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Characterizations and classification of paraconvex multimaps*

by

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Abstract: Paraconvex multimaps are revisited in normed vector space setting. A parallel is provided with the studies conducted for real valued paraconvex functions on generalized convexities and monotonicities. Several characterizations are then obtained. The links with some generalized convexities for multimaps are examined and a first classification is achieved. In addition, two representation results for 2-paraconvex multimaps are given.

Keywords: support function, paraconvex multimap, Lipschitz property, generalized convexities, submonotonicity

1. Introduction, preliminary notions and results

1.1. The preliminaries

Related to the needs of applications in variational analysis a wide literature has been devoted to extensions of convexity and/or smoothness for sets, functions and multifunctions. For real valued functions a robust extension of both convexity and smoothness is given by LC^k-functions, or *lower*-C^k ($k \in \mathbb{N}$) functions, introduced (for $k \geq 2$) by T. Rockafellar (1982) in finite dimension, then extended to infinite dimensional Hilbert space by J. P. Penot (1996). The *lower*-C^1 functions, due to J. E. Springarn (1981), are characterized by a submonotonicity property of subdifferentials. Then, J. P. Vial (1983) introduced in finite dimension the *weakly convex* functions and characterized them as difference of two convex functions (dc-functions). But Rockafellar (1982) and Penot (1996) showed that for all $k \geq 2$ the *lower*-C^k functions are *lower*-C^2 and coincide actually with *weakly convex* and locally Lipschitz functions. For the

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role of these classes in optimization we refer also to Georgiev (1997). Later, the uniform notion of ϵ-*convex* function, due to Jofré, Luc and Théra (1998), allowed Luc, Ngai and Théra (1999) to provide, in terms of a local version of this notion, the robust class of *approximately convex* functions, characterized in finite dimensions as LC^1-functions, and then, in Banach spaces by means of submonotonicity of *subdifferentials.* Another robust class, due to A. Daniilidis and J. Malick (2005) in finite dimensions is given by the α-*weakly convex* functions, or $LC^{1,\alpha}$-functions ($0 < \alpha \leq 1$) in the sense of LC^1-functions with α-*Hölder* derivatives. This class occurs as a strictly decreasing family under set inclusion with respect to α from the larger class of LC^1-functions ($\alpha = 0$) towards the smaller one of $LC^{1,1}$-functions ($\alpha = 1$), which, actually, coincides with LC^2-functions.

To be complete, we must cite the seminal papers by S. Rolewicz (1999, 2000, 2001, 2005) devoted, in infinite dimensional normed vector spaces (*n.v.s* in short), to paraconvex functions. Let us recall briefly that a function f from X (*n.v.s*) to $\mathbb{R}$ is paraconvex on a convex subset D, if there exists $\alpha()$ from $\mathbb{R}_+ := [\emptyset, +\infty[$ to $\mathbb{R}_+$ such that:

For all $t \in [0,1]$ and $a, b \in D$ there holds

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) + t(1-t)\alpha(\|a-b\|), \tag{1}$$

and f is said to be γ-paraconvex if $\alpha(\lambda) := \lambda^\gamma$and $\gamma \geq 1$.

Clearly, the notion (1) extends the convexity of functions and the corresponding class encompasses almost all the ones cited before. A challenge is then related to asking whether the notions mentioned above for functions can be extended to multimaps, so that they still satify similar properties. There are few studies that have been devoted to paraconvex multimaps, namely to the metric regularity of γ-paraconvex multimaps (see, e.g., Huang and Li, 2011; Huang, 2012). But a systematic investigation of paraconvex multimaps is motivated mainly by the active field of variational analysis, where the set-valuedness, the generalized convexities and monotonicities play eminent roles.

We revisit in this paper $\alpha()$.*paraconvex multimaps* in *n.v.s* setting. Then, we provide a parallel with the studies conducted for paraconvex functions.

Let X, Y be *n.v.s* and F be a multimap from X to Y (denoted $F : X \rightrightarrows Y$), whose domain and graph are, respectively,

$$dom(F) := \{x \in X : F(x) \neq \varnothing\} \text{ and } gh(F) := \{(x,y) \in X \times Y : y \in F(x)\}.$$

Let Θ be the set of functions $\alpha() : \mathbb{R}_+ \to \mathbb{R}_+$ nondecreasing with $\alpha(0) = 0$. Denote by $\mathcal{F}^\Theta(X,Y)$ the set of multimaps $F : X \rightrightarrows Y$ with closed, convex values and $\alpha()$-*paraconvex* on a convex subset $D \subseteq dom(F)$ (for some $\alpha \in \Theta$) in the sense that for all $x_1, x_2 \in D$, $t \in [0,1]$ and $x_t := tx_1 + (1-t)\,x_2$, there holds

$$tF(x_1) + (1-t)F(x_2) \subset cl\,[F(x_t) + t(1-t)\alpha(\|x_1 - x_2\|)B_Y] \tag{2}$$

where cl designates the closure operation and B_Y is the unit ball of Y. For $\alpha(\lambda) := \rho\lambda^\gamma : \rho \geq 0$ and $\lambda \geq 1$, we get the γ-paraconvex multimaps.

Observe that, in opposition to the to real valued functions, the relation (2) for $Y := \mathbb{R}$ and $\alpha() \equiv 0$ captures the affine multimaps, instead of the convex ones. But we will focus our attention essentially on the case of infinite dimensinal vector space Y.

The investigation of $\mathcal{F}^\Theta(X, Y)$ is organized in four parts and the realized progress will be the following:

(I). In the first part some functional characterizations of multimaps and several stability results are given. (Proposition 1; Lemmae 1 and 2; Proposition 2)

(II). The second part is devoted to Lipschitz properties and some characterizations of $\alpha()$.*paraconvex multimaps* (Theorems 1, 2, 3).

(III). In the third part, the links with some generalized convexities are characterized and a first classification in $\mathcal{F}^\Theta(X, Y)$ is established (Theorems 4, 5, 6).

(IV). The last part concerns two representation results for 2–paraconvex multimaps (Theorems 7 and 8).

Throughout, W, X, Y and Z are *n.v.s*, whose topological duals are W^*, X^*, Y^*, Z^*. By w (w^*) we indicate the weak (weak*) notions. The completeness and/or the reflexivity assumptions will be made explicit when needed. Without ambiguity, all the neutral elements are denoted by 0, while $\|.\|$ and $\langle . \rangle$ stand for the norm and the duality pairing. B_W (S_W) is the unit closed ball (sphere) of W and $\mathcal{L}(X, Y)$ is the *n.v.s* of linear, bounded operators from X to Y. We denote, respectively, by $cl(A)$, $int(A)$, $\overline{co}(A)$ the closure, the interior, the convex closure of A and by $span(A)$ (respectively $cone(A)$) the subspace (respectively cone) spanned by A.

The strong quasirelative interior (*sqri* in short) is given by

$$x \in sqri(A) \text{ iff } x \in A \text{ and } cone(A - x) \text{ is a closed linear subspace.}$$

The notations $\underline{\lim}_N$, $\overline{\lim}_N$ and $\lim_N$ designate the lower limit, the upper limit and the limit of sequences indexed in $N \subset \mathbb{N}$.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be proper; i.e, $domf := \{x \in X : f(x) < +\infty\} \neq \varnothing$.

