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ASYMPTOTIC ANALYSIS OF THE STEADY ADVECTION-DIFFUSION PROBLEM IN AXIAL DOMAINS

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Abstract. We present the asymptotic analysis of the steady advection-diffusion equation in a thin tube. The problem is modeled in a mixed-type variational formulation, in order to separate the phenomenon in the axial direction and a transverse one. Such formulation makes visible the natural separation of scales within the problem and permits a successful asymptotic analysis, delivering a limiting form, free from the initial geometric singularity and suitable for approximating the original one. Furthermore, it is shown that the limiting problem can be simplified to a significantly simpler structure.

Keywords: asymptotic analysis, mixed-type variational formulations, advection--diffusion.

Mathematics Subject Classification: 80A20, 35F15.

1. INTRODUCTION

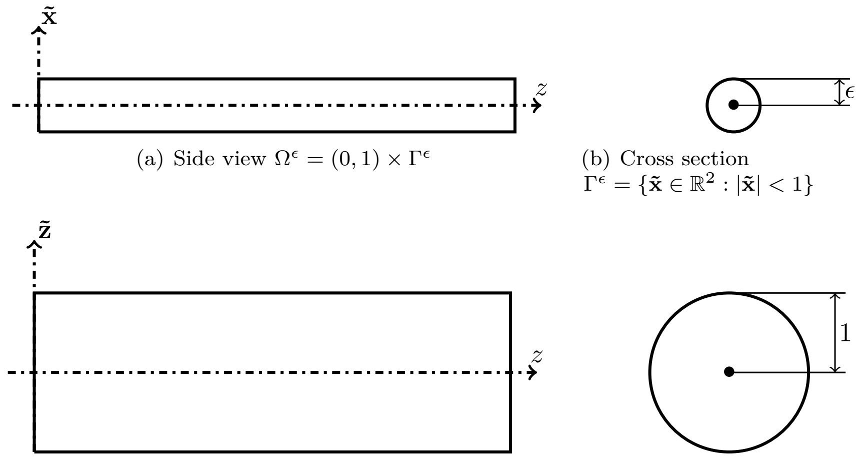
The analysis of a solute/pollutant dispersion within a flow field is a largely studied problem, as its importance is self evident. Among the numerous scenarios where this phenomenon takes place, we can count porous media flow and capillary flow, i.e., geometries where the axial direction is considerably large with respect the transverse section. However, little rigorous asymptotic analysis has been done in these scenarios, giving priority to other aspects such as adsorption, surface diffusion or chromatography (see [7]), preferential flow (see [5, 8, 9, 17]) or formal two-scale expansions (see [1, 4, 14]). In the present work, we introduce a mixed-type formulation for the diffusion-advection problem, that permits the separation of scales present in the phenomenon, when it takes place on a thin (capillary) tube. The unique source of singularity in our problem is the geometry of the domain (see Figure 1) hence, our formulation separating the *averaged* activity in the axial direction and its orthogonal complement (from the Hilbert space point of view), suffice for a complete analysis. In that sense, we radically differ from previous works (see [2, 10, 12, 13]), where they have to also use *constitutive scaling* in order to succeed.

Our approach starts with the usual diffusion-advection problem (Equation (1.5)) endowed with boundary conditions (Equation (1.6)). For clarity, we limit our analysis to a cylindrical type tube with straight horizontal axis, whose radius, namely $\epsilon > 0$, is very small with respect to the length, namely 1 (see Figure 1 (a) and (b)). Next, we give the customary direct variational formulation (2.2) and use it to, first scale the problem to a fixed geometry (see Figure 1 (c) and (d)) and then derive a mixed-type formulation of the diffusion advection-problem, tailored to treat the phenomenon according to its geometric singularity. Then the a-priori estimates on the model, together with weak convergence statements yield the limiting system/form. The *effective limiting system* (4.14)–(4.15) has the following properties.

1. It is free from the original *geometric singularity* or any other kind, which makes it an ideal form to approximate the original problem, as it is known that the singularities are a serious source of problems in numerical implementations: numerical instability, ill-conditioned matrices, etc.
2. It is a *one-way coupled system*, where the dominant part describes the behavior of the cross-section average. This feature allows to interpret the independent part as a diffusion-advection 1-D problem, similar to the original one, which reduces substantially the computational costs (aside from the numerical issues already mentioned). The dependent part (orthogonal in the Hilbert space sense), can be considered a second-order corrector (as it is understood in Homogenization Theory, see [16]) of the main part.
3. No *effective coefficients* are derived, because unlike the mainstream research of the field, our analysis does not need to include constitutive scaling for succeeding.

We close this section introducing the notation, the geometry and the original strong problem. Throughout the paper we denote vectors with bold characters, namely $\mathbf{x}, \mathbf{z}$, etc, with the exception of the canonical basis for $\mathbb{R}^3$, for which we use $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$ and the outer unitary normal vector, which we write $\hat{\boldsymbol{n}}$. We will use the standard identification $\mathbf{x} = (\tilde{\mathbf{x}}, z) \in \mathbb{R}^3$, where $\tilde{\mathbf{x}} \in \mathbb{R}^2$ indicates the first two coordinates of the vector $\mathbf{x}$. Similarly, we have for the gradient $\nabla = (\tilde{\nabla}, \partial_{\mathbf{z}})$, where $\tilde{\nabla} \overset{\textbf{def}}{=} \big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\big)$, i.e., the derivative with respect to the first two variables and $\partial_{\mathbf{z}}$ is the derivative with respect to the third variable z. Throughout the paper we work with the standard Lebesgue measure in $\mathbb{R}^i$, with i positive integer, to construct the standard real-valued Hilbert spaces. Given a Hilbert space X, we denote its topological dual by X^*. For an open region $\mathcal{U} \subseteq \mathbb{R}^i$, define the following spaces with corresponding inner products and norms:

$$\begin{gathered} L^2(\mathcal{U}) \overset{\textbf{def}}{=} \Big\{ p : \mathcal{U} \to \mathbb{R} \big| \int_{\mathcal{U}} p^2 d\mathbf{x} < \infty \Big\}, \\ \big\langle p, q \big\rangle_{L^2(\mathcal{U})} \overset{\textbf{def}}{=} \int_{\mathcal{U}} p\, q\, d\mathbf{x}, \quad \|p\|_{0,\mathcal{U}} \overset{\textbf{def}}{=} \big\langle p, p \big\rangle_{L^2(\mathcal{U})}^{1/2}. \end{gathered} \tag{1.1}$$



(a) Side view $\Omega^\epsilon = (0,1) \times \Gamma^\epsilon$

(b) Cross section $\Gamma^\epsilon = \{\tilde{\mathbf{x}} \in \mathbb{R}^2 : |\tilde{\mathbf{x}}| < 1\}$

(c) Side view $\Omega = (0,1) \times \Gamma$

(d) Cross section $\Gamma = \{\tilde{\mathbf{x}} \in \mathbb{R}^2 : |\tilde{\mathbf{x}}| < 1\}$

Fig. 1. Schematics of the ϵ-domain Ω^ϵ and the reference domain, when $\epsilon = 1$. Figure (a) shows a side view of the ϵ-domain, while figure (b) depicts its cross section. Figure (c) shows a side view of the reference domain, while figure (d) depicts its cross section

$$
\begin{aligned}
&H^1(\mathcal{U}) \overset{\mathbf{def}}{=} \Big\{ p \in L^2(\mathcal{U}) \big| \int\limits_{\mathcal{U}} |\nabla p|^2 d\mathbf{x} < \infty \Big\}, \\
&\big\langle p, q \big\rangle_{H^1(\mathcal{U})} \overset{\mathbf{def}}{=} \int\limits_{\mathcal{U}} \big(p\, q + \nabla p \cdot \nabla q \big)\, d\mathbf{x}, \quad \|p\|_{1,\mathcal{U}} \overset{\mathbf{def}}{=} \big\langle p, p \big\rangle_{H^1(\mathcal{U})}^{1/2}.
\end{aligned}
\tag{1.2}
$$

Given a manifold A contained in the boundary $\partial\mathcal{U}$, we denote by $|A|$ its Lebesgue measure, understood in the corresponding dimension; this notation will generate no confusion, as its meaning will be clear from the context. The letter K will denote a generic bounding positive constant.

Next we give the geometry of the family of domains, see Figure 1. For $\epsilon > 0$ define

$$
\Gamma^\epsilon \overset{\mathbf{def}}{=} \big\{ \tilde{\mathbf{x}} \in \mathbb{R}^2 : |\tilde{\mathbf{x}}| < \epsilon \big\}, \quad \Omega^\epsilon \overset{\mathbf{def}}{=} \Gamma^\epsilon \times (0,1). \tag{1.3}
$$

Also define the following boundary pieces

$$
G_0^\epsilon \overset{\mathbf{def}}{=} \Gamma^\epsilon \times \{0\}, \quad G_1^\epsilon \overset{\mathbf{def}}{=} \Gamma^\epsilon \times \{1\}, \quad O^\epsilon \overset{\mathbf{def}}{=} \partial\Omega^\epsilon - G_0^\epsilon - G_1^\epsilon. \tag{1.4}
$$

In the domain described above, we will analyze the asymptotic behavior of the diffusion-advection family of problems

$$
-\nabla \cdot D \nabla c^\epsilon - \mathbf{w}^\epsilon \cdot \nabla c^\epsilon = F^\epsilon \quad \text{in } \Omega^\epsilon = \big\{ (\tilde{\mathbf{x}}, z) \in \mathbb{R}^3 : |\tilde{\mathbf{x}}| < \epsilon, 0 < z < 1 \big\}, \tag{1.5}
$$

for $\epsilon > 0$. Here, the concentration of the solute c^ϵ is the unknown, $D > 0$ is the diffusion coefficient and $\mathbf{w}^\epsilon$ is an advective, conservative field in Ω^ϵ which is known. We complete the problem with the following Dirichlet and Neumann boundary conditions

$$c^\epsilon = 0 \text{ on } G_0^\epsilon \cup G_1^\epsilon, \quad \nabla c^\epsilon \cdot \hat{\boldsymbol{n}} = 0 \text{ on } O^\epsilon. \tag{1.6}$$

Finally, we are concerned with the most important case of flow within a capillary tube, hence the following hypothesis needs to be adopted.

