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On second-order derivatives of the efficient point multifunction in parametric vector optimization problems*

by

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Abstract: In this paper, we establish formulae for inner and outer evaluation of the second-order contingent derivative of index γ of the efficient point multifunction in parametric vector optimization problems. The results contained in this paper extend the results of Chuong (2013a) to the second-order sensitivity analysis case. On the other hand, examples are provided for purposes of analyzing and illustrating the obtained results. Concerning the potential domain of application, the functioning of the majority of economic systems depends on a set of indicators (criteria), i.e., the substance of economic systems includes multiple criteria and only the lack of mathematical methods in solving the problems of vector optimization is an obstacle to the effective use of the respective models. Therefore, the study of vector optimization problems is necessary and has practical significance.

Keywords: parametric vector optimization, efficient point multifunction, second-order contingent derivative of index γ, sensitivity analysis

1. Introduction

In parametric vector optimization problems, sensitivity analysis is the analysis of behavior of the efficient point multifunction. There are two main approaches in sensitivity analysis: the dual space approach and the primal space approach.

In the dual space approach, many important results in sensitivity analysis for parametric vector optimization problems via the coderivatives were given in

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Chuong and Yao (2009, 2013a) and the books by B.S. Mordukhovich (2006a,b; 2018).

Concerning the primal space approach, the first-order derivatives of the perturbation maps/the efficient solution maps have been studied in Chuong and Yao (2010, 2013b), Chuong (2013a), Kuk, Tanino and Tanaka (1996), Shi (1991, 1993), Tanino (1988a,b), and Tung and Pham (2020a,b). Some results in second-order sensitivity analysis for vector optimization problem have been considered in Li, Sun and Zhai (2012), Sun and Li (2014), Tung (2017), and Wang and Li (2011, 2012). Recently, new results in higher-order sensitivity analysis in parametric vector optimization problems/parametric set-valued optimization problems have been obtained in Anh and Khanh (2013), Anh (2017a,b), Diem, Khanh and Tung (2014), Sun and Li (2011), Thung (2017b), as well as Wang, Li and Teo (2010).

Another important topic in the primal space approach is the study of the protodifferentiability of perturbation maps. The important results on the first-order proto-differentiability/semi-differentiability of the perturbation maps/the efficient solution maps have been obtained in Huy and Lee (2007, 2008), Lee and Huy (2006), Levy and Rockafellar (1994), Luc, Soleimani-damaneh and Zamani (2018), and, first of all, Rockafellar (1989). Some results on the second-order proto-differentiability/second-order semi-differentiability of the perturbation maps/the efficient solution maps have been provided in Li and Liao (2012), Pham (2022), Pham and Nguyen (2022), and Tung (2018, 2021b). The higher-order proto-differen-tiability/higher-order semi-differentiability properties of the perturbation maps/ the proper perturbation maps/the weak perturbation maps have been investigated in Pham (2023) and Tung (2020, 2021a).

On the other hand, in the framework of the primal space approach, Chuong investigated first-order sensitivity analysis in parametric vector optimization problems via first-order S-derivative, see Chuong (2023a). In the present paper, we provide some new results for second-order sensitivity analysis in parametric vector optimization problems in terms of second-order contingent derivative of index γ.

The plan of the present paper is as follows. In Section 2, we recall several concepts of the derivatives of multifunctions and their properties, which are needed in the sequel. In Section 3, we establish formulae for inner and outer estimation of the second-order contingent derivative of index γ of the efficient point multifunction. An application to parametric vector optimization problem with finite constraints is given in Section 4. Finally, conclusions are given in Section 5.

2. Preliminaries

Throughout this paper, let P, X and Y be Euclidean spaces $\mathbb{R}^n$, equipped with the usual norms, where the space Y is partially ordered by closed convex pointed cone $K \subseteq Y$ with nonempty interior $\text{int}K$ and apex at the origin. The norms of all Euclidean spaces are denoted by $||\cdot||$. Index $\gamma \in \{0,1\}$. The origins of all Euclidean spaces are denoted by 0. B_X, B_Y stands for the closed unit balls in, respectively, X, Y. Closure and boundary of $A \subseteq X$ are denoted by $\text{cl}A$ and ∂A, respectively. Furthermore, $\text{cone}A = \{ka | k \geq 0, a \in A\}$. $\mathbb{N}, \mathbb{R}, \mathbb{R}_-$ and $\mathbb{R}_+$ are used for the sets of natural numbers, real numbers, negative real numbers, and nonnegative real numbers, respectively.

In this paper, we consider the second-order sensitivity analysis of parameterized vector optimization problems. Firstly, some notations and definitions are recollected. Let $f : P \times X \to Y$ be a vector function and $C : P \rightrightarrows X$ be a multifunction. We consider the following parameterized vector optimization problem

$$\min{}_K f(x,p) \quad \text{subject to} \quad x \in C(p),$$

where C is the constraint map and $\min_K$ indicates the minimum with respect to the ordering, induced by K. The cone K induces a partial order $\preceq_K$ on Y, i.e.,

$$y \preceq_K y' \qquad \Leftrightarrow \qquad y' - y \in K, \qquad y, y' \in Y.$$

Let $F : P \rightrightarrows Y$ be a multifunction defined by

$$F(p) := f(p, C(p)) = \{y \in Y \mid \exists x \in C(p), y = f(p,x)\}. \tag{1}$$

Definition 1 *(See Chuong, 2013a) We say that $y \in Y$ is an efficient point of a subset $A \subset Y$ with respect to K if and only if $(y - K) \cap A = \{y\}$. The set of efficient points of A is denoted by $\text{Eff}_K A$. We stipulate that $\text{Eff}_K \emptyset = \emptyset$. When K has a nonempty interior, read $\text{int}K \neq \emptyset$, an element $y \in A$ is called a weakly efficient point of A with respect to K, denoted by $\text{Eff}^w_K A$, if and only if $(y - \text{int}K) \cap A = \emptyset$. We stipulate that $\text{Eff}^w_K \emptyset = \emptyset$. From now on, when speaking of weakly efficient points, we always assume that $\text{int}K \neq \emptyset$.*

We consider the following parametric vector optimization problem:

$$\text{Eff}_K\{y \in Y \mid \exists x \in C(p), y = f(p,x)\} = \text{Eff}_K F(p), \tag{2}$$

where x is the decision variable, p is the perturbation parameter, f is the objective map, C is the constraint map and F is the feasible set map in the objective space.

The multifunction $\mathcal{F} : P \rightrightarrows Y$ assigns to p the set of efficient points of (2), i.e.,

$$\mathcal{F}(p) := \text{Eff}_K\{y \in Y \mid \exists x \in C(p), y = f(p,x)\} = \text{Eff}_K F(p), \tag{3}$$

is called the efficient point multifunction of (2).