Then, when f is locally Lipschitz on $D \subset domf$, the Clarke derivative of f at $a \in D$, given by

$$X \ni v \to f^0(a\ ; v) := \overline{\lim}_{(x,t)\to(a,0_+)} \left[\frac{1}{t}\left(f(x + tv) - f(x)\right)\right],$$

is sublinear, Lipschitz-continuous (Aubin and Ekeland, 1984) and the Clarke subdifferential of f at a is then

$$\partial^c f(a) := \left\{x^* \in X^* : \langle x^*, v\rangle \leq f^0(a\ ; v) \text{ for all } v \in X\right\}.$$

The domain of a function $\psi : X \times Y^* \to \mathbb{R} \cup \{\pm\infty\}$ is considered in the sense of saddle functions (see Barbu and Precupanu, 1978, pp. 130-140), or,

$$\begin{cases} dom\psi := dom_x(\psi) \times dom_p(\psi) \text{ where} \\ dom_x(\psi) := \{x \in X : \psi(x,p) > -\infty, \ \forall p \in Y^* \} \\ dom_p(\psi) := \{p \in Y^* : \psi(x,p) < +\infty, \ \forall x \in X\} \end{cases} \tag{3}$$

and ψ is said to be proper if $dom\psi \neq \varnothing$. The functions $x \to \psi(x,p)$, $p \to \psi(x,p)$ will be denoted by $\psi(.,p)$ and $\psi(x,.)$.

1.2. The support function of a multimap

Recall that for a multimap $F : X \rightrightarrows Y$, the domain and the graph are, respectively,

$$dom(F) := \{x \in X : F(x) \neq \varnothing\} \text{ and } gh(F) := \{(x,y) \in X \times Y : y \in F(x)\}.$$

Denote by ϑ_a a generic neighborhood of a. Then, F is said to be:

$*$ Proper if $dom(F) \neq \varnothing$ and $gh(F) \neq X \times Y$.

$*$ Strict if $dom(F) = X$.

$*$ A convex process if $gh(F)$ is a convex cone.

$*$ Closed (respectively convex) on a closed (respectively convex) subset $D \subset dom(F)$ if $gh(F) \bigcap (D \times Y)$ is closed (respectively convex) in $X \times Y$.

$*$ Upper semi-continuous ($u.s.c$ in short) at $a \in int(dom(F))$ if for every open subset $\theta \subset Z$, such that $\theta \supset F(a)$, there exists ϑ_a such that $F(\vartheta_a) \subset \theta$.

$*$ Lower-semi-continuous ($l.s.c$ in short) at $a \in int(dom(F))$ if for every open subset $\theta \subset Z$, $\theta \cap F(a) \neq \varnothing$ there exists ϑ_a, satisfying $F(v) \cap \theta \neq \varnothing$ $\forall v \in \vartheta_a$.

Let the duals X^* and Y^* be endowed with the weak* topology. Under the convention that for all $A \subset Y$, $A + \varnothing = \varnothing$, the support function of the multimap $F : X \rightrightarrows Y$ is the saddle function $s_F : X \times Y^* \to \overline{\mathbb{R}}$:

$$s_F(x,p) := inf\,\{\langle p, y\rangle : y \in F(x)\} \text{ if } F(x) \neq \varnothing \text{ and } +\infty \text{ otherwise.}$$

For some applications of this tool, see, e. g., Mokhtar-Kharroubi (1985, 1987, 2017). In this context, though, some facts are needed.

Proposition 1 *Assume that $F : X \rightrightarrows Y$ has convex and closed values. Then the following statements hold:*

(i). Let Y be complete. Then, s_F is proper iff F is proper and bounded valued.

(ii). Let C be a closed subset of $dom(F)$. Then, F is closed on C whenever for every $p \in Y^*$, $s_F(.,p)$ is $l.s.c$ on C.

(iii). The converse part of (ii) holds if, in addition, F is sequentially compact.

(iv). Let X and Y be complete and F be strict with weakly compact values. Then, F is $u.s.c$ whenever $s_F(.,p)$ is $l.s.c$ for every $p \in Y^*$.

Proof Clearly, for all $p \in Y^*$, $dom(s_F(.,p)) = dom(F)$. The easy proofs of *(i)-(ii)-(iii)* are omitted. The result *(iv)* is a consequence of Aubin and Ekeland (1984, Theorem 10, p. 128). ■

The well known Hörmander's Theorem establishes a correspondence between nonempty closed convex subsets of Y and proper, $l.s.c$ and sublinear functions, defined on Y^*. But by the following result we get a correspondence between the *multimaps* from X to Y with convex, closed and bounded values and *saddle functions* from $X \times Y^*$ to $\mathbb{R} \cup \{\pm\infty\}$, proper and closed in the sense of Barbu and Precupanu (1978, pp. 130-140).

Lemma 1 *Let X and Y be Banach spases, $D \subset X$ and $\psi : D \times Y^* \to \overline{\mathbb{R}}$. Then, ψ is the support function of a proper multimap $F_\psi : D \rightrightarrows Y$ with closed, convex and bounded values iff ψ is proper in the sense of (3) and for every $a \in D$ the function $\psi(a,.)$ is positively homogeneous, concave and u.s.c on Y^*.*

In that case, $dom(F_\psi) := D$ and for $x \in D$

$$F_\varphi(x) := \{y \in Y : \psi(x,p) \le \langle p, y\rangle \quad \forall\, p \in Y^*\}.$$

Proof Clearly, when $F : X \rightrightarrows Y$ has closed, convex and bounded values and $D \subset dom(F)$, then s_F satisfies the conditions of the lemma.

Conversely, such a function ψ allows for defining the multimap F_φ by

$$gh(F_\psi) := \{(x,y) \in D \times Y^* : \psi(x,p) \le \langle p, y\rangle \quad \forall\, p \in Y^*\}.$$

Then, $dom(F_\psi) = D$ and

$$s_{F_\psi}(x,p) \ge \psi(x,p) \text{ for all } (x,p) \in D \times Y^*.$$

But by Hörmander Theorem, for every $x \in D$ the function $\psi(x,.)$ is the lower support function of a closed and convex subset $C_x \subset Y$ and then, $F_\psi(x) \subset C_x$. We claim that the equality holds. Otherwise, by the Hahn-Banach separation theorem, we may argue easily by contradition. Hence $\psi = s_{F_\psi}$, which ends the proof. ■

The main operations on multimaps with later use are the following.

Let F_1, $F_2 : X \rightrightarrows Y$. Define λF_1 $(\lambda \ge 0)$, $\overline{co}F_1$ and $F_1 + F_2$ by :

$*$ $dom(\lambda F_1) := dom(F_1)$ and $\lambda F_1(x) := \lambda(F_1(x))$.

$*$ $dom(\overline{co}F) := dom(F)$ and $\overline{co}F(x) := \overline{co}(F(x))$.

$*$ $dom(F_1+F_2) := dom(F_1) \bigcap dom(F_2)$ and $(F_1+F_2)(x) := cl(F_1(x)+F_2(x))$.