Hypothesis 1.1. *In the sequel it will be assumed that the flow field satisfies*

$$\mathbf{w}^\epsilon = \big(\mathbf{w}^\epsilon \cdot \hat{\boldsymbol{k}}\big)\hat{\boldsymbol{k}} \quad \textit{for all } \epsilon > 0. \tag{1.7}$$

Observe that the conservative flow field together with the hypothesis give $\nabla \cdot \mathbf{w}^\epsilon = \partial_z(\mathbf{w}^\epsilon \cdot \hat{\boldsymbol{k}}) = 0$. Hence, denoting $w^\epsilon = \mathbf{w}^\epsilon \cdot \hat{\boldsymbol{k}}$, we have that $w^\epsilon = w^\epsilon(\tilde{\mathbf{x}})$ and the problem (1) reduces to

$$-\nabla \cdot D\nabla c^\epsilon - w^\epsilon\, \partial_{\mathbf{z}} c^\epsilon = F^\epsilon, \quad \text{in } \Omega^\epsilon. \tag{1.8}$$

$$c^\epsilon = 0 \text{ on } G_0^\epsilon \cup G_1^\epsilon, \quad \nabla c^\epsilon \cdot \hat{\boldsymbol{n}} = 0 \text{ on } O^\epsilon. \tag{1.9}$$

2. PRELIMINARIES

For the sake of completeness we present in the current section a summary of some classic general results that will be necessary along this work. It will be assumed that the reader is familiarized with the standard Hilbert space theory: the Cauchy-Schwarz-Bunyakovsky inequality, the projection theorem, equivalent norms, density of subspaces, direct sum decompositions, weak compactness and weak convergence methods. It will also be assumed that the reader is familiarized with the trace operator for functions in $H^1(\mathcal{U})$. We refer to [6] and [15] as a good source gathering all these results.

We will review in Theorem 2.5 below, the variational formulation and well-posedness of the problem (1.8), endowed with the boundary conditions (1.9). To that end first we recall the concept of coerciveness or ellipticity of bilinear forms, together with the Lax-Milgram Lemma and the Poincaré inequality.

Definition 2.1 (Coerciveness)**.** Let X be a Hilbert space and X^* its topological dual.

1. A continuous linear map $T : X \to X^*$ is said to be coercive or X-elliptic if there exists $K > 0$ such that $Tx(x) \geq K\|x\|_X^2$ for all $x \in X$.
2. A continuous bilinear map $t : X \times X \to \mathbb{R}$ is said to be coercive or X-elliptic if there exists $K > 0$ such that $t(x, x) \geq K\|x\|_X^2$ for all $x \in X$.

Remark 2.2. Notice that a continuous linear operator $T : X \to X^*$ defines a bilinear form $t : X \times X \to \mathbb{R}$ by $t(x, y) \overset{\mathbf{def}}{=} Tx(y)$. In the same fashion, a continuous bilinear form $t : X \times X \to \mathbb{R}$ defines an operator $T : X \to X^*$ by $Tx(\cdot) \overset{\mathbf{def}}{=} t(x, \cdot)$. In the present work we will identify both forms as it will cause no conflict.

Lemma 2.3 (Lax–Milgram Lemma)**.** *Let X be a Hilbert space and let $t : X \times X \to \mathbb{R}$ be a continuous X-elliptic, bilinear form. Then for each $f \in X^*$ there exists a unique $x_0 \in X$ such that $t(x_0, y) = f(y)$ for all $y \in X$.*

Theorem 2.4 (Poincaré Inequality)**.** *Let $\mathcal{U} \subseteq \mathbb{R}^i$ be a bounded open region and let $A \subseteq \partial\mathcal{U}$ be such that $|A| > 0$. Then there exists $K > 0$ such that*

$$\|u\|_{1,\mathcal{U}} \leq \|\nabla u\|_{0,\mathcal{U}}, \quad \textit{for all } u \in H^1(\mathcal{U}) \textit{ such that } u\big|_A = 0. \tag{2.1}$$

Theorem 2.5 (Well-posedness)**.** *The direct variational formulation of the problem* (1.8)*, endowed with the boundary conditions* (1.9)*, is given by*

$$c^\epsilon \in V^\epsilon : \quad \int\limits_{\Omega^\epsilon} \big(D\nabla c^\epsilon \cdot \nabla p - w^\epsilon \partial_{\mathbf{z}} c^\epsilon p\big) d\mathbf{x} = \int\limits_{\Omega^\epsilon} F^\epsilon p \, d\mathbf{x}, \quad \textit{for all } p \in V^\epsilon. \tag{2.2}$$

Here, the Hilbert space V^ϵ is defined by

$$V^\epsilon \stackrel{\mathbf{def}}{=} \big\{p \in H^1(\Omega^\epsilon) : p = 0 \textit{ on } G_0^\epsilon \cup G_1^\epsilon\big\}, \quad \textit{for all } \epsilon > 0. \tag{2.3}$$

Moreover, the problem (2.2) *is well-posed.*

Proof. In order to get the variational statement (2.2), we multiply the equation (1.8) with a test function $p \in V^\epsilon$ and integrate on the domain Ω^ϵ. Next, we integrate by parts the first summand as follows

$$\begin{aligned} \int\limits_{\Omega^\epsilon} -\nabla \cdot D\,\nabla c^\epsilon \, p \, d\mathbf{x} &= \int\limits_{\Omega^\epsilon} D\,\nabla c^\epsilon \cdot \nabla p \, d\mathbf{x} - \int\limits_{\partial\Omega^\epsilon} p\, D\,\nabla c^\epsilon \cdot \hat{\boldsymbol{n}} \, dS \\ &= \int\limits_{\Omega^\epsilon} D\,\nabla c^\epsilon \cdot \nabla p \, d\mathbf{x} - \int\limits_{G_0^\epsilon \cup G_1^\epsilon} p\, D\,\nabla c^\epsilon \cdot \hat{\boldsymbol{n}} \, dS \\ &\quad - \int\limits_{O^\epsilon} p\, D\,\nabla c^\epsilon \cdot \hat{\boldsymbol{n}} \, dS \\ &= \int\limits_{\Omega^\epsilon} D\,\nabla c^\epsilon \cdot \nabla p \, d\mathbf{x}. \end{aligned}$$

We apply the boundary conditions (1.9) on the second and third line of the expression above. The second summand of the second line vanishes due to the strong boundary conditions, applicable on the quantifier $p \in V^\epsilon$; while the summand ot the third line is null because of the weak boundary condition $\nabla c^\epsilon \cdot \hat{\boldsymbol{n}} = 0$ on O^ϵ, applicable only on the solution c^ϵ.

Next, proving that the variational problem (2.2) is well-posed, consists in showing that the form $\big(p, q\big)_{V^\epsilon} \stackrel{\mathbf{def}}{=} \int_{\Omega^\epsilon} \big(D\nabla p \cdot \nabla q - w^\epsilon p\, q\big)\, d\mathbf{x}$, defined on V^ϵ, satisfies the hypotheses of the Lax–Milgram Lemma 2.3. Proving that the form is bilinear and continuous is direct. The V^ϵ-ellipticity of the form is given by

$$\begin{aligned}
\big(p, p\big)_{V^\epsilon} &= \int\limits_{\Omega^\epsilon} D\nabla p \cdot \nabla p \, d\mathbf{x} - \int\limits_{\Omega} w^\epsilon (\partial_{\mathbf{z}} p) p \, d\mathbf{x} \\
&= \int\limits_{\Omega^\epsilon} D\big|\nabla p\big|^2 \, d\mathbf{x} - \int\limits_{\Gamma^\epsilon} w^\epsilon \int\limits_0^1 \frac{1}{2}\partial_{\mathbf{z}}(p^2) \, dz \, d\tilde{\mathbf{x}} \\
&= \int\limits_{\Omega^\epsilon} D\big|\nabla p\big|^2 \, d\mathbf{x} - \int\limits_{\Gamma^\epsilon} w^\epsilon \frac{1}{2} p^2 \big|_0^1 \\
&= \int\limits_{\Omega^\epsilon} D\big|\nabla p\big|^2 d\tilde{\mathbf{x}} \\
&\geq D \, K \, \|p\|_{1,\Omega^\epsilon}^2 \, d\mathbf{x}.
\end{aligned} \tag{2.4}$$

In the second summand of the third line the strong boundary conditions were applied, while in the fifth line $K > 0$ is the Poincaré constant, known to exist for H^1-subspaces defined on bounded domains and with strong boundary conditions, as in the case of V^ϵ (see Theorem 2.4).

Given that the bilinear form $\big(p, q\big)_{V^\epsilon}$ satisfies the hypothesis of the Lax-Milgram Lemma 2.3, it follows that the variational problem (2.2) is well-posed. $\square$

We close this section recalling some well-know results and a classic space.