DEFINITION 2 *(See Aubin and Frankowska, 1990; Bonnans and Shapiro, 2000) Let $f : X \to Y$ be a vector-valued map. f is said to be twice Fréchet differentiable at $\bar{x} \in X$, if there exist two linear continuous operators $\nabla f(\bar{x}) : X \to Y$ and $\nabla^2 f(\bar{x}) : X \times X \to Y$ such that*

$$f(x) = f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) + \frac{1}{2}\nabla^2 f(\bar{x})(x - \bar{x}, x - \bar{x}) + o\left(||x - \bar{x}||^2\right),$$

where $o(||x - \bar{x}||^2)$ satisfies $\dfrac{o(||x - \bar{x}||^2)}{||x - \bar{x}||^2} \to 0$ when $x \to \bar{x}$. $\nabla f(\bar{x})$ and $\nabla^2 f(\bar{x})$ are the Fréchet derivative and the second-order Fréchet derivative, respectively. f is said to be twice Fréchet differentiable on X if f is twice Fréchet differentiable at any $x \in X$. If $\nabla f(\bar{x})$ and $\nabla^2 f(\bar{x})$ are continuous at $\bar{x}$ then f is said to be twice continuously Fréchet differentiable at $\bar{x}$.

Let $H : P \rightrightarrows Y$ be a multifunction. The effective domain, graph, and epigraph of H are defined by

$$\begin{aligned} \mathrm{dom}H &:= \{p \in P \mid H(p) \neq \emptyset\}, \\ \mathrm{gph}H &:= \{(p, y) \in P \times Y \mid y \in H(p)\}, \\ \mathrm{epi}H &:= \{(p, y) \in P \times Y \mid p \in \mathrm{dom}H, y \in H(p) + K\}. \end{aligned}$$

DEFINITION 3 *(See Aubin and Frankowska, 1990) Let $M \subseteq Y, \bar{y}, \bar{v} \in Y$ and index $\gamma \in \{0, 1\}$. The second-order contingent set of index γ of M at $(\bar{y}, \bar{v})$ is*

$$\begin{aligned} T^2_\gamma(M, \bar{y}, \bar{v}) := \quad & \{y \in Y \mid \exists t_n \to 0^+, \exists r_n \to 0^+, \frac{t_n}{r_n} \to \gamma, \exists y_n \to y, \forall n \in \mathbb{N}, \\ & \text{such that } \bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n \in M\}. \end{aligned}$$

DEFINITION 4 *(See Aubin and Frankowska, 1990) Let $H : P \rightrightarrows Y$ be a set-valued map, $(\bar{p}, \bar{y}) \in \mathrm{gph}H$ and $(\bar{u}, \bar{v}) \in P \times Y$ and index $\gamma \in \{0, 1\}$. The second-order contingent derivative of index γ of H at $(\bar{p}, \bar{y})$ in the direction $(\bar{u}, \bar{v}) \in P \times Y$ is the set-valued map $D^2_\gamma H(\bar{p}, \bar{y}, \bar{u}, \bar{v}) : P \rightrightarrows Y$, defined by*

$$\begin{aligned} & D^2_\gamma H(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) := \{y \in Y \mid \exists t_n \to 0^+, \exists r_n \to 0^+, \frac{t_n}{r_n} \to \gamma, \exists (p_n, y_n) \to (p, y), \\ & \forall n \in \mathbb{N}, \text{ such that } \bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n \in H(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n)\}, \forall p \in P. \end{aligned}$$

DEFINITION 5 *(See Pham, 2023a) Let $H : P \rightrightarrows Y$ be the set-valued map, $(\bar{p}, \bar{y}) \in \mathrm{gph}H$ and $(\bar{u}, \bar{v}) \in P \times Y, \gamma \in \{0, 1\}$. H is said to be second-order directionally compact with index γ at $(\bar{p}, \bar{y})$ with respect to $(\bar{u}, \bar{v})$ in the direction $p \in P$ if for all sequences $t_n \to 0^+, r_n \to 0^+, \dfrac{t_n}{r_n} \to \gamma$ and $p_n \to p$, and every* *sequence* $\{y_n\}$ *with* $\bar{y} + t_n\bar{v} + \frac{1}{2}t_nr_ny_n \in H(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n)$*, there exists a convergent subsequence of* $\{y_n\}$*.*

Definition 6 *(See Chuong, 2013a)*

(i) The set $\Omega \subset Y$ *is said to satisfy the domination property if*

$$\Omega \subset \mathrm{Eff}_K\Omega + K.$$

(ii) We say that the domination property holds for $H : P \rightrightarrows Y$ *around* $\bar{p} \in P$ *if there exists a neighborhood* U *of* $\bar{p}$ *such that*

$$H(p) \subset \mathrm{Eff}_K H(p) + K, \forall p \in U.$$

Proposition 1 *Let* $H : P \rightrightarrows Y$ *be the set-valued map,* $(\bar{p}, \bar{y}) \in \mathrm{gph}H$ *and* $(\bar{u}, \bar{v}) \in P \times Y, \gamma \in \{0, 1\}$*. Suppose that* H *is said to be second-order directionally compact with index* γ *at* $(\bar{p}, \bar{y})$ *with respect to* $(\bar{u}, \bar{v})$ *in the direction* $p \in P$*. Then, one has*

$$D^2_\gamma(H + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = D^2_\gamma H(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) + K, \forall p \in P.$$

Proof Firstly, we prove that

$$D^2_\gamma(H + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \subseteq D^2_\gamma H(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) + K, \forall p \in P.$$

Let $y \in D^2_\gamma(H + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$. Then, there exist

$$t_n \to 0^+, r_n \to 0^+, \frac{t_n}{r_n} \to \gamma, (p_n, y_n) \to (p, y), k_n \in K$$

for all $n \in \mathbb{N}$ such that

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_nr_n(y_n - k_n) \in H(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n).$$

Denote $\bar{y}_n := y_n - k_n$. Because H is second-order directionally compact with index γ at $(\bar{p}, \bar{y})$ with respect to $(\bar{u}, \bar{v})$ in the direction $p \in P$, we can assume that $\bar{y}_n \to y' \in Y$. Then, we have $y' \in D^2_\gamma H(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$. Together with $k_n = y_n - \bar{y}_n \to y - y'$ and with K being closed, we have $k_n \to y - y' = k \in K$ and $y' = y - k$, which implies that $y - k = y' \in D^2_\gamma H(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$. Therefore, $y \in D^2_\gamma H(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) + K, \forall p \in P$.

Secondly, we prove that

$$D^2_\gamma H(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) + K \subseteq D^2_\gamma(H + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p), \forall p \in P.$$

Let $y \in D^2_\gamma H(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) + K$. Then, there exist $\hat{y} \in D^2_\gamma H(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$ and $k \in K$ such that $y = \hat{y} + k$. Thus, there exist

$$t_n \to 0^+, r_n \to 0^+, \frac{t_n}{r_n} \to \gamma, (p_n, y_n) \to (p, \hat{y})$$

such that

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n \in H(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n), \forall n \in \mathbb{N}.$$

Upon setting $y'_n := y_n + k$, one has $y'_n \to \hat{y} + k$ and

$$\bar{y}+t_n\bar{v}+\frac{1}{2}t_n r_n y'_n = \bar{y}+t_n\bar{v}+\frac{1}{2}t_n r_n y_n + t_n r_n k \in H(\bar{p}+t_n\bar{u}+\frac{1}{2}t_n r_n p_n)+K, \forall n \in \mathbb{N}.$$

Therefore, $y = \hat{y} + k \in D^2_\gamma(H + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$. ■

In Proposition 1, if H is not second-order directionally compact with index γ at $(\bar{p}, \bar{y})$ with respect to $(\bar{u}, \bar{v})$ in the direction $p \in P$, then Proposition 1 may not hold. The following example shows the case.