Then, clearly,

$$s_{\lambda F_1} = \lambda s_{F_1}, \; s_{\overline{co}F} = s_F \text{ and } s_{F_1+F_2} = s_{F_1} + s_{F_2}.$$

For the intersection operation the following holds:

Lemma 2 *Let F_1, $F_2 : X \rightrightarrows Y$ be proper, bounded valued such that*

$$dom(F_1) \cap dom(F_2) \neq \varnothing.$$

Define $F^{\cap} : X \rightrightarrows Y$ by $dom(F^{\cap}) := dom(F_1) \cap dom(F_2)$ and

$$F^{\cap}(x) := \overline{co}F_1(x) \bigcap \overline{co}F_2(x).$$

Then, for all x such that

$$0 \in sqri\left(\overline{co}F_1(x) - \overline{co}F_2(x)\right), \tag{4}$$

the sup-convolution

$$s_{F^{\cap}}(x,p) = \sup_{u \in Y^*} \left(s_{F_1}(x,u) + s_{F_2}(x,p-u)\right)$$

is valid and exact; or,

$$\forall p \in Y^* \ \exists q \in Y^* : s_{F^{\cap}}(x,p) = s_{F_1}(x,q) + s_{F_2}(x,p-q). \tag{5}$$

Proof Observe that F_1 and F_2 are not supposed to be convex valued.

Let x be fixed, satisfying (4). Then define the convex functions

$$f_1,\ f_2 : Y \to \overline{\mathbb{R}} : f_i(y) := \iota_{\overline{co}(F_i(x))}(y)$$

where i_A is the indicator function, i.e., $i_A(a) = 0$ if $a \in A$ and $+\infty$ otherwise. The Fenchel's Theorem (Bot and Csetnek, 2012, Theorem 13) then yields the thesis. ■

1.3. Stability of $\mathcal{F}^{\Theta}(X,Y)$

It is easy to check that $F : X \rightrightarrows Y$ is $\alpha()$-*paraconvex on* $D \subset dom(F)$ iff

$$s_F(.,p) \text{ is } \|p\|\,\alpha()\text{-}paraconvex\ on\ D.$$

Thus, an optimal $\alpha_F \in \Theta$ can be defined as follows. The quotient for $t \in \,]0,1[$

$$\Delta_F(x_1,x_2,p,t) := \tfrac{1}{t(1-t)}\left(s_F(x_t,p) - t.s_F(x_1,p) - (1-t).s_F(x_2,p)\right) : t \in \,]0,1[$$
$$\text{and } \ \Delta_F(x_1,x_2,p,0) = \Delta_F(x_1,x_2,p,1) := 0$$

allows for defining the *modulus of paraconvexity* (of F) from $\mathbb{R}_+ \times Y^*$ to $\mathbb{R}$ as

$$\eta_F(\lambda,p) := \sup_{\substack{x_1,x_2 \in D\\ \|x_1-x_2\|=\lambda\\ t\in[0,1]}} \left(\Delta_F(x_1,x_2,p,t)\right).$$

Then, $\eta_F\ (\lambda, .)$ is positively homogeneous and for all $\lambda > 0$ and $p \neq 0$,

$$0 < \eta_F(\lambda, p) \leq \|p\|\, \alpha(\lambda).$$

Hence, the optimal $\alpha_F \in \Theta$ is well defined as

$$\mathbb{R}_+ \ni \lambda \to \alpha_F(\lambda) := \inf\left\{\eta_F(\lambda, p) : p \in Y^*\right\}.$$

Proposition 2 *Let X, Y, Z be n.v.s. The main stability results of the set $\mathcal{F}^\Theta(X, Y)$ are the following.*

(i). Let $A \in \mathcal{L}(Z; X)$ and $F : X \rightrightarrows Y$ be $\alpha_F()$-paraconvex on $D \subseteq dom(F)$. Then $F \circ A : Z \rightrightarrows Y$ is $\beta()$-paraconvex on D for $\beta(\lambda) := \alpha_F(\lambda \|A\|)$.

(ii).Suppose that Y and Z are complete, $F : X \rightrightarrows Y$ is γ-paraconvex ($\alpha(\lambda) = \lambda^\gamma$) on $dom(F)$ for $\gamma \in\,]1, 2]$ and $G : Y \rightrightarrows Z$ is a strict convex process.

Then, for some $\delta > 0$, the superposition

$$GoF(x) := \bigcup_{y \in F(x)} G(y) \text{ is } \delta\alpha()\text{-paraconvex on } dom(F).$$

(iii). $\mathcal{F}^\Theta(X, Y)$ is stable by finite intersection $F := \bigcap_{1 \leq i \leq n} F_i$, given by :

$$dom(F) := \bigcap_{1 \leq i \leq n} dom(F_i) : F(x) := \bigcap_{1 \leq i \leq n} (F_i(x))$$

if for all $x \in dom(F)$ there holds

$$0 \in int \left(\bigcap_{1 \leq i \leq n} (F_i(x)) \right).$$

In that case,

$$\alpha_F \leq 2^{n-1} max_{1 \leq i \leq n} \left\{\alpha_{F_i}\right\}.$$

Proof The easy proof of (i) is omitted, while the result (ii) is essentially due to S. Rolewicz (2000). Let us prove (iii) by induction.

Let F_1, $F_2 \in \mathcal{F}^\Theta(X, Y)$. Then, clearly, F_1 and F_2 are $\alpha()$-paraconvex for

$$\alpha() := max\left\{\alpha_{F_1}, \alpha_{F_2}\right\}.$$

Let $F := F_1 \cap F_2$. Then, by Lemma 2 one has

$$\begin{aligned} s_F(x, p) &= sup_{u \in Y^*}\left(s_{F_1}(x, u) + s_{F_2}(x, p - u)\right) \text{ and} \\ \forall p, \exists q &: s_F(x, p) = s_{F_1}(x, q) + s_{F_2}(x, p - q). \end{aligned}$$

Let $p, q \in Y^*, a, b \in X$ and $x_t := ta + (1-t)b$. Then, for $\widehat{p} := p - q$ and $i = 1, 2$

$$\begin{array}{lll} s_{F_i}(x_t, q) & \leq & ts_{F_i}(a, q) + (1-t)s_{F_i}(b, q) + \ t(1-t)\|q\|\,\alpha(\|a-b\|), \quad (6) \\ s_{F_i}(x_t, \widehat{p}) & \leq & ts_{F_i}(a, \widehat{p}) + (1-t)s_{F_i}(b, \widehat{p}) + t(1-t)\|\widehat{p}\|\,\alpha(\|a-b\|). \end{array}$$

By adding the two relations for s_{F_1} and then for s_{F_2}, we obtain, with $\alpha() := max\{\alpha_{F_1}, \alpha_{F_2}\}$

$$s_F(x_t, p) \leq ts_F(a, p) + (1-t)s_F(b, p) + t(1-t)\|p\|\,(2\alpha(\|a-b\|)).$$

Now, let $\widetilde{F} := F_1 \cap F_2$. We can reiterate (6) with $F := \widetilde{F} \cap F_3$. The recurrence then works and we check easily that for $F := \bigcap_{1 \leq i \leq n} F_i$ we get

$$\alpha()_F \leq 2^{n-1} max_{1 \leq i \leq n}\{\alpha_{F_i}\}.$$

■

2. Lipschitz property and $\alpha()$-paraconvexity

2.1. The essential results

In order to avoid messy subscripts we take notation s for s_F and $\partial_x^c s(a, p)$ for the Clarke-subdifferential at a of $s(., p)$. Recall that F is Lipschitz on D if for some $\delta > 0$

$$F(x_1) \subset cl(F(x_2) + \delta\|x_1 - x_2\|\,B_Y) \ \text{ for all } x_1, x_2 \in D, \tag{7}$$

which amounts to saying that

$$\{s(., p) : p \in S_{Y^*}\} \ \text{ is equi-Lipschitz on } D \text{ of rank } \delta. \tag{8}$$

Theorem 1 *Let $F : X \rightrightarrows Y$ be with closed, convex and bounded values. If, additionally, F is $\alpha()$.paraconvex on an open convex subset $D \subset domF$, then F is locally Lipschitz on D if it is locally bounded on D.*

Proof Let us prove that (8) holds true.