Theorem 2.6. *Let X, Y be Hilbert spaces and let $X^*, Y^*, \|\cdot\|_X, \|\cdot\|_Y$ be their corresponding topological duals and norms. Let $\mathcal{A} : X \to X^*$, $\mathcal{B} : X \to Y^*$ and $\mathcal{C} : Y \to Y^*$ be continuous linear operators such that:*

1. *$\mathcal{A}$ is X-coercive,*
2. *$\mathcal{C}$ is Y-coercive.*

Then for each $x^* \in X^*$ and $y^* \in Y^*$ there exists a unique solution $[x_0, y_0] \in X \times Y$ to the following problem.

$$\text{Find } x \in X,\, y \in Y: \quad \mathcal{A}x + \mathcal{B}^* y = x^* \text{ in } X^*, \tag{2.5}$$

$$-\mathcal{B}x + \mathcal{C}y = y^* \text{ in } Y^*. \tag{2.6}$$

Moreover, the solution satisfies the estimate

$$\|x_0\|_X + \|y_0\|_Y \leq K\big(\|x^*\|_{X^*} + \|y^*\|_{Y^*}\big). \tag{2.7}$$

Proof. See Proposition 1.4 (p. 45) in [3]. $\square$

Definition 2.7. Denote by $H(\partial_{\mathbf{z}}, \Omega)$ the following space

$$H(\partial_{\mathbf{z}}, \Omega) \stackrel{\mathbf{def}}{=} \big\{p : \Omega \to \mathbb{R} \;\big|\; p \in L^2(\Omega), \partial_{\mathbf{z}} p \in L^2(\Omega)\big\}, \tag{2.8}$$

endowed with the natural inner product

$$\big\langle p, q \big\rangle_{H(\partial_{\mathbf{z}},\Omega)} \stackrel{\mathbf{def}}{=} \int\limits_{\Omega} \big(p\, q + \partial_{\mathbf{z}} p\, \partial_{\mathbf{z}} q\big)\, d\mathbf{x}. \tag{2.9}$$

Proposition 2.8. *Let* $\mathcal{H} \stackrel{\mathbf{def}}{=} \big\{p \in H(\partial_{\mathbf{z}}, \Omega) : p\big|_{G_0 \cup G_1} = 0\big\}$*. Then there exists a constant* $K > 0$ *such that*

$$\big\|p\big\|^2_{H(\partial_{\mathbf{z}},\Omega)} = \|p\|^2_{0,\Omega} + \|\partial_{\mathbf{z}} p\|^2_{0,\Omega} \le K^2 \big\|\partial_{\mathbf{z}} p\big\|_{0,\Omega}, \quad \textit{for all } p \in \mathcal{H}. \tag{2.10}$$

Proof. The proof uses the same arguments used to show the Poincaré inequality (2.1). □

3. A SCALED MIXED-TYPE FORMULATION OF THE DIFFUSION-ADVECTION PROBLEM

In the current section, we will present a mixed-type formulation for the problem (1.8)–(1.9). Our motivation is to separate the physical effects in two directions, the axial one (quantified by the variable z) and a *transverse one* (transverse in the geometry of the framework Hilbert space). Given that the physical problem has a clear dominant dimension, it is only natural to propose a formulation emphasizing on it.

We begin presenting the scaling hypotheses, together with an adequate change of variables on the ϵ-problems, to pose them in a general framework, with a common geometry, where the asymptotic analysis is possible. Next, in Subsection 3.1, the modeling subspaces are presented. Finally, in Subsection 3.2, we derive the mixed-type formulation of the problem and prove its well-posedness.

Notice that the family of solutions to the direct variational formulation (2.2), when $\epsilon > 0$, do not belong to the same function space. Therefore, we must reformulate the problem in a common geometric setting, independent from the parameter $\epsilon > 0$. To that end, we introduce a reference domain (see Figure 1) and a linear change of variables mapping bijectively Ω^ϵ onto such reference domain.

Definition 3.1 (Reference Domain and Function Space)**.**

1. The reference domain will be the one described in equations (1.3) and (1.4) for $\epsilon = 1$. In this case, we simply denote $\Omega \stackrel{\mathbf{def}}{=} \Omega^1$, $O \stackrel{\mathbf{def}}{=} O^1$, $G_i \stackrel{\mathbf{def}}{=} G_i^1$ for $i = 0, 1$, $\mathbf{w} \stackrel{\mathbf{def}}{=} \mathbf{w}^1$ and the space $V \stackrel{\mathbf{def}}{=} V^1$.
2. We define the change of variables

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \stackrel{\mathbf{def}}{=} \begin{bmatrix} \epsilon^{-1} & 0 & 0 \\ 0 & \epsilon^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ z \end{pmatrix}. \tag{3.1}$$

Analogously to the original variable $\mathbf{x} = (\tilde{\mathbf{x}}, z)$, we use the notation $\mathbf{z} = (\tilde{\mathbf{z}}, z)$.

Remark 3.2. Notice that the change of variables (3.1) gives the following relationship between derivatives

$$\begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_z \end{pmatrix} \overset{\mathbf{def}}{=} \begin{bmatrix} \epsilon^{-1} & 0 & 0 \\ 0 & \epsilon^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \partial_{z_1} \\ \partial_{z_2} \\ \partial_{z_3} \end{pmatrix}. \tag{3.2}$$

Before we can go any further, it becomes necessary to accept scaling hypotheses

Hypothesis 3.3 (Scaling Hypothesis)**.** *In the sequel, the following will be assumed.*

1. *There exists a flow field $w : \Omega \to \mathbb{R}$, such that*

$$\partial_{\mathbf{z}} w = 0, \quad w^\epsilon(\tilde{\mathbf{x}}) = w\Big(\frac{\tilde{\mathbf{x}}}{\epsilon}\Big), \quad \textit{for all } \epsilon > 0. \tag{3.3}$$

2. *There exists a function $F : \Omega \to \mathbb{R}$, such that the sequence of forcing terms $F^\epsilon : \Omega^\epsilon \to \mathbb{R}$ satisfy*

$$\big\| F^\epsilon(\epsilon\, \tilde{\mathbf{z}}, z) - F(\tilde{\mathbf{z}}, z) \big\|_{0,\Omega} \xrightarrow[\epsilon \to 0]{} 0. \tag{3.4}$$

In order to ease notation, the forcing terms will be denoted the same, before or after the change of variables.

Our next step is to write the problem (2.2) in the reference domain, using the change of variables (3.1). To that end, we begin writing the ϵ-problem, stressing the dependence on the variable $\mathbf{x} = (\tilde{\mathbf{x}}, z)$ defined on the domain Ω^ϵ; we have

$$\int_{\Omega^\epsilon} D \tilde{\nabla}_{\tilde{\mathbf{x}}} c^\epsilon \cdot \tilde{\nabla}_{\tilde{\mathbf{x}}} p \, d\tilde{\mathbf{x}} \, dz + \int_{\Omega^\epsilon} D \partial_{\mathbf{z}} c^\epsilon \, \partial_{\mathbf{z}} p \, d\tilde{\mathbf{x}} \, dz - \int_{\Omega^\epsilon} w^\epsilon \partial_{\mathbf{z}} c^\epsilon p \, d\tilde{\mathbf{x}} \, dz = \int_{\Omega^\epsilon} F^\epsilon p \, d\tilde{\mathbf{x}} \, dz.$$

Here, $\tilde{\nabla}_{\tilde{\mathbf{x}}}$ indicates the gradient with respect to the variables $\tilde{\mathbf{x}} = (x_1, x_2)$. Now, applying the change of variables (3.1) and the hypothesis 3.3 we get

$$\int_{\Omega} D \tilde{\nabla}_{\tilde{\mathbf{z}}} c^\epsilon \cdot \tilde{\nabla}_{\tilde{\mathbf{z}}} p \, d\tilde{\mathbf{z}} \, dz + \epsilon^2 \int_{\Omega} D \partial_{\mathbf{z}} c^\epsilon \, \partial_{\mathbf{z}} p \, d\tilde{\mathbf{z}} \, dz - \epsilon^2 \int_{\Omega} w \partial_{\mathbf{z}} c^\epsilon p \, d\tilde{\mathbf{z}} \, dz = \epsilon^2 \int_{\Omega} F^\epsilon p \, d\tilde{\mathbf{z}} \, dz,$$

where $\tilde{\nabla}_{\tilde{\mathbf{z}}}$ indicates the gradient with respect to the variables $\tilde{\mathbf{z}} = (z_1, z_2)$. In the sequel, we drop this notation as it will no longer be necessary. Hence, we will work with the following family of problems for the rest of the paper. For $\epsilon > 0$ arbitrary,

$$\begin{aligned} &\text{find } c^\epsilon \in V : \\ &\int_{\Omega} D \tilde{\nabla} c^\epsilon \cdot \tilde{\nabla} p \, d\mathbf{z} + \epsilon^2 \int_{\Omega} D \partial_{\mathbf{z}} c^\epsilon \, \partial_{\mathbf{z}} p \, d\mathbf{z} - \epsilon^2 \int_{\Omega} w \, \partial_{\mathbf{z}} c^\epsilon \, p \, d\mathbf{z} = \epsilon^2 \int_{\Omega} F^\epsilon p \, d\mathbf{z}, \\ &\text{for all } p \in V. \end{aligned} \tag{3.5}$$

3.1. INNER PRODUCT SUBSPACES AND PROJECTIONS

In the present section, we introduce the inner product and subspaces that will permit the mixed-type formulation that we seek and a necessary characterization of the orthogonal projections onto such spaces.

Proposition 3.4. *Let V be the reference space. Then the bilinear form*

$$\big\langle \cdot, \cdot \big\rangle_V : V \times V \to \mathbb{R}, \quad \big\langle p, q \big\rangle_V \overset{\textbf{def}}{=} \int\limits_{\Omega} \nabla p \cdot \nabla q \, d\mathbf{z}, \tag{3.6}$$

is equivalent to the standard inner product in $H^1(\Omega)$. In particular, there exists $K > 0$ such that

$$\|p\|_{1,\Omega} \leq K \, \|p\|_V, \quad \textit{for all } p \in V. \tag{3.7}$$

Proof. It follows directly from the Poincaré inequality (see Theorem 2.4) on bounded domains and with strong boundary conditions, as in the case of V. $\square$

Next we introduce the modeling subspaces.