EXAMPLE 1 *Let $P = \mathbb{R}^2, Y = \mathbb{R}, K = \mathbb{R}_+, \gamma \in \{0, 1\}$ and $H : P \rightrightarrows Y$ be defined by*

$$H(p) = \begin{cases} \{p_1^2 + p_1, -1\}, & \text{if } p_1 = p_2 \geq 0, \\ \{-2\}, & \text{otherwise,} \end{cases}$$

where $p = (p_1, p_2) \in \mathbb{R}^2$. Let $(\bar{p}, \bar{y}) = ((0, 0), 0) \in \text{gph}H$ and $(\bar{u}, \bar{v}) = ((1, 0), 1)$. We have, for all $p = (p_1, p_2) \in P$,

$$D^2_\gamma H(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = \begin{cases} \{y \in \mathbb{R} \mid y = 2\gamma + p_1\}, & \text{if } p_1 = p_2 \geq 0, \\ \emptyset, & \text{otherwise,} \end{cases}$$

and

$$(H + K)(p) = \begin{cases} \{y \in \mathbb{R} \mid y \geq -1\}, & \text{if } p_1 = p_2 \geq 0, \\ \{y \in \mathbb{R} \mid y \geq -2\}, & \text{otherwise.} \end{cases}$$

Thus, for all $p = (p_1, p_2) \in P$,

$$D^2_\gamma(H + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = \mathbb{R}.$$

Hence, for all $p = (p_1, p_2) \in P$,

$$D^2_\gamma(H + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \neq D^2_\gamma H(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) + K.$$

The reason is that the condition of being second-order directionally compact with index γ for H does not hold. Indeed, for the direction $p = (1, 1)$, for every

$t_n \to 0^+, r_n \to 0^+, \dfrac{t_n}{r_n} \to \gamma$ *for* $p_n = (p_{1n}, p_{2n}) \to (p_1, p_2) \to p = (1, 1)$. *Suppose that there exists a sequence* $\{y_n\}$ *with*

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n = -1 \in H(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n).$$

This implies that

$$y_n = -\frac{2}{t_n r_n} - \frac{2}{r_n},$$

which has no convergent subsequence.

3. Second-order contingent derivative of index γ of the efficient point multifunction

Firstly, we obtain inner and outer estimates of the second-order contingent derivative of index γ of the efficient point multifunction $\mathcal{F}$ defined in (3) at the reference point via the set of efficient/weakly efficient points of the second-order contingent derivative of index γ of F in (1) at the corresponding point.

Theorem 1 *Let* $(\bar{p}, \bar{y}) \in \mathrm{gph}\mathcal{F}$. *Suppose that the domination property holds for* F *defined in (1) around* $\bar{x}$. *Assume that* F *is said to be second-order directionally compact with index* γ *at* $(\bar{p}, \bar{y})$ *with respect to* $(\bar{u}, \bar{v})$ *in the direction* $p \in P$. *Then, one has*

$$D^2_\gamma\mathcal{F}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \supset \mathrm{Eff}_K D^2_\gamma F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p), \forall p \in P. \tag{4}$$

Proof Since $\mathcal{F}(p) \subset F(p)$ for all $p \in P$ and the domination property holds for F around $\bar{b}$, there exists a neighborhood U of $\bar{p}$ such that

$$\mathcal{F}(u) + K = F(u) + K, \forall u \in U.$$

Thus, one has

$$D^2_\gamma(\mathcal{F} + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = D^2_\gamma(F + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p), \forall p \in P. \tag{5}$$

Since F is second-order directionally compact with index γ at $(\bar{p}, \bar{y})$ with respect to $(\bar{u}, \bar{v})$ in the direction $p \in P$, so, one deduces that $\mathcal{F}$ is second-order directionally compact with index γ at $(\bar{p}, \bar{y})$ with respect to $(\bar{u}, \bar{v})$ in the direction $p \in P$. This implies, by Proposition 1, that

$$D^2_\gamma(\mathcal{F} + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = D^2_\gamma(\mathcal{F} + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p), \forall p \in P. \tag{6}$$

On the other hand, using Proposition 1 again, one has

$$D^2_\gamma(\mathcal{F} + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = D^2_\gamma(F + K)(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p), \forall p \in P. \tag{7}$$

By combining now (5) with (6) and (7), we obtain

$$D^2_\gamma(\mathcal{F}+K)(\bar p,\bar y,\bar u,\bar v)(p) = D^2_\gamma(F+K)(\bar p,\bar y,\bar u,\bar v)(p), \forall p\in P.$$

Hence,

$$\begin{aligned} D^2_\gamma\mathcal{F}(\bar p,\bar y,\bar u,\bar v)(p) &\supset \mathrm{Eff}_K D^2_\gamma\mathcal{F}(\bar p,\bar y,\bar u,\bar v)(p)\\ &= \mathrm{Eff}_K\left(D^2_\gamma\mathcal{F}(\bar p,\bar y,\bar u,\bar v)(p)+K\right)\\ &= \mathrm{Eff}_K\left(D^2_\gamma F(\bar p,\bar y,\bar u,\bar v)(p)+K\right)\\ &= \mathrm{Eff}_K D^2_\gamma F(\bar p,\bar y,\bar u,\bar v)(p), \forall p\in P.\end{aligned}$$

■

The following example demonstrates the importance of the domination property of F in Theorem 1, namely the inclusion in (4) may fail to hold if the assumption on the existence of the domination property of F around the point under consideration is omitted.

EXAMPLE 2 *Let $P=\mathbb{R}, Y=\mathbb{R}^2, K=\mathbb{R}^2_+, \gamma\in\{0,1\}$ and let $F: P\rightrightarrows Y$ be given as follows:*

$$\begin{aligned} F(p) &= \left\{(y_1,y_2)\in\mathbb{R}^2 \mid \frac{1}{2}p^2\le y_1\le p^2, -y_1+p^2\le y_2\le p^2\right\}\\ &\cup\left\{(y_1,y_2)\in\mathbb{R}^2 \mid 0<y_1\le \frac{1}{2}p^2, y_2=\frac{1}{2}p^2\right\}.\end{aligned}$$

For any $p\in P$,

$$\mathcal{F}(p)=\begin{cases}\left\{(y_1,y_2)\in\mathbb{R}^2 \mid \frac{1}{2}p^2<y_1\le p^2, y_2=-y_1+p^2\right\} & \text{if } p\neq 0,\\ \{(0,0)\}, & \text{if } p=0.\end{cases}$$

Let $(\bar p,\bar y)=(0,(0,0))\in \mathrm{gph}\mathcal{F}$ and $(\bar u,\bar v)=(1,(0,0))$. By a simple computation, for all $p\in P$,

$$\begin{aligned} D^2_\gamma F(\bar p,\bar y,\bar u,\bar v)(p) &= \left\{(y_1,y_2)\in\mathbb{R}^2 \mid \gamma\le y_1\le 2\gamma, -y_1+2\gamma\le y_2\le 2\gamma\right\}\\ &\cup\left\{(y_1,y_2)\in\mathbb{R}^2 \mid 0\le y_1\le\gamma, y_2=\gamma\right\}\end{aligned}$$

and

$$D^2_\gamma\mathcal{F}(\bar p,\bar y,\bar u,\bar v)(p)=\left\{(y_1,y_2)\in\mathbb{R}^2 \mid \gamma\le y_1\le 2\gamma, y_2=-y_1+2\gamma\right\}.$$

Thus, one has

$$\mathrm{Eff}_K D^2_\gamma F(\bar p,\bar y,\bar u,\bar v)(p)\not\subset D^2_\gamma\mathcal{F}(\bar p,\bar y,\bar u,\bar v)(p), \forall p\in P.$$

The reason is that the domination property does not hold for F around $\bar p$.