Clearly, F is locally bounded whenever it is locally Lipschitz, since it is bounded valued. For the converse part assume that for every $a \in D$ there exists an open, bounded and convex subset ω, such that $a \in \omega \subset D$ and $F(\omega) \subset \delta B_Y$ for some $\delta > 0$. Thus, for every $x \in \omega$ and $p \in Y^*$,

$$s(x, p) \geq -\delta\|p\|\,; i.e., -s(x, p) \leq \delta\|p\|$$

and then,

$$s(x, p) \leq \sup_{y \in F(x)} (\langle p, y\rangle) = -s(x, -p) \leq \delta\|-p\| = \delta\|p\|.$$

In this way,

$$\{s\,(.,p) : p \in S_{Y^*}\} \text{ is } \delta\text{-equibounded on } \omega.$$

For $r > 0 : a + rB_X \subset \omega$ select x and v in $a + rB_X$ such that

$$\|x - a\| < \frac{r}{2} \text{ and } \|v - a\| < \frac{r}{2}.$$

For $\epsilon > 0$ sufficiently small consider

$$\beta := \epsilon + \|x - v\| \text{ and } z := v + \frac{r}{2\beta}(v - x).$$

Then, $F(z) \subset \delta B_Y$, since

$$\|z - a\| < \|v - a\| + \frac{r}{2\beta}\,\|v - x\| < \frac{r}{2} + \frac{r}{2}(\frac{\|x - v\|}{\epsilon + \|x - v\|}) < r.$$

For $t := \frac{2\beta}{r+2\beta}$ we get $v = tz + (1 - t)x$. But by (8)

$$\{s\,(.,p)\,;\; p \in S_{Y^*}\} \text{ is equi-}\alpha()\text{-paraconvex on } \omega.$$

Having $t(1 - t) \leq t$, then,

$$s\,(v,p) = s\,(tz + (1 - t)x, p) \leq t.s\,(z,p) + (1 - t).s\,(x,p) + t\,\|p\|\,\alpha(\|x - z\|).$$

Hence,

$$s(x,p) \leq \delta \text{ for all } (x,p) \in D \times S_{Y^*}.$$

Because $\|x - z\| \leq 2r$, then, for $\eta := \frac{2}{r}(2\delta + \alpha(2r))$ and all $p \in S_{Y^*}$

$$\begin{aligned} s\,(v,p) - s\,(x,p) &\leq t\,(s\,(z,p) - s\,(x,p)) + t\alpha(\|x - z\|) \leq \\ & t(2\delta + \alpha(2r)) \leq \tfrac{2\beta}{r}(2\delta + \alpha(2r) \leq \eta(\epsilon + \|v - x\|). \end{aligned}$$

Exchanging the roles of x and v leads to

$$|s(v,p) - s(x,p)| \leq \eta(\epsilon + \|v - x\|).$$

Because ϵ is selected arbitrarily, then,

$$\{s\,(.,p) : p \in S_{Y^*}\} \text{ is equi-Lipschitz of rank } \eta \text{ on } a + rB_X.$$

The proof is complete. ■

Remark 1 *The Lipschitz property of closed, convex and locally bounded multimaps (a result due to Aubin and Ekeland, 1984, p. 132) holds true for the n.v.s setting. The completeness of the underlying spaces is useless.*

Recall that F is said to be sequentially compact on a closed subset $D \subset dom(F)$ if for every $a \in D$ and every sequence $(x_l, y_l)_{l\in\mathbb{N}} \subset gh(F)$ such that $(x_l)_{l\in\mathbb{N}} \to a$, there exists a convergent subsequence of $(y_l)_{l\in\mathbb{N}}$.

Theorem 2 *Assume that Y is complete, F is $\alpha()$.paraconvex and bounded valued on an open, convex subset $D \subset domF$. If, in addition, F is sequentially compact on D, then F is locally Lipschitz on D.*

Proof Let us prove that F is locally bounded. Then, Theorem 1 yields.

Under the condition of sequential compactness Proposition 2.(i) applies and then, for every $p \in S_{Y^*}$, the function $s(.,p)$ is $l.s.c$ on D. Hence,

$$x \to h(x) = sup_{p\in S_{Y^*}} (s(x,p)) \text{ is } l.s.c \text{ on } D. \tag{9}$$

Observe that sequential compactness is assumed to ensure only that h is $l.s.c$ (see also Remark 2 below).

We claim that $h(x)$ is finite for every $x \in D$. Indeed, the function $s(x,.)$ is concave and $u.s.c.$ Because B_{Y^*} is weak*-compact, then,

$$-\infty < h(x) = sup_{p\in S_{Y^*}}(s(x,p)) \leq sup_{p\in B_{Y^*}}(s(x,p)).$$

In this way, for every $n \in \mathbb{N}$ the subset

$$U_n := \{x \in D : h(x) \leq n\}$$

is closed.

Having that $D = \bigcup\{U_n : n \in \mathbb{N}\}$ then, by Baire's category theorem,

$$\exists\ n_0 : int\ U_{n_0} \neq \varnothing.$$

Let $a \in D$, $x_0 \in int\ U_{n_0}$ and $\delta > 1$ satisfying

$$z_0 := x_0 + \delta(a - x_0) \in int\ U_{n_0}.$$

Select some number $\rho > 0$ such that

$$z = x_0 + \delta(x - x_0) \in int\ U_{n_0} \text{ for all } x \in a + \rho B_X.$$

Because

$$x = \delta^{-1}z + (1-\delta^{-1})x_0,$$

then for $p \in S_{Y^*}$ the relation (8) leads to

$$s(z,p) \leq n_0$$

and

$$\begin{aligned} s(x,p) = {} & s(\delta^{-1}z + (1-\delta^{-1})x_0, p) \leq \\ & \delta^{-1}n_0 + (1-\delta^{-1})s(x_0,p) + \delta^{-1}(1-\delta^{-1})\alpha(\|x_0 - z\|). \end{aligned}$$

But, as $\alpha()$ is proper and nondecreasing, then there exist $\sigma > 0$, $\beta > 0$:

$$\alpha(\|x_0 - z\|) \leq \beta \text{ whenever } \|x_0 - z\| \leq \sigma.$$

Hence

$$\|x_0 - z\| = \delta \|x - x_0\|, \ \ x \in a + \rho B_X$$

and

$$z = x_0 + \delta(x - x_0) \in int\, U_{n_0}.$$

Thus, for all $x \in a + \rho B_X$

$$\|x - x_0\| \leq \|a - x_0\| + \|x - a\| \leq 2\rho.$$

By a suitable choice of δ and ρ

$$s(x,p) \leq \delta^{-1} n_0 + (1 - \delta^{-1}) s\,(x_0, p) + \delta^{-1}(1 - \delta^{-1})\beta.$$

Because $s\,(x_0, .)$ is concave and $u.s.c$, then, for some $M_1 > 0$,

$$s(x_0, p) \leq M_1 \text{ for all } p \in B_{Y^*}.$$

Thereby, for some $M_2 > 0$ we have the upper bound

$$s(x,p) \leq M_2 \text{ for all } (x,p) \in (a + \rho B_X) \times S_{Y^*} \ . \tag{10}$$