Definition 3.5 (Modeling Subspaces)**.** Let V be the reference space, endowed with the inner product (3.6). We define the following orthogonal decomposition $V = M \oplus Q$, where the subspaces are given by

$$M \overset{\textbf{def}}{=} \big\{ \lambda \in V : \tilde{\nabla} \lambda = 0 \big\}. \tag{3.8}$$

$$Q \overset{\textbf{def}}{=} M^{\perp} = \Big\{ r \in V : \int\limits_{\Omega} \nabla \lambda \cdot \nabla r \, d\mathbf{z} = 0 \text{ for all } \lambda \in M \Big\}. \tag{3.9}$$

Next we define the "transverse-average" operator.

Definition 3.6. Let p be in $L^2(\Omega)$, define its transverse-average by

$$\bar{p}(z) \overset{\textbf{def}}{=} \frac{1}{|\Gamma|} \int\limits_{\Gamma} p(\tilde{\mathbf{z}}, z) \, d\tilde{\mathbf{z}}. \tag{3.10}$$

Remark 3.7. Recall, on one hand, that Definition 1.3 implies $|\Gamma| > 0$. On the other hand, if $p \in L^2(\Omega)$, then $\int_{\Gamma} p^2(\tilde{\mathbf{z}}, z) \, d\tilde{\mathbf{z}}$ must be finite z-almost everywhere. Consequently $\bar{p}$ in Definition 3.6 above, is well-defined, z-almost everywhere.

Theorem 3.8. *With the definitions above let $P_M, P_Q : V \to V$ be the orthogonal projections onto M and Q, respectively. Then for every $p \in V$ the following assertions hold.*

1.
$$P_M p = \bar{p}. \tag{3.11}$$

2.
$$P_Q p = p - \bar{p}. \tag{3.12}$$

3.
$$Q = \big\{ p \in V : \bar{p} = 0 \big\}. \tag{3.13}$$

Proof. 1. Let $p \in V$ and $\lambda \in M$ be arbitrary. Then

$$\begin{aligned}\big\langle p - \bar{p}, \lambda \big\rangle_V &= \int\limits_{\Omega} \nabla\big(p - \bar{p}\big) \cdot \nabla \lambda \, d\mathbf{z} \\ &= \int\limits_{\Omega} \partial_{\mathbf{z}}\big(p - \bar{p}\big)\, \partial_{\mathbf{z}} \lambda \, d\mathbf{z} \\ &= \int\limits_0^1 \int\limits_{\Gamma} \partial_{\mathbf{z}}\big(p - \bar{p}\big)\, \partial_{\mathbf{z}} \lambda \, d\tilde{\mathbf{z}}\, dz \\ &= \int\limits_0^1 \partial_{\mathbf{z}} \int\limits_{\Gamma} p \, d\tilde{\mathbf{z}}\; \partial_{\mathbf{z}} \lambda \, dz - \int\limits_0^1 \int\limits_{\Gamma} d\tilde{\mathbf{x}}\; \partial_{\mathbf{z}} \bar{p}\, \partial_{\mathbf{z}} \lambda \, dz = 0.\end{aligned}$$

Given that the orthogonality above holds for all $\lambda \in M$ and due to the characterization of the orthogonal projection on Hilbert spaces, the first part is complete.

2. It follows immediately from the previous part.
3. Due to the first part it is clear that $p \in Q$ if and only if $P_Q p = \bar{p} = 0$ and the proof is complete.

□

3.2. THE MIXED-TYPE VARIATIONAL FORMULATION

We begin this section presenting in Definition 3.9, a mixed variational formulation of the problem (1.8); which is based on the direct sum decomposition $V = M \oplus Q$ introduced in Definition 3.5. Next, in Theorem 3.10, we prove the well-posedness of the mixed formulation, using Theorem 2.6. We close this section characterizing in Theorem 3.11, the solution of the newly introduced formulation in terms of the solution c^ϵ to the direct variational formulation problem (3.5).

Definition 3.9 (The Mixed Variational Formulation)**.** Let M, Q be the spaces introduced in (3.8) and (3.9) respectively. For every $\epsilon > 0$, we define the following mixed formulation problem.

$$\begin{gathered}\text{Find } \mu^\epsilon \in M,\ q^\epsilon \in Q: \\ \int\limits_{\Omega} D\, \partial_{\mathbf{z}} \mu^\epsilon\, \partial_{\mathbf{z}} \lambda \, d\mathbf{z} - \int\limits_{\Omega} |\Gamma|\, \bar{w}\, \partial_{\mathbf{z}} \mu^\epsilon\, \lambda \, d\mathbf{z} + \int\limits_{\Omega} w\, q^\epsilon\, \partial_{\mathbf{z}} \lambda \, d\mathbf{z} = \int\limits_{\Omega} F^\epsilon\, \lambda \, d\mathbf{z},\end{gathered} \tag{3.14}$$

$$\begin{gathered}- \int\limits_{\Omega} w\, \partial_{\mathbf{z}} \mu^\epsilon\, r \, d\mathbf{z} + \frac{1}{\epsilon^2} \int\limits_{\Omega} D\, \tilde{\nabla} q^\epsilon \cdot \tilde{\nabla} r \, d\mathbf{z} + \int\limits_{\Omega} D\, \partial_{\mathbf{z}} q^\epsilon\, \partial_{\mathbf{z}} r \, d\mathbf{z} - \int\limits_{\Omega} w\, \partial_{\mathbf{z}} q^\epsilon\, r \, d\mathbf{z} \\ = \int\limits_{\Omega} F^\epsilon\, r \, d\mathbf{z}, \text{ for all } \lambda \in M,\ r \in Q.\end{gathered} \tag{3.15}$$

Alternatively, the system above is equivalent to the following problem.

$$\text{Find } \mu^\epsilon \in M,\, q^\epsilon \in Q: \quad \mathcal{A}^\epsilon \mu^\epsilon + \mathcal{B}^* q^\epsilon = f_M^\epsilon \text{ in } M^*, \tag{3.16}$$

$$-\mathcal{B}\mu^\epsilon + \mathcal{C}^\epsilon q^\epsilon = f_Q^\epsilon \text{ in } Q^*. \tag{3.17}$$

Here, $\epsilon > 0$ is arbitrary and $f_M^\epsilon \in M^*, f_Q^\epsilon \in Q^*$ are defined as

$$f_M^\epsilon(\lambda) \overset{\mathbf{def}}{=} \int_\Omega F^\epsilon\, \lambda\, d\mathbf{z}, \quad \forall \lambda \in M, \quad f_Q^\epsilon(r) \overset{\mathbf{def}}{=} \int_\Omega F^\epsilon\, r\, d\mathbf{z}, \quad \text{for all } r \in Q.$$

The operators $\mathcal{A}^\epsilon : Q \to Q^*$, $\mathcal{B} : M \to Q^*$, $\mathcal{C}^\epsilon : M \to M^*$ are defined as follows:

$$\mathcal{A}^\epsilon \nu(\lambda) \overset{\mathbf{def}}{=} \int_\Omega D\, \partial_{\mathbf{z}}\nu\, \partial_{\mathbf{z}}\lambda\, d\mathbf{z} - \int_\Omega |\Gamma|\, \bar{w}\, \partial_{\mathbf{z}}\nu\, \lambda\, d\mathbf{z}, \tag{3.18}$$

$$\mathcal{B}\lambda(r) \overset{\mathbf{def}}{=} \int_\Omega w\, \partial_{\mathbf{z}}\lambda\, r\, d\mathbf{z}, \tag{3.19}$$

$$\mathcal{C}^\epsilon s(r) \overset{\mathbf{def}}{=} \frac{1}{\epsilon^2} \int_\Omega D\, \tilde{\nabla} s \cdot \tilde{\nabla} r\, d\mathbf{z} + \int_\Omega D\, \partial_{\mathbf{z}} s\, \partial_{\mathbf{z}} r\, d\mathbf{z} - \int_\Omega w\, \partial_{\mathbf{z}} s\, r\, d\mathbf{z}. \tag{3.20}$$

Next we show the mixed formulation problem (3.16)–(3.17) is well-posed.

Theorem 3.10 (Well-Posedness of the Mixed Formulation)**.** *The system* (3.16)–(3.17) (*and equivalently the system* (3.14)–(3.15)) *is well-posed.*

Proof. We have to that the operators $\mathcal{A}^\epsilon, \mathcal{B}$ and $\mathcal{C}^\epsilon$, defined in (3.18), (3.19), (3.20) respectively, verify the hypotheses of Theorem 2.6. Firstly, it is clear that the operators $\mathcal{A}^\epsilon, \mathcal{B}$ and $\mathcal{C}^\epsilon$ are bilinear and continuous due to the Cauchy–Schwartz–Bunyakovsky inequality and the norms of the involved spaces.

Secondly, we prove the M-coerciveness of $\mathcal{A}^\epsilon$.

$$\begin{aligned}
\mathcal{A}^\epsilon \lambda(\lambda) &= \int_\Omega D\, \partial_{\mathbf{z}}\lambda\, \partial_{\mathbf{z}}\lambda\, d\mathbf{z} - \int_\Omega |\Gamma|\, \bar{w}\, \partial_{\mathbf{z}}\lambda\, \lambda\, d\mathbf{z} \\
&= \int_\Omega D\, |\partial_{\mathbf{z}}\lambda|^2\, d\mathbf{z} - \int_\Gamma |\Gamma|\, \bar{w} \int_0^1 \frac{1}{2}\, \partial_{\mathbf{z}}\big(\lambda^2\big)\, dz d\tilde{\mathbf{z}} \\
&= D\, \|\lambda\|_V^2 - \int_\Gamma |\Gamma|\, \bar{w} \frac{1}{2}\, \lambda^2\Big|_0^1 d\tilde{\mathbf{z}} \\
&= D\, \|\lambda\|_V^2 .
\end{aligned}$$

In the third line for the first summand we simply recalled that $\tilde{\nabla}\lambda = 0$ for every $\lambda \in M$, while in the second summand the strong boundary conditions of the space V were applied.

