable multiobjective fractional variational control problem and also under univexity hypotheses several duality results have been established between them and the considered nonsmooth multiobjective fractional continuous-time problem. Our results apparently generalize a fairly large number of optimality conditions and duality results previously derived for multiobjective variational control problems with the nondifferentiable terms in the numerators and in the denominators established under others generalized convexity notions.

In fact, there are the following special cases of the nondifferentiable multiobjective fractional variational control problem (MFP) considered in the paper, which can be found in the literature:

(i) If $B_i(t) = E_i(t) = 0$, $i = 1,\ldots,p$, then we obtain a nondifferentiable multiobjective fractional variational control problem. Such multiobjective fractional continuous-time problems have been considered by Park and Jeong [35]

(ii) If $A_i(t) = C_i(t) = 0$, $B_i(t) = E_i(t) = 0$, $i = 1,\ldots,p$, then we obtain a differentiable multiobjective fractional variational control problem. Various types of such multiobjective fractional continuous-time problems have been considered by Bhatia and Mehra [10], Nahak [33], Nahak and Nanda [34], Mishra and Mukherjee [23], Mititelu and Stancu-Minasian [29]. The optimality and duality results established in the paper generalize similar results proved in the foregoing papers.

(iii) If $C_i(t) = 0$, $B_i(t) = E_i(t) = 0$, $i = 1, \ldots, p$, then we obtain a nondifferentiable multiobjective fractional variational control problem in which nondifferentiability enter due having a term of square root a quadratic form in each numerators of objective functionals. Various types of such nondifferentiable multiobjective fractional continuous-time problems have been considered by Ding *et al.* [11], Mishra and Mukherjee [25], Park and Jeong [35].

(iv) If all denominators of objective functions in (MFP) all equal to 1, then, in fact, (MFP) reduces to a nondifferentiable continuous-time vector optimization problem in which nondifferentiability enter due having a term of square root a quadratic form in each component of the vector-valued integrand of objective functional. Various types of such multiobjective variational problems have been considered by Husain and Jain [15], Husain and Mattoo [16], Kim and Kim [19], Mishra and Mukherjee [24].

(v) If all denominators of objective functions in (MFP) all equal to 1 and $A_i(t) = C_i(t) = 0$, then we obtain a differentiable multiobjective variational problem. Various types of such smooth vector continuous-time optimization problems were considered by Ahmad and Sharma [2], Aghezzaf and Khazafi [1], Antczak [4], Bhatia and Kumar [9], Bhatia and Mehra [10], Gramatovici [12], Lee *et al.* [20], Mishra and Mukherjee [26], Mititelu [27].

(vi) If all denominators of objective functions in (MFP) all equal to 1, $A_i(t) = C_i(t) = 0$ and, moreover, omitting the boundary conditions for the fixed end points as was done by Mond and Hanson [32], we obtain the so-called Multiobjective Natural Boundary Value Problem considered, for example, by Bhatia and Mehra [10].

The optimality and duality results established in the foregoing papers can be generalized and derived easily on the lines of the analysis of this research.

The question arises as to whether the results developed in the paper hold for various classes of nondifferentiable multiobjective fractional variational control problem involving the nondifferentiable terms under other generalized convexity hypotheses.

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