The next example proves that if F is not second-order directionally compact with index γ at $(\bar{p},\bar{y})$ with respect to $(\bar{u},\bar{v})$ in the direction $p \in P$, then Theorem 1 may not hold.

EXAMPLE 3 *Let* $P = Y = \mathbb{R}, K = \mathbb{R}_+, \gamma \in \{0,1\}$ *and* $F : P \rightrightarrows Y$ *be defined by*

$$F(p) = \begin{cases} \{0\}, & \text{if } p \leq 0, \\ \{p, -\sqrt{p}\}, & \text{otherwise.} \end{cases}$$

For any $p \in \mathbb{R}$,

$$\mathcal{F}(p) = \begin{cases} \{0\}, & \text{if } p \leq 0, \\ \{-\sqrt{p}\}, & \text{otherwise.} \end{cases}$$

Let $(\bar{p},\bar{y}) = (0,0) \in \text{gph}\mathcal{F}$ *and* $(\bar{u},\bar{v}) = (1,1)$. *We have, for all* $p \in P$,

$$D^2_\gamma F(\bar{p},\bar{y},\bar{u},\bar{v})(p) = \begin{cases} \mathbb{R}, & \text{if } p \leq 0, \\ \{p\}, & \text{if } p > 0, \end{cases}$$

and

$$D^2_\gamma \mathcal{F}(\bar{p},\bar{y},\bar{u},\bar{v})(p) = \begin{cases} \mathbb{R}, & \text{if } p \leq 0, \\ \emptyset, & \text{if } p > 0. \end{cases}$$

It is easy to see that the domination property holds for F *around* $\bar{p}$. *Meanwhile,*

$$\text{Eff}_K D^2_\gamma F(\bar{p},\bar{y},\bar{u},\bar{v})(p) \not\subset D^2_\gamma \mathcal{F}(\bar{p},\bar{y},\bar{u},\bar{v})(p), \forall p > 0.$$

The reason is that the condition of being second-order directionally compact with index γ *for* F *does not hold. Indeed, for the direction* $p = 1$, *for every* $t_n \to 0^+, r_n \to 0^+, \dfrac{t_n}{r_n} \to \gamma$ *for* $p_n \to p = 1$. *Suppose that there exists a sequence* $\{y_n\}$ *with*

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n = -\sqrt{p} = -1 \in F(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n).$$

This implies that

$$y_n = -\frac{2}{t_n r_n} - \frac{2}{r_n},$$

which has no convergent subsequence.

THEOREM 2 *Let* $(\bar{p},\bar{y}) \in \text{gph}\mathcal{F}$ *and* $(\bar{u},\bar{v}) \in P \times Y$. *Suppose that for each* $(p,y) \in T^2_\gamma(\text{gph}\mathcal{F},(\bar{p},\bar{y}),(\bar{u},\bar{v}))$ *such that*

$$D^2_\gamma F(\bar{p},\bar{y},\bar{u},\bar{v})(p) \cup (y - \text{int}K) \subset \{v \in Y \mid \forall t_n \to 0^+, \forall r_n \to 0^+, \frac{t_n}{r_n} \to \gamma,$$
$$\forall p_n \to p, \exists y_n \to v, \forall n \in \mathbb{N}, \bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n \in F(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n)\}. \quad (8)$$

We have

$$D^2_\gamma \mathcal{F}(\bar{p},\bar{y},\bar{u},\bar{v})(p) \subset \text{Eff}^w_K D^2_\gamma F(\bar{p},\bar{y},\bar{u},\bar{v})(p), \forall p \in P. \quad (9)$$

PROOF Take any $p \in P$ and let $y \in D^2_\gamma \mathcal{F}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$. Then, there exist $t_n \to 0^+, r_n \to 0^+, \dfrac{t_n}{r_n} \to \gamma$ and $(p_n, y_n) \to (p, y)$ such that

$$\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y_n \in \mathcal{F}(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n).$$

Assume, to the contrary, that $y \notin \mathrm{Eff}^w_K D^2_\gamma F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$. Then, there is $y \in D^2_\gamma F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$ such that

$$y - y' \in \mathrm{int} K.$$

Since $y \in D^2_\gamma F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$, one deduces that there exist

$$t_n \to 0^+, r_n \to 0^+, \frac{t_n}{r_n} \to \gamma \text{ and } (p_n, y_n) \to (p, y)$$

such that

$$\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y_n \in \mathcal{F}(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n) \subseteq F(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n).$$

It follows by (2) that for any $\forall t_n \to 0^+, \forall r_n \to 0^+, \dfrac{t_n}{r_n} \to \gamma$ and for all $p_n \to p$ there exists $y'_n \to y'$ such that

$$\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y'_n \in F(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n), \forall n \in \mathbb{N}.$$

It follows from

$$y'_n - y_n \to y' - y \in -\mathrm{int} K$$

and $-\mathrm{int}K$ being an open cone that

$$\frac{\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y'_n - \left(\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y_n\right)}{\frac{1}{2} t_n r_n} = y'_n - y_n \to y' - y \in -\mathrm{int} K,$$

for all $n \in \mathbb{N}$ sufficiently large. Thus, it results that

$$\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y'_n - \left(\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y_n\right) \in -\mathrm{int} K,$$

for all $n \in \mathbb{N}$ sufficiently large. Consequently,

$$\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y_n \notin \mathcal{F}\left(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n\right),$$

for all $n \in \mathbb{N}$ sufficiently large, which is impossible. Therefore,

$$y \in \text{Eff}_K^w D_\gamma^2 F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p), \forall p \in P.$$

The proof is complete. ■

Note that relation (2) in Theorem 2 is essential for having (9). To see this, let us recall Example 3. Let $(\bar{p}, \bar{y}) = (0, 0) \in \text{gph}\mathcal{F}$ and $(\bar{u}, \bar{v}) = (1, 1)$. Observe that relation (2) does not hold for $(p, y) = (0, 0) \in T_\gamma^2(\text{gph}\mathcal{F}, (\bar{p}, \bar{y}), (\bar{u}, \bar{v}))$. Indeed, choose

$$v = -1 \in (-\infty, 0) = D_\gamma^2 F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(0) \cup (0 - \text{int}K)$$

and

$$t_n = \frac{\gamma}{n}, r_n = \frac{1}{n}, \frac{t_n}{r_n} = \gamma, p_n = \frac{1}{n^2}, \forall n \in \mathbb{N}.$$

Then, $t_n \to 0^+, r_n \to 0^+, \dfrac{t_n}{r_n} = \gamma$ and $(p_n \to p = 0$. Since $F(p_n) = \left\{ \dfrac{1}{n^2}, -\dfrac{1}{n} \right\}$ for all $n \in \mathbb{N}$, it follows that for any sequence $\{y_n\}$ such that $y_n \to v$ there is

$$\bar{y} + t_n \bar{v} + \frac{1}{2} t_n r_n y_n \notin F(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n), \forall n \in \mathbb{N}.$$

Thus, (9) does not hold. Indeed, one has

$$\mathbb{R} = D_\gamma^2 \mathcal{F}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(0) \not\subset \text{Eff}_K^w D_\gamma^2 F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(0) = \emptyset.$$

The following auxiliary result gives a formula for computing the second-order contingent derivatives of index γ of F in (1) at a given point via the second-order contingent derivatives of index γ of the constraint mapping C and the second-order Fréchet derivative of the objective function f at the corresponding points.