Third, we prove the Q-coerciveness of $\mathcal{C}^\epsilon$.

$$\begin{aligned}
\mathcal{C}^\epsilon r(r) &= \frac{1}{\epsilon^2}\int\limits_{\Omega} D\,\tilde{\nabla} r\cdot\tilde{\nabla} r\,d\mathbf{z} + \int\limits_{\Omega} D\,\partial_{\mathbf{z}} r\,\partial_{\mathbf{z}} r\,d\mathbf{z} - \int\limits_{\Omega} w\,\partial_{\mathbf{z}} r\,r\,d\mathbf{z}\\
&\geq K\int\limits_{\Omega} D\,|\nabla r|^2\,d\mathbf{z} - \int\limits_{\Omega} w\frac{1}{2}\,\partial_{\mathbf{z}}\big(r^2\big)\,d\mathbf{z}\\
&= \int\limits_{\Omega} D\,|\nabla r|^2\,d\mathbf{z} - \int\limits_{\Gamma} w\int\limits_0^1 \frac{1}{2}\,\partial_{\mathbf{z}}\big(r^2\big)\,dzd\tilde{\mathbf{z}}\\
&= K\,D\,\|r\|_V^2 - \int\limits_{\Gamma} |\Gamma|\,\bar{w}\frac{1}{2}\,r^2\big|_0^1 d\tilde{\mathbf{z}}\\
&= K\,D\,\|r\|_V^2.
\end{aligned}$$

In the second line of the expression above we took $K \overset{\mathbf{def}}{=} \min\big\{\epsilon^{-2}, 1\big\} > 0$, in order to recover $\|\nabla r\|_{0,\Omega}^2 = \|r\|_V^2$. In the fourth line, second summand the strong boundary conditions were applied (as before).

Finally, given that the operators $\mathcal{A}^\epsilon, \mathcal{B}$ and $\mathcal{C}^\epsilon$ satisfy the hypotheses of Theorem 2.6, the system (3.16)–(3.17) is well-posed. $\square$

Theorem 3.11 (Characterization of the Mixed Variational Problem Solution)**.** *Let $\epsilon > 0$ be fixed and let $c^\epsilon \in V$ be the unique solution to the problem* (3.5)*. Then $c^\epsilon - \bar{c^\epsilon} = P_Q c^\epsilon \in Q$ and $\bar{c^\epsilon} = P_M c^\epsilon \in M$ are the unique solution to the system* (3.14)*–*(3.15)*.*

Proof. We start proving that $P_Q c^\epsilon, P_M c^\epsilon$ satisfy the equality (3.14). To that end, test (3.5) with $\lambda \in M$ arbitrary and get

$$\epsilon^2\int\limits_{\Omega} F^\epsilon\,\lambda\,d\mathbf{z} = \epsilon^2\int\limits_{\Omega} D\,\partial_{\mathbf{z}} c^\epsilon\,\partial_{\mathbf{z}}\lambda\,d\mathbf{z} - \epsilon^2\int\limits_{\Omega} w\partial_{\mathbf{z}} c^\epsilon\lambda\,d\mathbf{z}. \tag{3.21}$$

We treat the advective term in the following way. Since

$$\int\limits_{\Gamma}\big(w - |\Gamma|\,\bar{w}\big)\partial_{\mathbf{z}}\bar{c^\epsilon}\,d\tilde{\mathbf{z}} = \int\limits_{\Gamma}\big(w - |\Gamma|\bar{w}\big)d\tilde{\mathbf{z}}\,\partial_{\mathbf{z}}\bar{c^\epsilon} = 0,$$

then we conclude

$$\int\limits_{\Gamma}\big(w - |\Gamma|\,\bar{w}\big)\big(\partial_{\mathbf{z}} c^\epsilon - \partial_{\mathbf{z}}\bar{c^\epsilon}\big)\,d\tilde{\mathbf{z}} = \int\limits_{\Gamma}\big(w\,\partial_{\mathbf{z}} c^\epsilon\,d\tilde{\mathbf{x}} - |\Gamma|^2\,\bar{w}\,\partial_{\mathbf{z}}\bar{c^\epsilon}\big)\,d\tilde{\mathbf{z}}. \tag{3.22}$$

Hence, applying (3.22) in the expression (3.21) yields

$$\begin{aligned}
&\epsilon^2\int\limits_{\Omega} D\,\partial_{\mathbf{z}}\bar{c^\epsilon}\,\partial_{\mathbf{z}}\lambda\,d\mathbf{z} - \epsilon^2\int\limits_{\Omega}|\Gamma|\,\bar{w}\,\partial_{\mathbf{z}}\bar{c^\epsilon}\,\lambda\,d\mathbf{z}\\
&- \epsilon^2\int\limits_{\Omega}\big(w - |\Gamma|\,\bar{w}\big)\,d\mathbf{z}\big(\partial_{\mathbf{z}} c^\epsilon - \partial_{\mathbf{z}}\bar{c^\epsilon}\big)\,\lambda\,d\mathbf{z} = \epsilon^2\int\limits_{\Omega} F^\epsilon\,\lambda\,d\mathbf{z}.
\end{aligned}$$

Here, the first summand used the identity $\int_\Omega D\,\partial_{\mathbf{z}} c^\epsilon\,\partial_{\mathbf{z}}\lambda\,d\mathbf{z} = \int_\Omega D\,\partial_{\mathbf{z}}\bar{c^\epsilon}\,\partial_{\mathbf{z}}\lambda\,d\mathbf{z}$, because $\lambda \in M$. Since $\partial_{\mathbf{z}} c^\epsilon - \partial_{\mathbf{z}}\bar{c^\epsilon} \in Q$, it holds that

$$\int_\Omega |\Gamma|\,\bar{w}\big(\partial_{\mathbf{z}} c^\epsilon - \partial_{\mathbf{z}}\bar{c^\epsilon}\big)\,\lambda\,d\mathbf{z} = \int_0^1 |\Gamma|\,\bar{w}\lambda \int_\Gamma \big(\partial_{\mathbf{z}} c^\epsilon - \partial_{\mathbf{z}}\bar{c^\epsilon}\big)d\tilde{\mathbf{z}}\,dz = 0.$$

Hence, the equality above is equivalent to

$$\int_\Omega D\,\partial_{\mathbf{z}}\bar{c^\epsilon}\,\partial_{\mathbf{z}}\lambda\,d\mathbf{z} - \int_\Omega |\Gamma|\,\bar{w}\,\partial_{\mathbf{z}}\bar{c^\epsilon}\,\lambda\,d\mathbf{z} - \int_\Omega w\big(\partial_{\mathbf{z}} c^\epsilon - \partial_{\mathbf{z}}\bar{c^\epsilon}\big)\,\lambda\,d\mathbf{z} = \int_\Omega F^\epsilon\,\lambda\,d\mathbf{z}.$$

Next, we integrate by parts the third summand of the left hand side

$$\begin{aligned}
-\int_\Omega w\,\partial_{\mathbf{z}}\big(c^\epsilon - \bar{c^\epsilon}\big)\,\lambda\,d\mathbf{z} &= \int_\Omega \big(c^\epsilon - \bar{c^\epsilon}\big)\,\partial_{\mathbf{z}}\big(w\,\lambda\big)\,d\mathbf{z} - \int_{\partial\Omega} w\big(c^\epsilon - \bar{c^\epsilon}\big)\lambda\,\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}\,dS \\
&= \int_\Omega w\big(c^\epsilon - \bar{c^\epsilon}\big)\partial_{\mathbf{z}}\lambda\,d\mathbf{z} - \int_{G_0\cup G_1} w\big(c^\epsilon - \bar{c^\epsilon}\big)\lambda\,\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}\,dS \\
&\quad - \int_O w\big(c^\epsilon - \bar{c^\epsilon}\big)\lambda\,\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}\,dS.
\end{aligned}$$

In the expression above we used the fact that $\partial_{\mathbf{z}} w = 0$. The boundary term on $G_0 \cup G_1$ vanishes because $\lambda = 0$ here and the integral over O vanishes because $\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}} = 0$ in this part of the boundary. Hence, $\bar{c^\epsilon} = P_M c^\epsilon \in M$ and $c^\epsilon - \bar{c^\epsilon} = P_Q c^\epsilon \in Q$ verify the statement (3.14).

Next we prove that $P_Q c^\epsilon, P_M c^\epsilon$ verify the statement (3.15), first recall that $\tilde{\nabla}\bar{c^\epsilon} = 0$ because $\bar{c^\epsilon} \in M$. Test (3.5) with $r \in Q$, this gives

$$\begin{aligned}
\epsilon^2\int_\Omega F^\epsilon\,r\,d\mathbf{z} &= \int_\Omega D\,\tilde{\nabla} c^\epsilon\cdot\tilde{\nabla} r\,d\mathbf{z} + \epsilon^2\int_\Omega D\,\partial_{\mathbf{z}} c^\epsilon\,\partial_{\mathbf{z}} r\,d\mathbf{z} - \epsilon^2\int_\Omega w\,\partial_{\mathbf{z}} c^\epsilon\,r\,d\mathbf{z} \\
&= \int_\Omega D\,\tilde{\nabla}\big(c^\epsilon - \bar{c^\epsilon}\big)\cdot\tilde{\nabla} r\,d\mathbf{z} + \epsilon^2\int_\Omega D\,\partial_{\mathbf{z}} c^\epsilon\,\partial_{\mathbf{z}} r\,d\mathbf{z} - \epsilon^2\int_\Omega w\,\partial_{\mathbf{z}} c^\epsilon\,r\,d\mathbf{z} \\
&= \int_\Omega D\,\tilde{\nabla}\big(c^\epsilon\,d\mathbf{z} - \bar{c^\epsilon}\big)\cdot\tilde{\nabla} r\,d\mathbf{z} + \epsilon^2\int_\Omega D\,\partial_{\mathbf{z}} c^\epsilon\,\partial_{\mathbf{z}} r\,d\mathbf{z} \\
&\quad - \epsilon^2\int_\Omega w\big(\partial_{\mathbf{z}} c^\epsilon - \partial_{\mathbf{z}}\bar{c^\epsilon}\big)\,r\,d\mathbf{z} - \epsilon^2\int_\Omega w\,\partial_{\mathbf{z}}\bar{c^\epsilon}\,r\,d\mathbf{z}.
\end{aligned} \tag{3.23}$$