To show the boundedness from below, note that

$$(2a - x) \in a + \rho B_X \text{ for all } x \in a + \rho B_X$$

and then,

$$s(a,p) \leq 2^{-1} s(x,p) + 2^{-1} s(2a - x, p) + 2^{-1} \alpha(\|x - a\|).$$

Once again, $\alpha()$ being nondecreasing, there exists $M_3 > 0$ such that

$$s(a,p) \leq 2^{-1} s(x,p) + 2^{-1} s(2a - x, p) + 2^{-1} M_3.$$

Therefore, for all $x \in a + \rho B_X$

$$s(x,p) \geq 2s(a,p) - M_2 - M_3. \tag{11}$$

By (10) and (11) we obtain for some $M > 0$ the bound

$$|s(x,p)| \leq M \text{ for all } (x,p) \in (a + \rho B_X) \times S_{Y^*}. \tag{12}$$

Hence

$$\{s\,(.,p) : p \in S_{Y^*}\} \text{ is uniformly locally bounded on } D.$$

Which ends the proof. ∎

Remark 2 *When F is convex, closed, bounded valued, then $s(.,p)$ is convex, u.s.c (hence continuous) on $int(dom(F))$. Then, $x \to h(x) = \sup_{p \in S_{Y^*}} s(x,p)$ is l.s.c without assuming the compactness condition. In this way, Theorem 2 improves the Lipschitz property result, obtained for the first time by the author for the complete spaces X and Y (Mokhtar-Kharroubi, 1987, Theorem 4.1, p. 77).*

The following results have a later use.

Lemma 3 *Let X and Y be complete. Assume that F has closed and bounded values and is $\alpha()$-paraconvex on a nonempty open and convex subset $D \subset dom(F)$. Then, whenever F is locally bounded or sequentially compact, the following holds.*

$$\partial_x^c s(x,p) \neq \varnothing \text{ for all } (x,p) \in D \times Y^*$$

and for all $x, v \in D$, $p \in Y^*$ and $x^* \in \partial_x^c s(x,p)$ one has

$$\langle x^*, v - x \rangle \leq s(v,p) - s(x,p) + \|p\| \, \alpha(\|v - x\|). \tag{13}$$

Proof In view of Theorem 1 (or Theorem 2) F is locally Lipschitz on an open and convex subset $D \subset domF$. Thus, $\{s\,(.,p) : p \in S_{Y^*}\}$ is (locally) equi-Lipschitz on D and then, $\partial_x^c s(x,p) \neq \varnothing$ for all $(x,p) \in D \times Y^*$.

To prove (13) observe that for all $p \in Y^*$, $x_1, x_2 \in D$, $t \in [0,1]$ and $x_t := tx_1 + (1-t)\,x_2$, one has

$$s\,(x_t, p) \leq ts\,(x_1, p) + (1-t)s\,(x_2, p) + t(1-t)\,\|p\|\,\alpha(\|x_1 - x_2\|).$$

Let (x,p) be fixed in $D \times Y^*$. For all $y, v \in D$, $w := v - y$ and $t \in [0,1]$,

$$y + tw = (1-t)y + t(y+w) \in D$$

and then

$$\begin{gathered} s(y+tw,p) = s\,((1-t)y + t(y+w), p) \leq \\ (1-t)s(y,p) + ts(y+w,p) + t(1-t)\,\|p\|\,\alpha(\|w\|). \end{gathered}$$

Thus,

$$\frac{1}{t}\,(s(y+tw,p) - s(y,p)) \leq s(y+w,p) - s(y,p) + (1-t)\,\|p\|\,\alpha(\|w\|). \tag{14}$$

Let $h := v - x$ and $u = h + x - y$. Then, clearly

$$y \to x \text{ iff } u \to h$$

and (14) leads to

$$s(y+th,p) - s(y,p) = s(y + t\,u + t(y-x), p) - s(y,p).$$

Therefore, for all $p \in S_{Y^*}$

$$s^o\left((x;h),p\right) = \overline{lim}_{(y,t)\to(x,0_+)}\tfrac{1}{t}\left(s(y+th,p)-s(y,p)\right) = \\ \overline{lim}_{(y,t)\to(x,0_+)}\tfrac{1}{t}(s(y+tu+t(y-x),p)-s(y,p)).$$

But having

$$s(y+t\ u+t(y-x),p)-s(y,p) = \\ s(y+tu,p)-s(y,p)+s(y+tu+t(y-x),p)-s(y+tu,p),$$

we obtain that

$$s^o_x\left(((x;h),p\right) \le \overline{lim}_{(y,t)\to(x,0_+)}\tfrac{1}{t}\left(s(y+tu,p)-s(y,p)\right)+ \\ \overline{lim}_{(y,t)\to(x,0_+)}\tfrac{1}{t}(s(y+tu+t(y-x),p)-s(y+tu,p))$$

and since $s(.,p)$ is locally Lipschitz with some rank δ_p, we get

$$\overline{lim}_{(y,t)\to(x,0_+)}\tfrac{1}{t}\left(s(y+tu+t(y-x),p)-s(y+tu,p)\right) \\ \le \overline{lim}_{(y,t)\to(x,0_+)}\left(\tfrac{\delta_p}{t}\ \|t(y-x)\|\right) = 0.$$

Then,

$$s^o_x\left((x;h),p\right) \le \overline{lim}_{(y,t)\to(x,0_+)}\frac{1}{t}\left(s(y+tu,p)-s(y,p)\right)$$

which, combined with (14), leads to

$$s^o_x\left((x;h),p\right) \le s(x+h,p)-s(x,p)+\|p\|\,\alpha(\|h\|).$$

The result follows, since for all $p \in Y^*$, $h = v-x$ and $x^* \in \partial^c_x s(x,p)$

$$\langle x^*,h\rangle \le s^o_x\left((x;h),p\right).$$

■

2.2. Characterizations in $\mathcal{F}^{\Theta}(X,Y)$

Recall that $T : X \rightrightarrows X^*$ is said to be $\alpha()$-*hypomonotone* ($\alpha \in \Theta$) on $D \subset X$ if for all $x_1, x_2 \in D$ and all $x_1^* \in T(x_1)$, $x_2^* \in T(x_2)$

$$\langle x_1^*-x_2^*, x_1-x_2\rangle \ge -\alpha(\|x_1-x_2\|). \tag{15}$$

Theorem 3 *Assume that X and Y are complete and F has closed and bounded values. If, in addition, F is locally bounded or sequentially compact, then among the following assertions the implications $(i) \Longrightarrow (ii) \Longrightarrow (iii)$ are always valid.*

(i). F is $\alpha()$-paraconvex on a convex open subset $D \subset dom(F)$.

(ii). For all $x, v \in D,\ p \in S_{Y^*}$ *and* $x^* \in \partial_x^c s(x,p)$

$$\langle x^*, v-x\rangle \le s_F(v,p) - s(x,p) \ + \alpha(\|v-x\|). \tag{16}$$

(iii). The subdifferentials $\{\partial_x^c s\,(.,p) : p \in S_{Y^*}\}$ *are* $2\alpha()$*-hypomonotone on* D*, or for all* $x_1, x_2 \in D$ *and all* $x_1^* \in \partial_x^c s\,(x_1,p)\,,\ x_2^* \in \partial_x^c s\,(x_2,p)$

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \ge -2\alpha(\|x_1 - x_2\|). \tag{17}$$

Furthermore,

(iv). when, additionally, the function $\lambda \to \frac{\alpha(\lambda)}{\lambda}$ *is nondecreasing, then* $(iii) \Longrightarrow (i)$ *holds with* 2α *in place of* α*.*

Proof The implication $(i) \Longrightarrow (ii)$ hods by Lemma 3.