Proposition 2 *Let $\bar{p} \in P, \bar{x} \in C(\bar{p}), \gamma \in \{0, 1\}$ and $\bar{y} = f(\bar{p}, \bar{x}), (\bar{u}, \bar{w}, \bar{v}) \in P \times X \times Y$. Suppose that the following conditions hold:*

(i) C is second-order directionally compact with index γ at $(\bar{p}, \bar{x})$ with respect to $(\bar{u}, \bar{w})$ in the direction $p \in P$;

(ii) f is twice continuously Fréchet differentiable at $(\bar{p}, \bar{x})$ and $\bar{v} = \nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w})$.

Then, for all $p \in P$,

$$\begin{aligned} D_\gamma^2 F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = \quad & \{y \in Y \mid x \in D_\gamma^2 C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p), y = \nabla f(\bar{p}, \bar{x})(p, x) \\ & + \gamma \nabla^2 f(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w}))\}. \end{aligned} \tag{10}$$

Proof Firstly, we will prove that

$$\begin{aligned} D_\gamma^2 F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \subset \{ & y \in Y \mid x \in D_\gamma^2 C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p), y = \nabla f(\bar{p}, \bar{x})(p, x) \\ & + \gamma \nabla^2 f(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w}))\}, \forall p \in P. \end{aligned} \tag{11}$$

Let $y \in D^2_\gamma F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$. Then, there exist $t_n \to 0^+, r_n \to 0^+, \dfrac{t_n}{r_n} \to \gamma$ and $(p_n, y_n) \to (p, y)$ such that

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n \in F(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n).$$

Then, there exists a sequence $\{x_n\} \subseteq X$ such that $x_n \in C(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n)$ and

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n = f(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n, x_n).$$

By setting $x'_n := \dfrac{x_n - \bar{x} - t_n\bar{w}}{\frac{1}{2}t_n r_n}$, we get

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_n r_n y_n = f\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n, \bar{x} + t_n\bar{w} + \frac{1}{2}t_n r_n x'_n\right) \tag{12}$$

and

$$x_n = \bar{x} + t_n\bar{w} + \frac{1}{2}t_n r_n x'_n \in C(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n).$$

By combining this with (i), we can suppose that $x'_n \to x'$. Then,

$$x' \in D^2_\gamma C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p).$$

Moreover, since f is twice continuously Fréchet differentiable at $(\bar{p}, \bar{x})$, $\bar{y} = f(\bar{p}, \bar{x}), \bar{v} = \nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w})$, and (12), we obtain

$$y_n \to y := \nabla f(\bar{p}, \bar{x})(p, x') + \gamma\nabla^2 f(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w})).$$

Therefore, it follows that (3) holds.

Now, we will prove that

$$\begin{aligned}&\{y \in Y \mid x \in D^2_\gamma C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p), y = \nabla f(\bar{p}, \bar{x})(p, x) + \gamma\nabla^2 f(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w}))\}\\ &\subset D^2_\gamma F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p), \forall p \in P.\end{aligned} \tag{13}$$

Take any $p \in P$ and let $x \in D^2_\gamma C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p)$. Upon putting

$$y := \nabla f(\bar{p}, \bar{x})(p, x) + \gamma\nabla^2 f(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w})),$$

we have to show that $y \in D^2_\gamma F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$. Since $x \in D^2_\gamma C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p)$, there exist

$$t_n \to 0^+, r_n \to 0^+, \frac{t_n}{r_n} \to \gamma \text{ and } (p_n, x_n) \to (p, x)$$

such that

$$\bar{x} + t_n\bar{w} + \frac{1}{2}t_nr_nx_n \in C(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n),$$

which implies that,

$$f(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n, \bar{x} + t_n\bar{w} + \frac{1}{2}t_nr_nx_n) \in F(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n). \quad (14)$$

We set

$$y_n := \frac{f\left(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n, \bar{x} + t_n\bar{w} + \frac{1}{2}t_nr_nx_n\right) - f(\bar{p}, \bar{x}) - t_n\bar{v}}{\frac{1}{2}t_nr_n}. \quad (15)$$

Then, by (14), one has,

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_nr_ny_n \in F(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n).$$

Since f is twice continuously Fréchet differentiable at $(\bar{p}, \bar{x})$, $\bar{y} = f(\bar{p}, \bar{x}), \bar{v} = \nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w})$, and (15), one has

$$y_n \to y := \nabla f(\bar{p}, \bar{x})(p, x) + \gamma\nabla^2 f(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w})).$$

Thus, there exist $t_n \to 0^+, r_n \to 0^+, \dfrac{t_n}{r_n} \to \gamma$ and $(p_n, y_n) \to (p, y)$ such that

$$\bar{y} + t_n\bar{v} + \frac{1}{2}t_nr_ny_n \in F(\bar{p} + t_n\bar{u} + \frac{1}{2}t_nr_np_n).$$

Consequently, $y \in D^2_\gamma F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p)$. It follows that (13) holds. The proof is complete. ■

Our first main result in this section provides an inner estimate for evaluating the second-order contingent derivatives of index γ of the efficient point multifunction $\mathcal{F}$ via the second-order contingent derivatives of index γ of the constraint mapping C and the second-order Fréchet derivative of the objective function f.

THEOREM 3 *Let $\bar{p} \in P, \bar{x} \in C(\bar{p}), \gamma \in \{0, 1\}$ and $\bar{y} = f(\bar{p}, \bar{x}), (\bar{u}, \bar{w}, \bar{v}) \in P \times X \times Y$. Suppose that the following conditions hold:*

(i) C is second-order directionally compact with index γ at $(\bar{p}, \bar{x})$ with respect to $(\bar{u}, \bar{w})$ in the direction $p \in P$;

(ii) f is twice continuously Fréchet differentiable at $(\bar{p}, \bar{x})$ and $\bar{v} = \nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w})$;

(iii) The domination property holds for F defined in (1) around $\bar{p}$ and (10) holds true.

Then, for all $p \in P$,

$$D^2_\gamma \mathcal{F}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \supset \mathrm{Eff}_K\{y \in Y \mid x \in D^2_\gamma C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p), y = \nabla f(\bar{p}, \bar{x})(p, x) + \gamma \nabla^2 f(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w}))\}.$$

PROOF By applying Theorem 1 and (10), we obtain the desired result. ■

Next, we provide an outer estimate for evaluating the second-order contingent derivative of index γ of the efficient point multifunction $\mathcal{F}$ via the second-order contingent derivative of index γ of the constraint mapping C and the second-order Fréchet derivative of the objective function f.