We modify the second and fourth summand in last line. Recalling that $r \in Q = M^\perp$ implies $|\Gamma|\,\bar{r} = \int_\Gamma r d\tilde{\mathbf{z}} = 0$ we get

$$\int_\Omega \partial_{\mathbf{z}}\bar{c^\epsilon}\,\partial_{\mathbf{z}} r\,d\mathbf{z} = \int_\Omega \nabla\bar{c^\epsilon}\cdot\nabla r\,d\mathbf{z} = 0.$$

Hence, the equality (3.23) transforms in

$$\int_{\Omega} D\,\tilde{\nabla}\big(c^{\epsilon}-\bar{c^{\epsilon}}\big)\cdot\tilde{\nabla}r\,d\mathbf{z}+\epsilon^{2}\int_{\Omega} D\,\partial_{\mathbf{z}}\big(c^{\epsilon}-\bar{c^{\epsilon}}\big)\partial_{\mathbf{z}}r\,d\mathbf{z}$$
$$-\epsilon^{2}\int_{\Omega} w\,\partial_{\mathbf{z}}\big(c^{\epsilon}-\bar{c^{\epsilon}}\big)r\,d\mathbf{z}-\epsilon^{2}\int_{\Omega} w\,\partial_{\mathbf{z}}\bar{c^{\epsilon}}\,r\,d\mathbf{z}=\epsilon^{2}\int_{\Omega} F^{\epsilon}\,r\,d\mathbf{z}.$$

Dividing the expression above over ϵ^2 and rearranging the terms, it follows that $c^{\epsilon}-\bar{c^{\epsilon}}\in Q$ and $\bar{c^{\epsilon}}\in M$ satisfy the statement (3.15).

Finally, given that $c^{\epsilon}-\bar{c^{\epsilon}}=P_Q c^{\epsilon}\in Q$, $\bar{c^{\epsilon}}=P_M c^{\epsilon}\in M$ are a solution to the system (3.14)–(3.15), we conclude it is the unique solution due to the well-posedness of the problem shown in Theorem 3.11. □

4. THE ASYMPTOTIC ANALYSIS

Our goal in the current section is to present the asymptotic analysis of the ϵ-problems (3.14)–(3.15) and derive a limiting form, which permits to approximate the original one, for small values of ϵ. In the subsection 4.1 we present the Hilbert space that is naturally deduced from the asymptotic analysis and consequently, part of the functional setting where the limiting problem is posed. Next, the subsection 4.2 delivers the a-priori estimates analysis, the limiting form and even strong convergence statements.

4.1. A PARTICULAR FUNCTION SPACE

In the current section we introduce the Hilbert space for modeling the limiting problem (4.14)–(4.15). We also prove some of its features, needed to show the well-posedness of the limiting form.

Definition 4.1. Define the set of null transverse weighted average maps

$$\mathcal{G}\overset{\mathbf{def}}{=}\Big\{u\in H^{1}(\Gamma):\int_{\Gamma} u\,d\tilde{\mathbf{z}}=0\Big\}. \tag{4.1}$$

Next we recall a familiar result, its proof uses standard weak topology compactness arguments for Hilbert spaces and the compact embedding $H^1(\Gamma)\hookrightarrow L^2(\Gamma)$, given by the Rellich–Kondrachov theorem for bounded domains, such as Γ.

Lemma 4.2. *There exists a positive constant K depending only on the domain Γ such that*

$$\big\|u\big\|_{1,\Gamma}\leq K\,\big\|\tilde{\nabla}u\big\|_{0,\Gamma},\quad \textit{for all } u\in\mathcal{G}. \tag{4.2}$$

In particular the application $u\mapsto\big\|\tilde{\nabla}u\big\|_{0,\Gamma}$ is a norm, equivalent to the standard $H^1(\Gamma)$-norm in $\mathcal{G}$.

Remark 4.3 (Notation of the Gradient)**.** A slight abuse of notation was made in Lemma 4.2, denoting the gradient of a function in $\mathcal{G}$ by $\tilde{\nabla} u$ instead of ∇u. This was done in order to ease the notation in the proof of Lemma 4.5 below and it will cause no conflict.

Next, we introduce a space function where the *limiting problem* takes place.

Definition 4.4. Denote by $\mathcal{Q}$ the space

$$\mathcal{Q} \stackrel{\mathbf{def}}{=} \Big\{ p \in L^2(\Omega) : \bar{p} = 0, \int\limits_{\Omega} \big|\tilde{\nabla} p\big|^2 d\mathbf{z} < +\infty \Big\}, \tag{4.3}$$

endowed with the natural inner product

$$\big\langle p, q \big\rangle_{\mathcal{Q}} \stackrel{\mathbf{def}}{=} \int\limits_{\Omega} \big(p\, q + \tilde{\nabla} p \cdot \tilde{\nabla} q\big) d\mathbf{z}. \tag{4.4}$$

It is trivial to see that $\mathcal{Q}$ is a Hilbert space. Now we prove the following lemma.

Lemma 4.5. *The application $p \mapsto \|\tilde{\nabla} p\|_{0,\Omega}$ is a norm in $\mathcal{Q}$ equivalent to the induced by the inner product* (4.4). *Moreover, if K is the positive constant such that $\|u\|_{1,\Gamma} \leq K\, \|\tilde{\nabla} u\|_{0,\Gamma}$ for all $u \in \mathcal{G}$ given by Lemma 4.2, then*

$$\|p\|_{\mathcal{Q}}^2 = \|p\|_{0,\Omega}^2 + \|\tilde{\nabla} p\|_{0,\Omega}^2 \leq K^2\, \|\tilde{\nabla} p\|_{0,\Omega}^2, \quad \forall p \in \mathcal{Q}. \tag{4.5}$$

Proof. It is direct to see that the application $p \mapsto \|\tilde{\nabla} p\|_{0,\Omega}$ is a semi-norm in $\mathcal{Q}$. Due to the geometry of the domain and the definition of $\mathcal{Q}$ the map $p(\cdot, z)$ is in $\mathcal{G}$ for z-almost everywhere. Therefore, the inequality (4.2) applies, i.e.

$$\int\limits_{\Gamma} \big(|p(\tilde{\mathbf{z}}, z)|^2 + |\tilde{\nabla} p(\tilde{\mathbf{z}}, z)|^2\big) d\tilde{\mathbf{z}} \leq K^2 \int\limits_{\Gamma} \big|\tilde{\nabla} p(\tilde{\mathbf{z}}, z)\big|^2 d\tilde{\mathbf{z}}.$$

Now, integrating over z, the inequality (4.5) follows. The latter implies the homogeneity of the map $p \mapsto \|\tilde{\nabla} p\|_{0,\Omega}$, as well as its equivalence to the norm induced by (4.4). $\square$

Finally, we have a density result.

Theorem 4.6. *The space Q is dense in $\mathcal{Q}$ with the norm $\|\cdot\|_{\mathcal{Q}}$.*

Proof. Clearly the space Q is included in $\mathcal{Q}$. It is also direct to see that

$$\Big\{ \varphi \in C^\infty(\Omega) : \varphi = 0 \text{ on } G_0 \cup G_1 \text{ and } \int\limits_{\Gamma} \varphi\, d\tilde{\mathbf{z}} = 0 \Big\},$$

is included in H. Therefore, the result follows by standard arguments. $\square$

4.2. A-PRIORI ESTIMATES AND CONVERGENCE

Lemma 4.7 (Global Boundedness of the Solutions). *Let* $\big\{(q^\epsilon, \mu^\epsilon)\big\}$ *be the sequence of solutions to the family of systems* (3.14)–(3.15)*. Then there exists a constant* $K > 0$ *such that*

$$\Big(\big\|\mu^\epsilon\big\|_V^2 + \big\|\frac{1}{\epsilon} q^\epsilon\big\|_{\mathcal{Q}}^2 + \big\|q^\epsilon\big\|_{H(\partial_{\mathbf{z}},\Omega)}^2\Big)^{1/2} \le K \quad \textit{for all } \epsilon > 0. \tag{4.6}$$

Proof. Test the statement (3.14) with μ^ϵ and (3.15) with q^ϵ, add them together and get

$$\begin{aligned}&\int_\Omega D\left|\partial_{\mathbf{z}}\mu^\epsilon\right|^2 d\mathbf{z} - \int_\Omega |\Gamma|\,\bar{w^\epsilon}\,\partial_{\mathbf{z}}\mu^\epsilon\,\mu^\epsilon\,d\mathbf{z} + \frac{1}{\epsilon^2}\int_\Omega D\left|\tilde{\nabla} q^\epsilon\right|^2 d\mathbf{z} + \int_\Omega D\left|\partial_{\mathbf{z}} q^\epsilon\right|^2 d\mathbf{z}\\ &- \int_\Omega \epsilon\, w^\epsilon\,\partial_{\mathbf{z}} q^\epsilon\, q^\epsilon\, d\mathbf{z} = \int_\Omega F^\epsilon\,\mu^\epsilon\,d\mathbf{z} + \int_\Omega F^\epsilon\, q^\epsilon\, d\mathbf{z}.\end{aligned} \tag{4.7}$$