To prove the implication $(ii) \Longrightarrow (iii)$ let $x,\ v \in D$ be such that (16) holds. Let $x_1, x_2 \in D,\ p \in S_{Y^*}$ and $x_1^* \in \partial_x^c s_F(x_1,p)$, $x_2^* \in \partial_x^c s_F(x_2,p)$. Then

$$\begin{aligned}
\langle x_1^*, x_2 - x_1\rangle &\le s(x_2,p) - s(x_1,p) + \alpha(\|x_1 - x_2\|),\\
\langle x_2^*, x_1 - x_2\rangle &\le s(x_1\ ,p) - s(x_2,p) + \alpha(\|x_1 - x_2\|).
\end{aligned}$$

Upon adding the two inequalities the relation (17) follows.

For the implication $(iii) \Longrightarrow (i)$ let $x_1,\ x_2\ \in D$, $t \in [0,1]$ and $x_t = tx_1 + (1-t)x_2$. Then, by Lebourg's Mean Value Theorem (Lebourg, 1979, Theorem 1.7) applied on $[x_1, x_t]$, there exist $v_1 \in [x_1, x_t[$ and some $v_1^* \in \partial_x^c s_F(v_1,p)$, such that

$$\langle v_1^*, x_t - x_1\rangle = s(x_t,p) - s(x_1,p). \tag{18}$$

Similarly, there exist $v_2 \in [x_t, x_2[$ and $v_2^* \in \partial_x^c s(v_2,p)$ such that

$$\langle v_2^*, x_t - x_2\rangle = s(x_t,p) - s(x_2,p). \tag{19}$$

Since $x_t - x_1 = (1-t)(x_2 - x_1)$ and $x_t - x_2 = t(x_1 - x_2)$, then, by multiplying relations (18) by t and (19) by $(1-t)$ and adding the resulting equalities, we get

$$s(x_t,p) = ts(x_1,p) + (1-t)s(x_2,p) - t(1-t)\langle v_1^* - v_2^*, x_1 - x_2\rangle.$$

Because

$$\frac{x_1 - x_2}{\|x_1 - x_2\|} = \frac{v_1 - v_2}{\|v_1 - v_2\|}$$

the condition (17) of (iii) leads to

$$\langle v_1^* - v_2^*, v_1 - v_2\rangle \ge -2\alpha(\|v_1 - v_2\|).$$

Then

$$\langle v_1^* - v_2^*, v_1 - v_2\rangle \left(\tfrac{\|x_1-x_2\|}{\|v_1-v_2\|}\right) \geq -2\alpha(\|v_1 - v_2\|) \left(\tfrac{\|x_1-x_2\|}{\|v_1-v_2\|}\right),$$

or,

$$\langle v_1^* - v_2^*, x_1 - x_2\rangle \geq -2\alpha(\|v_1 - v_2\|) \left(\tfrac{\|x_1-x_2\|}{\|v_1-v_2\|}\right).$$

Hence, for all $p \in S_{Y^*}$

$$\begin{aligned} s(x_t, p) \leq{}& ts(x_1, p) + (1-t)s(x_2, p)+ \\ & 2t(1-t)\alpha(\|v_1 - v_2\|) \left(\tfrac{\|x_1-x_2\|}{\|v_1-v_2\|}\right). \end{aligned} \tag{20}$$

Finally, when $\lambda \to \frac{\alpha(\lambda)}{\lambda}$ is nondecreasing, and having v_1, $v_2 \in [x_1, x_2]$, then

$$\frac{\alpha(\|v_1 - v_2\|)}{\|v_1 - v_2\|} \leq \frac{\alpha(\|x_1 - x_2\|)}{\|x_1 - x_2\|},$$

i.e.,

$$\alpha(\|v_1 - v_2\|) \left(\frac{\|x_1 - x_2\|}{\|v_1 - v_2\|}\right) \leq \alpha(\|x_1 - x_2\|),$$

which, in view of (20), leads to

$$s(x_t, p) \leq ts(x_1, p) + (1-t)s(x_2, p) + t(1-t)\left(2\alpha(\|x_1 - x_2\|)\right).$$

Hence

$$\{s\,(.,p) : p \in S_{Y^*}\} \text{ is equi-}2\alpha()\text{-paraconvex on } D$$

and F is $2\alpha()$-paraconvex on D as well. The proof is complete. ∎

COROLLARY 1 *The equivalences (i) $\iff$ (ii) $\iff$ (iii) are always valid for γ-paraconvex multimaps when $\gamma \geq 1$.*

PROOF Because $\alpha(\lambda) := \rho\lambda^\gamma$ (for some $\rho > 0$), then, $\lambda \to \frac{\alpha(\lambda)}{\lambda}$ is nondecreasing when $\gamma \geq 1$ and Theorem 3 yields the corollary. ∎

3. The links with some convexities

The links with generalized convexities are examined in this section. Then, a first classification is achieved in $\mathcal{F}^\Theta(X,Y)$. To this end, observe that the *approximate convexity* for functions (Jofré, Luc and Théra, 1998) and its uniform version (Rolewicz, 2001) allow for defining the notion of *uniform approximately convex* multimap.

Definition 1 *A proper multimap $F : X \rightrightarrows Y$ with closed, convex and bounded values is said to be uniform approximately convex on a convex subset $D \subset dom(F)$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x_1, x_2 \in D$: $\|x_1 - x_2\| \leq \delta$ and $x_t := tx_1 + (1-t)\,x_2 : t \in [0,1]$ there holds*

$$tF(x_1) + (1-t)F(x_2) \subset cl\left(F(x_t) + \epsilon t(1-t)\,\|x_1 - x_2\|\, B_Y\right). \tag{21}$$

Observe that for $\alpha(\lambda) := \epsilon\lambda : \epsilon > 0$ we get the notion of ϵ-convex multimap.

Clearly, all locally γ-paraconvex multimaps with $\gamma > 1$ are uniform approximately convex. But the following characterization holds true.

Theorem 4 *Let X and Y be n.v.s and $F : X \rightrightarrows Y$ be proper with closed, convex and bounded values. Then F is uniform approximately convex on an open, convex subset $D \subset dom(F)$ iff F is $\alpha()$-paraconvex on D for some $\alpha \in \Theta$ satisfying*

$$\underline{lim}_{\lambda \to 0_+}\left(\frac{\alpha(\lambda)}{\lambda}\right) = 0, . \tag{22}$$

Proof Clearly, by (21), the functions $\{s\,(.,p) : p \in S_{Y^*}\}$ are equi-uniform approximately convex on D in the sense that for every $\epsilon > 0$ there exist $\delta > 0$ such that for all $p \in S_{Y^*}$, $t \in [0,1]$, $x_1, x_2 \in D$: $\|x_1 - x_2\| \leq \delta$ and $x_t := tx_1 + (1-t)\,x_2$

$$s\,(x_t, p) - \epsilon t(1-t)\,\|x_1 - x_2\| \leq t\; s\,(x_1, p) + (1-t)s\,(x_2, p). \tag{23}$$

Then, we check easily by the Hahn-Banach theorem the sufficiency of (22). Let us prove the converse part.