THEOREM 4 *Let* $\bar{p} \in P, \bar{x} \in C(\bar{p}), \gamma \in \{0, 1\}$ *and* $\bar{y} = f(\bar{p}, \bar{x}), (\bar{u}, \bar{w}, \bar{v}) \in P \times X \times Y$. *Suppose that* f *is twice continuously Fréchet differentiable at* $(\bar{p}, \bar{x})$ *and* $\bar{v} = \nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w})$. *Assume further that (2) and (10) hold true. Then, for all* $p \in P$,

$$D^2_\gamma \mathcal{F}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \subset \mathrm{Eff}^w_K\{y \in Y \mid x \in D^2_\gamma C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p), y = \nabla f(\bar{p}, \bar{x})(p, x) + \gamma \nabla^2 f(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w}))\}.$$

PROOF The proof follows from Theorem 2 and (10). ■

Now, we present an example to explain the results given in Theorem 3 and Theorem 4.

EXAMPLE 4 *Let* $P = \mathbb{R}, X = Y = \mathbb{R}^2, K = \mathbb{R}^2_+, \gamma \in \{0, 1\}$ *and* $f : P \times X \to Y, C : P \rightrightarrows X$ *be defined by:*

$$f(p, x) = (2p + x_1, x_2), \forall p \in \mathbb{R},\ x = (x_1, x_2) \in \mathbb{R}^2,$$

$$C(p) = \{(x_1, x_2) \in \mathbb{R}^2 \mid -p + x_1 - 2x_2 \le 0,\ 2p - 2x_1 + x_2 \le 0\}.$$

Taking $(\bar{p}, \bar{x}) = (0, (0, 0)), (\bar{u}, \bar{w}) = (0, (0, 0))$ *and* $(\bar{y}, \bar{v}) = ((0, 0), (0, 0))$, *we obtain*

$$D^2_\gamma C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p) = \{(x_1, x_2) \in \mathbb{R}^2 \mid -p + x_1 - 2x_2 \le 0, 2p - 2x_1 + x_2 \le 0\}.$$

We have $\bar{y} = f(\bar{p}, \bar{x}) = (0, 0)$,

$$\nabla f(p, x) = (\nabla_p f(p, x), \nabla_x f(p, x)) = \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \nabla^2 f(p, x) = 0,$$

$$\nabla f(\bar{p}, \bar{x}) = \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \nabla^2 f(\bar{p}, \bar{x}) = 0,$$

and $\bar{v} = \nabla f(\bar{p}, \bar{x})(\bar{u}, \bar{w}) = \left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)(0, (0, 0)) = (0, 0)$. *By direct calculation, for any* $p \in P$,

$$F(p) = \{(y_1, y_2) \in \mathbb{R}^2 \mid -3p + y_1 - 2y_2 \le 0, 6p - 2y_1 + y_2 \le 0\},$$

$$\mathcal{F}(p) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 3p, y_2 = 0\}.$$

It is easy to prove that the conditions of Theorem 3 and Theorem 4 are satisfied. One has, for any $p \in P$

$$D^2_\gamma F(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = \{(y_1, y_2) \in \mathbb{R}^2 \mid 3p + y_1 - 2y_2 \le 0, 6p - 2y_1 + y_2 \le 0\},$$

and

$$D^2_\gamma \mathcal{F}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 3p, y_2 = 0\} = \{(3p, 0)\}.$$

On the other hand, we have,

$$\nabla f(\bar{p}, \bar{x})(p, x) + \frac{1}{2}\nabla^2 f(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w})) = (2p + x_1, x_2),$$

$$\begin{aligned}
&\mathrm{Eff}_K\{y \in Y \mid x \in D^2_\gamma C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p),\\
&y = \nabla f(\bar{p}, \bar{x})(p, x) + \frac{1}{2}\nabla^2 f(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w}))\}\\
&= \mathrm{Eff}_K\{(y_1, y_2) \in \mathbb{R}^2 \mid 3p + y_1 - 2y_2 \le 0, 6p - 2y_1 + y_2 \le 0\}\\
&= \{(y_1, y_2) \in \mathbb{R}^2 \mid 3p + y_1 - 2y_2 = 0, 6p - 2y_1 + y_2 = 0\}\\
&= \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 3p, y_2 = 0\}\\
&= \{(3p, 0)\},
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Eff}^w_K\{y \in Y \mid x \in D^2_\gamma C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p),\\
&y = \nabla f(\bar{p}, \bar{x})(p, x) + \frac{1}{2}\nabla^2 f(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w}))\}\\
&= \mathrm{Eff}_K\{(y_1, y_2) \in \mathbb{R}^2 \mid 3p + y_1 - 2y_2 \le 0, 6p - 2y_1 + y_2 \le 0\}\\
&= \{(y_1, y_2) \in \mathbb{R}^2 \mid 3p + y_1 - 2y_2 = 0, 6p - 2y_1 + y_2 = 0\}\\
&= \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 3p, y_2 = 0\}\\
&= \{(3p, 0)\}.
\end{aligned}$$

Finally, by applying Theorem 3 and Theorem 4, we obtain, respectively, that

$$D^2_\gamma \mathcal{F}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \supset \{(3p, 0)\} \text{ and } D^2_\gamma \mathcal{F}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) \subset \{(3p, 0)\}, \forall p \in P.$$

Hence, one has,

$$D^2_\gamma \mathcal{F}(\bar{p}, \bar{y}, \bar{u}, \bar{v})(p) = \{(3p, 0)\}, \forall p \in P.$$

4. Application to optimization problems with finite constraints

In this section, we apply the results obtained in the previous section to the consideration of problem (2) with the constraint mapping $C : P \rightrightarrows X$ being defined by

$$C(p) := \{x \in X \mid g_i(p, x) \le 0, i \in I\}, \tag{16}$$

where $I := \{1, 2, \cdots, m\}$ is an arbitrary index set and, for each $i \in I, g_i : P \times X \to \mathbb{R}$ is a twice continuously Fréchet differentiable map. Constraints of type (16) are known as finite inequality constraints. Denote by $T(\bar{p}, \bar{x}, \bar{u}, \bar{w}) := \{i \in I \mid g_i(\bar{p}, \bar{x}) = 0 \text{ and } \nabla g_i(\bar{p}, \bar{x})(\bar{u}, \bar{w}) = 0\}$ the index set of all active constraints at $(\bar{p}, \bar{x}) \in P \times X$ in direction $(\bar{u}, \bar{w}) \in P \times X$.