In the expression above, we simplify the second and fifth summands of the left hand side in the following way:

$$\int_\Omega |\Gamma|\,\bar{w}\,\partial_{\mathbf{z}}\mu^\epsilon\,\mu^\epsilon\,d\mathbf{z} = \int_\Gamma |\Gamma|\,\bar{w}\int_0^1 \frac{1}{2}\partial_{\mathbf{z}}(\mu^\epsilon)^2 dz\,d\tilde{\mathbf{z}} = \int_\Gamma |\Gamma|\,\bar{w}\frac{1}{2}(\mu^\epsilon)^2\Big|_0^1 d\tilde{\mathbf{z}} = 0, \tag{4.8}$$

$$\int_\Omega w\,\partial_{\mathbf{z}} q^\epsilon\, q^\epsilon\, d\mathbf{z} = \int_\Gamma w\int_0^1 \frac{1}{2}\partial_{\mathbf{z}}(q^\epsilon)^2 dz\,d\tilde{\mathbf{z}} = \int_\Gamma w\frac{1}{2}(q^\epsilon)^2\Big|_0^1 d\tilde{\mathbf{z}} = 0. \tag{4.9}$$

Here we used the fact that $p\big|_{G_0\cup G_1} = 0$ for all $p \in V$, in particular, for q^ϵ and μ^ϵ. Hence, rearranging (4.7) and using the simplifications above, we get

$$\begin{aligned}\int_\Omega D\left|\partial_{\mathbf{z}}\mu^\epsilon\right|^2 d\mathbf{z} &+ \frac{1}{\epsilon^2}\int_\Omega D\big|\tilde{\nabla} q^\epsilon\big|^2 d\mathbf{z} + \int_\Omega D\left|\partial_{\mathbf{z}} q^\epsilon\right|^2 d\mathbf{z}\\ &= \int_\Omega F^\epsilon\,\mu^\epsilon\,d\mathbf{z} + \int_\Omega F^\epsilon\, q^\epsilon\, d\mathbf{z}\\ &\le \big\|F^\epsilon\big\|_{0,\Omega}\big\|\mu^\epsilon\big\|_{0,\Omega} + \big\|F^\epsilon\big\|_{0,\Omega}\big\|q^\epsilon\big\|_{0,\Omega}\\ &\le \sqrt{2}\,\big\|F^\epsilon\big\|_{0,\Omega}\Big(\big\|\mu^\epsilon\big\|_V^2 + \big\|\frac{1}{\epsilon} q^\epsilon\big\|_{\mathcal{Q}}^2 + \big\|q^\epsilon\big\|_{H(\partial_{\mathbf{z}},\Omega)}^2\Big)^{1/2}.\end{aligned} \tag{4.10}$$

Due to the hypothesis 3.3 (ii), the sequence $\big\{F^\epsilon\big\}$ is bounded in $L^2(\Omega)$. Finally, the estimate (3.7) is applicable on μ^ϵ because $\mu^\epsilon \in M$ and then $\tilde{\nabla}\mu^\epsilon = 0$. From here, the inequality (4.6) follows. □

With the a-priori estimate (4.6) we prove the existence of weakly convergent subsequences.

Corollary 4.8 (Weak Convergence of Sequences)**.** *There exist $q^* \in \mathcal{Q}$, $\mu^* \in M$ and subsequences, still denoted $\{\mu^\epsilon\}, \{\frac{1}{\epsilon} q^\epsilon\}, \{q^\epsilon\}$, such that*

$$\mu^\epsilon \to \mu^*, \quad \textit{weakly in } H^1(\Omega) \textit{ strongly in } L^2(\Omega), \tag{4.11}$$

$$\frac{1}{\epsilon^2} q^\epsilon \to q^*, \quad \textit{weakly in } \mathcal{Q}, \tag{4.12}$$

$$q^\epsilon \to 0, \quad \textit{weakly in } H(\partial_{\mathbf{z}}, \Omega) \textit{ strongly in } L^2(\Omega). \tag{4.13}$$

Proof. First, from the estimate (4.6) and standard Hilbert space theory, there must exist weakly convergent subsequences $\{\mu^\epsilon\}, \{\frac{1}{\epsilon} q^\epsilon\}$, $\{q^\epsilon\}$ and limits $\mu^* \in M$, $q_0 \in \mathcal{Q}$, $q_{00} \in H(\partial_{\mathbf{z}}, \Omega)$, in their corresponding spaces.

Next, the topology of $M \leq V$ is equivalent to the $H^1(\Omega)$ standard topology (see Proposition 3.4), consequently the Rellich-Kondrachov theorem applies, hence the strong convergence of $\{\mu^\epsilon\}$ in $L^2(\Omega)$ follows.

Since $\{\frac{1}{\epsilon} q^\epsilon\}$ is bounded in $\mathcal{Q}$, in particular it is bounded in $L^2(\Omega)$. Therefore $\{q^\epsilon\}$ converges to 0, strongly in $L^2(\Omega)$. Given that the weak limits are unique, the statement (4.13) follows.

Next, let $\epsilon \to 0$ in (3.15) and get

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int\limits_\Omega D\, \tilde{\nabla} q^\epsilon \cdot \tilde{\nabla} r \, d\mathbf{z} - \int\limits_\Omega w\, \partial_{\mathbf{z}} \mu^* \, r \, d\mathbf{z} = \int\limits_\Omega F \, r \, d\mathbf{z},$$

for all $r \in Q$. Due to the density of Q in $\mathcal{Q}$ (see Theorem 4.6), the statement above is extended to all $r \in \mathcal{Q}$. Hence, the sequence of functionals

$$\ell^\epsilon : \mathcal{Q} \to \mathbb{R}, \quad \ell^\epsilon(r) \overset{\mathbf{def}}{=} \frac{1}{\epsilon^2} \int\limits_\Omega \tilde{\nabla} q^\epsilon \cdot \tilde{\nabla} r \, d\mathbf{z},$$

is bounded. Due to Riez Representation Lemma and the inequality (4.5), we conclude that the sequence $\{\epsilon^{-2} q^\epsilon\}$ is bounded in $\mathcal{Q}$. Again, from standard Hilbert space theory, the statement (4.12) follows and the result is complete. □

With the previous results, we are ready to deliver the limiting problem and prove its well-posedness independently from the convergence of the subsequences.

Theorem 4.9 (The Limiting Problem)**.** *The limit functions $\mu^* \in M$, $q^* \in \mathcal{Q}$ given by Corollary 4.8 satisfy the following statements.*

$$\textit{Find } \mu \in M: \quad \int\limits_\Omega D\, \partial_{\mathbf{z}} \mu \, \partial_{\mathbf{z}} \lambda \, d\mathbf{z} - \int\limits_\Omega |\Gamma| \, \bar{w} \, \partial_{\mathbf{z}} \mu \, \lambda \, d\mathbf{z} = \int\limits_\Omega F \, \lambda \, d\mathbf{z}, \quad \textit{for all } \lambda \in M. \tag{4.14}$$

$$\textit{Find } q \in \mathcal{Q}: \quad \int\limits_\Omega D\, \tilde{\nabla} q \cdot \tilde{\nabla} r \, d\mathbf{z} - \int\limits_\Omega w \, \partial_{\mathbf{z}} \mu \, r \, d\mathbf{z} = \int\limits_\Omega F \, r \, d\mathbf{z}, \quad \textit{for all } r \in \mathcal{Q} \textit{ and } \mu \textit{ solution of (4.14)}. \tag{4.15}$$

Furthermore, the system (4.14)–(4.15) *is well-posed.*

Proof. In order to prove that the limit functions solve the system (4.14)–(4.15) it suffices to let $\epsilon \to 0$ in the equations (3.14), (3.15) and apply the weak convergence statements (4.11), (4.12), (4.13), delivered by Corollary 4.8.

Next we prove the well-posedeness. Observe that the problem (4.14) is independent from q and it has a very similar structure to that of Problem (2.2) but on a different functional setting. Hence, proving its well-posedness in M is identical to the proof of Theorem 2.5. Now that the existence and uniqueness of $\mu \in M$ is assured, we rewrite the statement (4.15), regarding $\partial_{\mathbf{z}}\mu$ as a forcing term, i.e.,

$$\int_\Omega D\,\tilde{\nabla} q \cdot \tilde{\nabla} r\, d\mathbf{z} = \int_\Omega F\, r\, d\mathbf{z} + \int_\Omega w\, \partial_{\mathbf{z}}\mu\, r\, d\mathbf{z}, \quad \forall r \in \mathcal{Q}.$$

Due to the lemma 4.5 and Lax-Milgram 2.3, it follows that (4.15) is well-posed. Hence, the system (4.14)–(4.15) is well-posed. □

Remark 4.10. Notice that the limiting problem (4.14)–(4.15) is only one-way coupled i.e., q depends on μ but not the other way around. This is a substantial difference with respect to the system (3.14)–(3.15), corresponding to the ϵ-problem, which is two-way coupled.

Finally, we prove that the convergence of the ϵ-solutions occurs in the strong sense, furthermore, the whole sequence (not only a subsequence) converges.

Theorem 4.11 (Strong Convergence of the Solutions)**.** *Let $\mu^* \in M$ and $q^* \in \mathcal{Q}$ be the limit functions given by Corollary 4.8 and let $\{\mu^\epsilon\}, \{q^\epsilon\}$ be the sequence of solutions to the ϵ-systems* (3.14)–(3.15)*. Then*

$$\left\|\mu^\epsilon - \mu^*\right\|_M \xrightarrow[\epsilon \to 0]{} 0, \tag{4.16}$$

$$\left\|\frac{1}{\epsilon^2} q^\epsilon - q^*\right\|_{\mathcal{Q}} \xrightarrow[\epsilon \to 0]{} 0. \tag{4.17}$$

Proof. First, we show that the full sequences $\{\mu^\epsilon\}, \{\epsilon^{-2} q^\epsilon\}$ converge weakly to μ^* and q^* respectively. To that end, take any subsequence and apply Corollary 4.8 to get corresponding weak limits μ^{**} and q^{**}. Due to the Theorem 4.9, these new limits solve the system (4.14)–(4.15) and due to the problem well-posedness, the solution is unique. Hence, $\mu^* = \mu^{**}, q^* = q^{**}$. Consequently, every subsequence of the original family of ϵ-solutions has yet another subsequence weakly convergent to μ^* and q^*. From standard Hilbert space theory, we conclude that the whole sequence is weakly convergent.