For every $\epsilon > 0$, let $\widehat{\delta}(\epsilon)$ denote the supremum of the numbers δ such that (23) holds for all $p \in S_{Y^*}$. It is easy to check that $\epsilon \to \widehat{\delta}(\epsilon)$ is non-decreasing. Now, let $\epsilon \to \delta(\epsilon)$ be an arbitrary continuous increasing function such that

$$\delta(\epsilon) < \widehat{\delta}(\epsilon) \text{ for } \epsilon > 0 \text{ and } lim_{\epsilon \to 0_+}\delta(\epsilon) = 0. \tag{24}$$

Let σ be the inverse function of $\delta : \tau \to \sigma(\tau) := \delta^{-1}(\tau)$. Then, σ is continuous, increasing and

$$lim_{\tau \to 0_+}\sigma(\tau) = 0.$$

The function $\alpha : \tau \to \alpha(\tau) := \tau\sigma(\tau)$ satisfies (22).

Furthermore, if $x_1, x_2 \in D$ and $\|x_1 - x_2\| = \tau$, then the functions $\delta()$ and $\sigma()$ ensure that for all $p \in S_{Y^*}$,

$$\left\{\begin{array}{l} s\,(x_t, p) \leq t\; s\,(x_1, p) + (1-t)s\,(x_2, p) + t(1-t)\tau\sigma(\tau) \\ \qquad = t\; s\,(x_1, p) + (1-t)s\,(x_2, p) + t(1-t)\alpha(\tau) \\ \quad = t\; s\,(x_1, p) + (1-t)s\,(x_2, p) + t(1-t)\alpha(\|x_1 - x_2\|) \end{array}\right. .$$

This ends the proof. ∎

The link with *convex* multimaps is characterized. Indeed, Theorem 5 *Let X and Y be n.v.s and $F : X \rightrightarrows Y$ be bounded valued and $\alpha()$ paraconvex for some $\alpha \in \Theta$ on a convex subset $D \subset dom(F)$. Assume, additionally that F is locally bounded. Then the following equivalence holds*

$$F \text{ is convex on } D \text{ iff } \underline{lim}_{\lambda \to 0^+} \left(\frac{\alpha(\lambda)}{\lambda^2}\right) = 0. \tag{25}$$

Proof Observe that completness of the spaces is useless.

When F is convex, then F is 0-*paraconvex* and (25) holds true.

For the converse part let us prove that for every $p \in S_{Y^*}$ the function $s\,(.,p)$ is convex. But this amounts to showing that for all $a, b \in D$ and $v := b - a$ the function

$$[0,1] \ni t \to \psi(t) := s\,(a + tv, p) \text{ is convex on } [0,1]. \tag{26}$$

Let $(t_n)_{n\in\mathbb{N}} \subset [0,1]$ be such a sequence that

$$\underline{lim}_{\lambda \to 0^+} \left(\frac{\alpha(t_n)}{t_n^2}\right) = 0. \tag{27}$$

By extracting a subsequence we may suppose that

$$lim_{\lambda \to 0^+} \left(\frac{\alpha(t_n)}{t_n^2}\right) = 0.$$

Because $\{s\,(.,p) : p \in S_{Y^*}\}$ is equi-$\alpha()$-paraconvex, then it is easy to check that for every $k \in \mathbb{N}$ satisfying $(k+1)\,t_n < 1$

$$\psi(kt_n) \leq \frac{1}{2}\left[\psi((k+1)t_n) + \psi((k-1)t_n)\right] + \alpha(2t_n). \tag{28}$$

Thus, for $(k+2)\,t_n < 1$ we have also

$$\psi((k+1)t_n) \leq \frac{1}{2}\left[\psi((k+2)t_n) + \psi(kt_n)\right] + \alpha(2t_n). \tag{29}$$

By adding (28) and (29) we get for every k such that $(k+1)\,t_n < 1$

$$\begin{aligned} 2\psi(kt_n) + 2\psi((k+1)t_n) \quad \leq \quad & \psi((k+1)t_n + \psi((k-1)t_n + \\ & \psi((k+2)t_n) + \psi(kt_n) + 2\alpha(2t_n), \end{aligned} \tag{30}$$

which leads to

$$[\psi(kt_n) - \psi((k-1)t_n)] - [\psi((k+2)t_n) - \psi((k+1)t_n)] \leq 2\alpha(2t_n). \tag{31}$$

As we divide both sides of (31) by t_n, we get

$$\begin{aligned} \tfrac{1}{t_n}\left[\psi(kt_n) - \psi((k-1)t_n)\right] - \tfrac{1}{t_n}\left[\psi((k+m+1)t_n) - \psi((k+m)t_n)\right] \leq \\ \tfrac{1}{t_n} 2(m-1)\alpha(2t_n). \end{aligned} \tag{32}$$

Then, for $m_n \leq \frac{1}{t_n}$ relations (32) lead to

$$\frac{1}{t_n}\left[\psi(k_n t_n) - \psi((k_n-1)t_n)\right] - \frac{1}{t_n}\left[\psi((k_n+m_n+1)t_n) - \psi((k_n+m_n)t_n)\right]$$

$$\leq 8\left(\frac{\alpha(2t_n)}{(2t_n)^2}\right).$$

Taking (27) into account and passing to the limit as $n \to +\infty$, we get

$$\psi'(\tau_1) \leq \psi'(\tau_2).$$

Hence, $t \to \psi(t)$ is convex. Then, $s_F(.,p)$ is convex, too, and F is convex as well. ■

COROLLARY 2 *Let X and Y be n.v.s and $F : X \rightrightarrows Y$ be bounded valued and $\alpha()$-paraconvex for some $\alpha \in \Theta$ on a convex subset $D \subset dom(F)$. If, in addition, Y is complete and F is sequentially compact, then the following holds.*

$$F \textit{ is convex on } D \textit{ iff } \underline{lim}_{\lambda\to 0^+}\left(\frac{\alpha(\lambda)}{\lambda^2}\right) = 0.$$

PROOF Clearly, F is locally bounded (by Theorem 2). Then, Theorem 5 yields the result. ■

A first classification in $\mathcal{F}^{\Theta}(X,Y)$ then follows.

THEOREM 6 *Let $F \in \mathcal{F}^{\Theta}(X,Y)$ or be $\alpha()$-paraconvex for some $\alpha \in \Theta$. Then, the following hold*

(i). F is ϵ-convex iff $\underline{\lim}_{\lambda\to 0_+} \frac{\alpha(\lambda)}{\lambda} > 0$.

(ii). F is uniform approximately convex iff $\lim_{\lambda\to 0_+} \frac{\alpha(\lambda)}{\lambda} = 0$.

(iii). F is weakly convex iff $\underline{lim}_{\lambda\to 0^+} \frac{\alpha(\lambda)}{\lambda^2} > 0$.

(iv). F is convex iff $\underline{lim}_{\lambda\to 0^+} \frac{\alpha(\lambda)}{\lambda^2} = 0$.

PROOF We know (by Theorems 4 and 5) that all γ-paraconvex multimaps are

(i). ϵ-convex for $\gamma = 1$.

(ii). uniform approximately convex for $\gamma \in [1,2]$ (Theorem 4)

(iii). weakly convex for $\gamma = 2$ and convex for $\gamma > 2$. (Theorem 5).

Then, again, Theorems 4 and 5 yield the result. ■

4. Representations of weakly-convex multimaps

Clearly, γ-*paraconvexity* deals with $\gamma \in [1,2]$, since when $\gamma > 2$, the multimaps are convex. But for $\gamma = 2$ (or weak-convexity) the following holds.