In the line of Definition 4.1 (see Chuong, 2013a), we propose the following definition:

DEFINITION 7 *Let C be defined as in (16) and let $(\bar{p}, \bar{x}) \in \text{gph}C$ and $(\bar{u}, \bar{w}) \in P \times X, \gamma \in \{0, 1\}$. We say that C satisfies the second-order constraint qualification (CQ) at $(\bar{p}, \bar{x})$ in the direction $(\bar{u}, \bar{w})$ if*

$$T^2_\gamma(\text{gph}C, (\bar{p}, \bar{x}), (\bar{u}, \bar{w})) \supset \{(p, x) \in P \times X \mid \nabla g_i(\bar{p}, \bar{x})(p, x) + \gamma \nabla^2 g_i(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w})) \leq 0, \forall i \in T(\bar{p}, \bar{x}, \bar{u}, \bar{w})\}. \tag{17}$$

The following proposition gives us a criterion for computing the second-order contingent derivative of index γ of the constraint mapping C in (16).

PROPOSITION 3 *Let $(\bar{p}, \bar{x}) \in \text{gph}C$ and $(\bar{u}, \bar{w}) \in P \times X, \gamma \in \{0, 1\}$. Suppose that C in* (16) *satisfies the condition* (CQ) *at $(\bar{p}, \bar{x})$ in the direction $(\bar{u}, \bar{w})$* (see (17)) *and, for each $i \in I$, g_i is twice continuously Fréchet differentiable at $(\bar{p}, \bar{x})$. Then*

$$D^2_\gamma C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p) = \big\{x \in X \mid \nabla g_i(\bar{p}, \bar{x})(p, x) + \gamma \nabla^2 g_i(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w})) \leq 0, \forall i \in T(\bar{p}, \bar{x}, \bar{u}, \bar{w})\big\}, \forall p \in P.$$

PROOF Let $p \in P$ and $x \in D^2_\gamma C(\bar{p}, \bar{x}, \bar{u}, \bar{w})(p)$. Then, there exist

$$t_n \to 0^+, r_n \to 0^+, \frac{t_n}{r_n} \to \gamma \text{ and } (p_n, x_n) \to (p, x)$$

such that

$$\bar{x} + t_n \bar{w} + \frac{1}{2} t_n r_n x_n \in C(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n), \forall n \in \mathbb{N},$$

leading to

$$g_i(\bar{p} + t_n \bar{u} + \frac{1}{2} t_n r_n p_n, \bar{x} + t_n \bar{w} + \frac{1}{2} t_n r_n x_n) \leq 0, \forall n \in \mathbb{N}, \forall i \in I. \tag{18}$$

We deduce from the twice continuously Fréchet differentiability of g_i that

$$g_i(\bar{p} + t_n\bar{u} + \frac{1}{2}t_n r_n p_n, \bar{x} + t_n\bar{w} + \frac{1}{2}t_n r_n x_n) = g_i(\bar{p}, \bar{x}) + t_n\nabla g_i(\bar{p}, \bar{x})(\bar{u}, \bar{w})$$
$$+\frac{1}{2}t_n r_n \nabla g_i(\bar{p}, \bar{x})(p_n, x_n)$$
$$+\frac{1}{2}t_n^2\nabla^2 g_i(\bar{p}, \bar{x})\left(\left(\bar{u} + \frac{1}{2}r_n p_n, \bar{w} + \frac{1}{2}r_n x_n\right), \left(\bar{u} + \frac{1}{2}r_n p_n, \bar{w} + \frac{1}{2}r_n x_n\right)\right)$$
$$+o\left(\left\|\left(t_n\bar{u} + \frac{1}{2}t_n r_n p_n, t_n\bar{w} + \frac{1}{2}t_n r_n x_n\right)\right\|^2\right), \forall n \in \mathbb{N}, \forall i \in I. \tag{19}$$

From (18) and (4), one has

$$g_i(\bar{p}, \bar{x}) + t_n\nabla g_i(\bar{p}, \bar{x})(\bar{u}, \bar{w}) + \frac{1}{2}t_n r_n \nabla g_i(\bar{p}, \bar{x})(p_n, x_n)$$
$$+\frac{1}{2}t_n^2\nabla^2 g_i(\bar{p}, \bar{x})\left(\left(\bar{u} + \frac{1}{2}r_n p_n, \bar{w} + \frac{1}{2}r_n x_n\right), \left(\bar{u} + \frac{1}{2}r_n p_n, \bar{w} + \frac{1}{2}r_n x_n\right)\right)$$
$$+o\left(\left\|\left(t_n\bar{u} + \frac{1}{2}t_n r_n p_n, t_n\bar{w} + \frac{1}{2}t_n r_n x_n\right)\right\|^2\right) \leq 0, \forall n \in \mathbb{N}, \forall i \in I.$$

Since $g_i(\bar{p}, \bar{x}) = 0$ and $\nabla g_i(\bar{p}, \bar{x})(\bar{u}, \bar{w}) = 0$ for all $i \in T(\bar{p}, \bar{x}, \bar{u}, \bar{w})$, we deduce that

$$\frac{1}{2}t_n r_n \nabla g_i(\bar{p}, \bar{x})(p_n, x_n)$$
$$+\frac{1}{2}t_n^2\nabla^2 g_i(\bar{p}, \bar{x})\left(\left(\bar{u} + \frac{1}{2}r_n p_n, \bar{w} + \frac{1}{2}r_n x_n\right), \left(\bar{u} + \frac{1}{2}r_n p_n, \bar{w} + \frac{1}{2}r_n x_n\right)\right)$$
$$+o\left(\left\|\left(t_n\bar{u} + \frac{1}{2}t_n r_n p_n, t_n\bar{w} + \frac{1}{2}t_n r_n x_n\right)\right\|^2\right) \leq 0, \forall n \in \mathbb{N}, \forall i \in T(\bar{p}, \bar{x}, \bar{u}, \bar{w}).$$

Consequently,

$$\nabla g_i(\bar{p}, \bar{x})(p_n, x_n)$$
$$+\frac{t_n}{r_n}\nabla^2 g_i(\bar{p}, \bar{x})\left(\left(\bar{u} + \frac{1}{2}r_n p_n, \bar{w} + \frac{1}{2}r_n x_n\right), \left(\bar{u} + \frac{1}{2}r_n p_n, \bar{w} + \frac{1}{2}r_n x_n\right)\right)$$
$$+\frac{o\left(\left\|\left(t_n\bar{u} + \frac{1}{2}t_n r_n p_n, t_n\bar{w} + \frac{1}{2}t_n r_n x_n\right)\right\|^2\right)}{\frac{1}{2}t_n r_n} \leq 0, \forall n \in \mathbb{N}, \forall i \in T(\bar{p}, \bar{x}, \bar{u}, \bar{w}).$$

Let $n \to \infty$, one obtains

$$\nabla g_i(\bar{p}, \bar{x})(p, x) + \gamma\nabla^2 g_i(\bar{p}, \bar{x})((\bar{u}, \bar{w}), (\bar{u}, \bar{w})) \leq 0, \forall i \in T(\bar{p}, \bar{x}, \bar{u}, \bar{w}).$$

Hence,

$$D^2_\gamma C(\bar p,\bar x,\bar u,\bar w)(p) \subset \left\{x\in X \mid \nabla g_i(\bar p,\bar x)(p,x)+\nabla^2 g_i(\bar p,\bar x)((\bar u,\bar w),(\bar u,\bar w))\le 0, \forall i\in T(\bar p,\bar x,\bar u,\bar w)\right\}, \forall p\in P.$$

This, together with condition (CQ), implies that

$$D^2_\gamma C(\bar p,\bar x,\bar u,\bar w)(p) = \left\{x\in X \mid \nabla g_i(\bar p,\bar x)(p,x)+\gamma\nabla^2 g_i(\bar p,\bar x)((\bar u,\bar w),(\bar u,\bar w))\le 0, \forall i\in T(\bar p,\bar x,\bar u,\bar w)\right\}, \forall p\in P.$$

The proof is complete. ■

The first result in this section provides an inner estimate for evaluating the second-order contingent derivative of index γ of the efficient point multifunction $\mathcal{F}$ in (3) via the second-order Fréchet derivative of the objective function f and of the constraint functions $g_i, i\in I$, given by (16) at the reference point.