Next, we prove the strong convergence of $\{\mu^\epsilon\}$. Test (3.14) with μ^ϵ and get

$$\int_\Omega D\left(\partial_{\mathbf{z}}\mu^\epsilon\right)^2 d\mathbf{z} - \int_\Omega |\Gamma|\, \bar{w}\, \partial_{\mathbf{z}}\mu^\epsilon\, \mu^\epsilon\, d\mathbf{z} + \int_\Omega w\, q^\epsilon\, \partial_{\mathbf{z}}\mu^\epsilon\, d\mathbf{z} = \int_\Omega F^\epsilon\, \mu^\epsilon\, d\mathbf{z}.$$

In the expression above notice that the second summand vanishes as shown in the equality (4.8). In the third summand $\{q^\epsilon\}$ converges strongly to zero and $\{\mu^\epsilon\}$ converges weakly to μ^*. The right hand side converges because the forcing term converges strongly. Hence, letting $\epsilon \to 0$ gives

$$\lim \int_\Omega D\left(\partial_{\mathbf{z}}\mu^\epsilon\right)^2 d\mathbf{z} = \int_\Omega F^\epsilon\,\mu^*\,d\mathbf{z}.$$

On the other hand, testing (4.14) on μ^* and repeating the technique of Equation (4.8), we get that

$$\int_\Omega D\left(\partial_{\mathbf{z}}\mu^*\right)^2 d\mathbf{z} = \int_\Omega F^\epsilon\,\mu^*\,d\mathbf{z}.$$

Consequently, $\lim \left\|\mu^\epsilon\right\|_M = \lim \left\|\mu^\epsilon\right\|_V = \left\|\mu^*\right\|_V = \left\|\mu^*\right\|_M$ which, together with the weak convergence implies the statement (4.16).

Finally, we prove the strong convergence statement (4.17). Test (3.15) with $\left\{\epsilon^{-2}q^\epsilon\right\}$ and get

$$\begin{aligned}
-\int_\Omega w\,\partial_{\mathbf{z}}\mu^\epsilon\,\frac{1}{\epsilon^2}q^\epsilon\,d\mathbf{z} + \int_\Omega D\left|\tilde{\nabla}\left(\frac{1}{\epsilon^2}q^\epsilon\right)\right|^2 d\mathbf{z} + \frac{1}{\epsilon^2}\int_\Omega D\left(\partial_{\mathbf{z}}q^\epsilon\right)^2 d\mathbf{z} - \frac{1}{\epsilon^2}\int_\Omega w\,\partial_{\mathbf{z}}q^\epsilon\,q^\epsilon\,d\mathbf{z} \\
= \int_\Omega F^\epsilon\,\frac{1}{\epsilon^2}q^\epsilon\,d\mathbf{z}.
\end{aligned}$$

In the latest expression the fourth summand vanishes as shown in the equality (4.9). The first summand converges due to the strong convergence of $\left\{\mu^\epsilon\right\}$ (already stated) and the weak convergence of $\left\{\epsilon^{-2}q^\epsilon\right\}$. The right hand side converges, due to the strong convergence of the forcing terms and the weak convergence of the testing sequence. Hence, taking the upper limit

$$\begin{aligned}
\limsup \frac{1}{\epsilon^2}\int_\Omega D\left(\partial_{\mathbf{z}}q^\epsilon\right)^2 d\mathbf{z} &\le \limsup\left\{\int_\Omega D\left|\tilde{\nabla}\left(\frac{1}{\epsilon^2}q^\epsilon\right)\right|^2 d\mathbf{z} + \frac{1}{\epsilon^2}\int_\Omega D\left(\partial_{\mathbf{z}}q^\epsilon\right)^2 d\mathbf{z}\right\} \\
&= \int_\Omega F\,q^*\,d\mathbf{z} + \int_\Omega w\,\partial_{\mathbf{z}}\mu^*\,q^*\,d\mathbf{z}.
\end{aligned}$$

On the other hand, testing (4.15) with q^* gives

$$\int_\Omega D\left|\tilde{\nabla}q^*\right|^2 d\mathbf{z} - \int_\Omega w\,\partial_{\mathbf{z}}\mu^*\,q^*\,d\mathbf{z} = \int_\Omega F\,q^*\,d\mathbf{z}.$$

Due to the last observations we conclude that $\limsup\left\|\epsilon^{-2}q^\epsilon\right\|_{\mathcal{Q}} \le \left\|q^*\right\|_{\mathcal{Q}}$. From standard Hilbert space theory, we know that the weak convergence of $\left\{\epsilon^{-2}q^\epsilon\right\}$ yields $\liminf\left\|\epsilon^{-2}q^\epsilon\right\|_{\mathcal{Q}} \ge \left\|q^*\right\|_{\mathcal{Q}}$. Then $\lim\left\|\epsilon^{-2}q^\epsilon\right\|_{\mathcal{Q}} = \left\|q^*\right\|_{\mathcal{Q}}$ which, combined with the weak convergence gives the statement (4.17). $\square$

We close this section giving a dimensional-reduced model of the limiting problem (4.14)–(4.15). Its main motivation is to furnish a more advantageous formulation from the computational cost point of view.

Theorem 4.12 (A Dimensional Reduction). *The following system is well-posed.*

$$\text{Find } \mu \in H_0^1(\Omega): \int_0^1 D\,\partial_{\mathbf{z}}\mu\,\partial_{\mathbf{z}}\lambda\,dz - \int_0^1 |\Gamma|\,\bar{w}\,\partial_{\mathbf{z}}\mu\,\lambda\,dz = \int_0^1 \bar{F}\,\lambda\,dz, \tag{4.18}$$
$$\text{for all } \lambda \in H_0^1(\Omega).$$

$$\text{Find } q \in \mathcal{Q}: \int_\Omega D\,\tilde{\nabla} q \cdot \tilde{\nabla} r\,d\mathbf{z} - \int_0^1 \int_\Gamma w\,r\,d\tilde{\mathbf{z}}\,\partial_{\mathbf{z}}\mu\,dz = \int_\Omega F\,r\,d\mathbf{z}, \tag{4.19}$$
$$\text{for all } r \in \mathcal{Q} \text{ and } \mu \text{ solution of (4.18)}.$$

Moreover, its solution $\mu \in H_0^1, q \in \mathcal{Q}$ *satisfies*

$$\mu^*(\tilde{\mathbf{z}}, z) = \mu(z) \text{ for all } \tilde{\mathbf{z}} \in \Gamma, z \in (0,1), \quad q^* = q, \tag{4.20}$$

where $\mu^* \in M$, $q^* \in \mathcal{Q}$ *are the solution to the system* (4.14)–(4.15).

Proof. To derive the expression (4.18), it suffices to develop the integrals in (4.14), recalling that the functions do not depend on $\tilde{\mathbf{z}}$. The statement (4.19) is attained, using the same observation on the second summand of the left hand side in the statement (4.15). The well-posedness follows from the equivalence between both systems. □

5. CONCLUSIONS AND FINAL REMARKS

The present work delivers important conclusions which we list below.

1. We have performed the asymptotic analysis of the advection-diffusion problem (1.8)–(1.9), decoupling its solution c^ϵ as the sum of two effects $\mu^\epsilon + q^\epsilon$, where $(\mu^\epsilon, q^\epsilon)$ is the solution the system (3.14)–(3.15). The system (3.14)–(3.15) is a mixed formulation of the original problem (1.8)–(1.9) and it is introduced, to permit the separation of effects in the asymptotic analysis. While μ^ϵ indicates the transverse average of c^ϵ, q^ϵ is its orthogonal complement in the sense of the modeling Hilbert space.
2. The asymptotic analysis showed that the solution c^ϵ of an advection-diffusion problem (1) in a thin tube (see Figure 1), can be well approximated by $c^\epsilon = \mu^\epsilon + q^\epsilon \sim \mu + \epsilon^2 q$, for $\epsilon > 0$ small. Here (μ, q) is the solution of the system (4.18)–(4.19), which is far simpler than the original one. As before, μ is the transverse average and q is the orthogonal complement.
3. It has been observed that the approximation $c^\epsilon \sim \mu + \epsilon^2 q$, sets q as a second order corrector factor in the sense of Homogenization Theory, see [16]. Hence, depending on the accuracy level that the application context demands, the summand q could be computed directly, or approximated by a small random variable or simply be dropped.

From the generalization point of view (for potential applications), we highlight the following.

4. It is possible to extend the same analysis to a more general geometry, namely, where the cross-section is different from a disk, but the domain is a prismatic tube. More specifically, the change of variables (3.1) still applies. The analysis is essentially the same for such variant.
5. It is also possible to adopt a more general diffusion coefficient $D = D(\tilde{\mathbf{z}})$. A practical scenario for such consideration would be a tube having parallel internal strata with different diffusion coefficient values. In addition, parallel laminar flow should be included in the hypotheses, in order to satisfy the hypothesis 1.1. Such assumptions are reasonable in the context of porous media flow, see [11] for example.
6. In the case where $D = D(\tilde{\mathbf{z}})$, the main change with respect to the analysis presented here, is that the Hilbert space should be endowed with the inner product $\langle p, q \rangle_V \overset{\mathbf{def}}{=} \int_\Omega D(\tilde{\mathbf{z}}) p(\mathbf{z}) q(\mathbf{z})\, d\mathbf{z}$. In addition, the analysis should be done on weighted averages, more specifically, the projection of $\bar{p}$, presented in (3.10), should be replaced by $\left(\int_\Omega D(\tilde{\mathbf{z}})\, d\mathbf{z}\right)^{-1} \int_\Omega D(\tilde{\mathbf{z}}) p(\mathbf{z})\, d\mathbf{z}$.

The generalizations mentioned above were not exposed in this work because they contribute little to the central concepts of the present analysis, but introduce a level of complexity (specially in the mathematical expressions) that would obscure the key ideas: the priority was clarity over generality.

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