THEOREM 7 *Let $F : X \rightrightarrows Y$ be 2-paraconvex on a convex subset $D \subset domF$. If, in addition, X is a Hilbert space, then there exist $\rho > 0$ and a multimap $H : D \rightrightarrows Y$ convex, closed and bounded valued, such that*

$$\begin{cases} domH = D \text{ and for all } (x,p) \in D \times Y^*, \\ \quad s_F(x,p) = s_H(x,p) + \rho \|p\| \|x\|^2 \end{cases}. \tag{33}$$

Thereby, one gets the representation

$$H(x) = F(x) + (\rho \|x\|^2) B_Y \quad \text{for all } x \in D. \tag{34}$$

PROOF Given a and b in D and $t \in [0,1]$, we apply the following identity

$$\|x\|^2 = \|y\|^2 + 2\langle x-y, y\rangle + \|x-y\|^2$$

to $x = a$ and $y = x_t := ta + (1-t)b$, and then to $x = b$ and $y = x_t$. Multiplying the first identity thus obtained by t and the second by $(1-t)$ and summing them one gets (after some easy calculus) the identity

$$\begin{cases} \|x_t\|^2 = t\|a\|^2 + (1-t)\|b\|^2 - t(1-t)\|a-b\|^2, \text{ or,} \\ \quad t(1-t)\|a-b\|^2 = t\|a\|^2 + (1-t)\|b\|^2 - \|x_t\|^2. \end{cases} \tag{35}$$

But for 2-*paraconvex multimaps* one has for some $\rho > 0$ and all $p \in Y^*$

$$s_F(x_t,p) \leq t s_F(a,p) + (1-t) s_F(b,p) + \|p\| \rho t(1-t)\|a-b\|^2.$$

Hence, by (35) we get

$$s_F(x_t,p) \leq t s_F(a,p) + (1-t) s_F(b,p) - \|p\| \rho(-\|x_t\|^2 + t\|a\|^2 + (1-t)\|b\|^2).$$

In this manner,

$$s_F(x_t,p) - \rho\|p\|\|x_t\|^2 \leq t(s_F(a,p) - \rho\|p\|\|a\|^2) + (1-t)(s_F(b,p) - \rho\|p\|\|b\|^2),$$

and then the function $h : D \times Y^* \to \mathbb{R}$:

$$h(x,p) := s_F(x,p) - \rho\|p\|\|x\|^2$$

is such that $h(.,p)$ is convex and $h(x,.)$ is concave $u.s.c.$ Hence (by Lemma 1) h is the support function of a multimap H convex, closed with bounded values. Furthermore $domH = D$. The representation (34) then follows. ■

COROLLARY 3 *Assume that X is a Hilbert space, Y is complete and F is proper with closed, convex and bounded values and 2-paraconvex on an open and convex subset $D \subset int(dom(F))$. Then, F is locally Lipschitz on D.*

PROOF Observe that the assuptions of local boundedness (in Theorem 1) or sequential conpactness (in Theorem 2) are useless. Indeed by Theorem 7 one has

$$s_F(x,p) = s_H(x,p) + \rho\|p\|\|x\|^2$$

and H is from D to Y proper, convex and bounded valued, then locally Lipschitz (see Remark 3). Thus, $\{s_H(.,p) : p \in S_{Y^*}\}$ is locally equi-Lipschitz on D and so is $\{s_F(.,p) : p \in S_{Y^*}\}$. ■

Theorem 8 *Let X be a Hilbert space and F be bounded valued and 2-paraconvex on a convex subset $D \subset dom(F)$. Then, there exist a convex neighbourhood $\vartheta \subset D$, a compact topological space K and a continuous function*

$$\vartheta \times Y^* \times K \ni (x, p, \kappa) \to g(x, p, \kappa) \in \mathbb{R}$$

such that $g(\ .,p,\kappa)$ is C^2-Frechet differentiable on ϑ, the derivatives g'_x and g''_x are continuous on $\vartheta \times Y^* \times K$ and

$$s_F(x,p) = \max_{\kappa \in K} g(x,p,\kappa). \tag{36}$$

Thereby, for all $p \in Y^*$, $s_F(.,p)$ is lower-C^2.

In addition, there exists a multimap $G : \vartheta \times K \rightrightarrows Y$ closed with convex and bounded values, satisfying the representation

$$F(x) = \bigcap_{\kappa \in K} G(x,\kappa) \text{ for all } x \in \vartheta. \tag{37}$$

Proof Under the assumptions, for every $p \in Y^*$, $s_F(.,p)$ is $\alpha()$-paraconvex with

$$\alpha(\lambda) := \rho \|p\| \lambda^2 \quad \text{for some } \rho > 0.$$

In view of Penot (1996, Theorem 3.2) there exists a convex open subset $\vartheta \subset dom(F)$, a compact topological space K and a function

$$\vartheta \times Y^* \times K \ni (x, p, \kappa) \to g(x, p, \kappa) \in \mathbb{R}$$

such that $g(\ .,p,\kappa)$ is C^2-Frechet differentiable on ϑ, the derivatives g'_x and g''_x are continuous on $\vartheta \times Y^* \times K$ and

$$s_F(x,p) = \max_{\kappa \in K} g(x,p,\kappa). \tag{38}$$

Hence, $s_F(.,p)$ is lower-C^2..

Because $s_F(x,.)$ is concave $u.s.c$ and positively homogeneous, then by (38) the concave and positively homogeneous hull of the function $p \to g(x,p,\kappa)$, denoted $cc_p(g(x,.,s))$, exists and is well defined. Having

$$g(x,p,\kappa) \le cc_p(g(x,p,\kappa)) \le s_F(x,p),$$

then

$$s_F(x,p) = \max_{\kappa \in K} cc_p(g(x,p,\kappa)). \tag{39}$$

But by Lemma 2 the function $p \to cc_p(g(x,.,\kappa))$ is the support function of a multimap $G(.,\kappa)\colon X \rightrightarrows Y$ with convex, closed and bounded values. Hence,

$$cc_p(g(x,p,\kappa)) = \ s_{G(.,\kappa)}\,(x,p) \ \text{ for all } (x,p) \in \vartheta \times Y^*$$

and then

$$s_F\,(x,p) \geq s_{G(.,\kappa)}\,(x,p) \ \text{ for all } \kappa \in K.$$

In this way,

$$F(x) \subset \bigcap_{\kappa \in K} G(x,\kappa) \text{ for every } x \in \vartheta.$$

We claim that equality holds. Otherwise, there will exist $x \in \vartheta$ and $y \in \bigcap_{\kappa \in K} G(x,\kappa)$ such that $y \notin F(x)$. Then, by separation arguments

$$\text{for some } \epsilon > 0 \text{ and } \overline{p} \in Y^* \text{ one has } \langle \overline{p}, y \rangle < s_F\,(x,\overline{p}) - \epsilon.$$

Because $\langle \overline{p}, y \rangle \geq s_{G(x,\kappa)}\,(\overline{p})$ for all $\kappa \in K$, we get

$$s_F\,(x,\overline{p}) = \max_{\kappa \in K} [\overline{cc}_p(g(x,\overline{p},\kappa))] = \max_{\kappa \in K} s_{G(x,\kappa)}\,(\overline{p}) \leq \langle \overline{p}, y \rangle < s_F\,(x,\overline{p}) - \epsilon.$$

The contradiction ends the proof. ■

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