THEOREM 5 *Let $\mathcal{F}$ be the efficient point multifunction of (2) with the constraint mapping C given by (16). Let $\bar p\in P$, $\bar x\in C(\bar p)$, $\gamma\in\{0,1\}$ and $\bar y = f(\bar p,\bar x), (\bar u,\bar w,\bar v)\in P\times X\times Y$. Suppose that the following conditions hold:*

(i) C is second-order directionally compact with index γ at $(\bar p,\bar x)$ with respect to $(\bar u,\bar w)$ in the direction $p\in P$;

(ii) f is twice continuously Fréchet differentiable at $(\bar p,\bar x)$ and $\bar v = \nabla f(\bar p,\bar x)$ $(\bar u,\bar w)$;

(iii) The domination property holds for F defined in (1) around $\bar p$ and (10) holds true;

(iv) C satisfies the second-order constraint qualification (CQ) at $(\bar p,\bar x)$ in the direction $(\bar u,\bar w)$ in (17).

Then, for all $p\in P$,

$$\begin{aligned} D^2_\gamma\mathcal{F}(\bar p,\bar y,\bar u,\bar v)(p)\supset \mathrm{Eff}_K\{y\in Y \mid \nabla g_i(\bar p,\bar x)(p,x)+\gamma\nabla^2 g_i(\bar p,\bar x)((\bar u,\bar w),\\ (\bar u,\bar w))\le 0,\\ \forall i\in T(\bar p,\bar x,\bar u,\bar w),\\ y=\nabla f(\bar p,\bar x)(p,x)+\gamma\nabla^2 f(\bar p,\bar x)((\bar u,\bar w),(\bar u,\bar w))\}.\end{aligned}$$

PROOF By Theorem 3, one has

$$D^2_\gamma\mathcal{F}(\bar p,\bar y,\bar u,\bar v)(p)\supset \mathrm{Eff}_K\{y\in Y \mid x\in D^2_\gamma C(\bar p,\bar x,\bar u,\bar w)(p), y=\nabla f(\bar p,\bar x)(p,x) +\gamma\nabla^2 f(\bar p,\bar x)((\bar u,\bar w),(\bar u,\bar w))\}.$$

It follows from Proposition 3 that

$$\begin{aligned} D^2_\gamma\mathcal{F}(\bar p,\bar y,\bar u,\bar v)(p)\supset \mathrm{Eff}_K\{y\in Y \mid \nabla g_i(\bar p,\bar x)(p,x)+\gamma\nabla^2 g_i(\bar p,\bar x)((\bar u,\bar w),\\ (\bar u,\bar w))\le 0,\\ \forall i\in T(\bar p,\bar x,\bar u,\bar w),\\ y=\nabla f(\bar p,\bar x)(p,x)+\gamma\nabla^2 f(\bar p,\bar x)((\bar u,\bar w),(\bar u,\bar w))\}.\end{aligned}$$

This concludes the proof. ■

As an immediate consequence of Theorem 4 and Proposition 3, we have the following result, which gives an outer estimate for evaluating the second-order contingent derivative of index γ of the efficient point multifunction $\mathcal{F}$ in (3) via the second-order Fréchet derivative of the objective function f and of the constraint functions $g_i, i \in I$, given by (16) at the point under consideration.

THEOREM 6 *Let $\mathcal{F}$ be the efficient point multifunction of (2) with the constraint mapping C given by (16). Let $\bar{p} \in P, \bar{x} \in C(\bar{p}), \gamma \in \{0,1\}$ and $\bar{y} = f(\bar{p},\bar{x}), (\bar{u},\bar{w},\bar{v}) \in P \times X \times Y$. Suppose that f is twice continuously Fréchet differentiable at $(\bar{p},\bar{x})$ and $\bar{v} = \nabla f(\bar{p},\bar{x})(\bar{u},\bar{w})$. Assume that (2) and (10) hold true. C satisfies the second-order constraint qualification (CQ) at $(\bar{p},\bar{x})$ in the direction $(\bar{u},\bar{w})$ in (17). Then, for all $p \in P$,*

$$\begin{aligned}&D^2_\gamma \mathcal{F}(\bar{p},\bar{y},\bar{u},\bar{v})(p) \subset \\ &\mathrm{Eff}^w_K\{y \in Y \mid \nabla g_i(\bar{p},\bar{x})(p,x) + \gamma\nabla^2 g_i(\bar{p},\bar{x})((\bar{u},\bar{w}),(\bar{u},\bar{w})) \leq 0, \\ &\forall i \in T(\bar{p},\bar{x},\bar{u},\bar{w}), y = \nabla f(\bar{p},\bar{x})(p,x) + \gamma\nabla^2 f(\bar{p},\bar{x})((\bar{u},\bar{w}),(\bar{u},\bar{w}))\}.\end{aligned}$$

PROOF By Theorem 4, one has

$$\begin{aligned}D^2_\gamma \mathcal{F}(\bar{p},\bar{y},\bar{u},\bar{v})(p) \subset \mathrm{Eff}^w_K\{y \in Y \mid x \in D^2_\gamma C(\bar{p},\bar{x},\bar{u},\bar{w})(p), y = \nabla f(\bar{p},\bar{x})(p,x) \\ +\gamma\nabla^2 f(\bar{p},\bar{x})((\bar{u},\bar{w}),(\bar{u},\bar{w}))\}.\end{aligned}$$

It follows from Proposition 3 that

$$\begin{aligned}D^2_\gamma \mathcal{F}(\bar{p},\bar{y},\bar{u},\bar{v})(p) \subset \mathrm{Eff}^w_K\{y \in Y \mid \nabla g_i(\bar{p},\bar{x})(p,x) + \gamma\nabla^2 g_i(\bar{p},\bar{x})((\bar{u},\bar{w}), \\ (\bar{u},\bar{w})) \leq 0, \\ \forall i \in T(\bar{p},\bar{x},\bar{u},\bar{w}), \\ y = \nabla f(\bar{p},\bar{x})(p,x) + \gamma\nabla^2 f(\bar{p},\bar{x})((\bar{u},\bar{w}),(\bar{u},\bar{w}))\}.\end{aligned}$$

■

5. Conclusion

In this paper, we have investigated the second-order sensitivity in vector optimization problems. We have established the formulae for inner and outer evaluation of the second-order contingent derivative of index γ of the efficient point multifunction of parametric vector optimization problems. These estimating formulae have been presented via the set of efficient/weakly efficient points of the second-order contingent derivative of index γ of a composite multifunction of the objective function and the constraint mapping. An application to vector optimization problems with finite constraints has also been given.

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