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Post-optimal analysis for multicriteria integer linear programming problem with parametric optimality*

Vladimir Emelichev
Belarusian State University, Faculty of Mechanics and Mathematics,
BLR-220030 Minsk, Belarus
vemelichev@gmail.com

Yury Nikulin
University of Turku, Department of Mathematics and Statistics,
FIN-20014 Turku, Finland
yurnik@utu.fi

Abstract: This paper addresses a multicriteria problem of integer linear programming with parametric optimality. Parameterizations is introduced by dividing a set of objectives into a family of disjoint subsets, within each Pareto optimality is used to establish dominance between alternatives. The introduction of this principle allows us to connect such classical optimality sets as Pareto and extreme. The parameter space of admissible perturbations in such problem is formed by a set of additive matrices, with arbitrary Hölder's norms specified in the solution and criterion spaces. The attainable lower and upper bounds for the radii of quasistability are obtained.

Keywords: post-optimal analysis; multiple criteria; quasistability radius; integer linear programming; parametric optimality

1. Introduction

In multicriteria optimization and decision making, we sometimes deal with the choice functions different from the well-known Pareto optimality principle. Such functions play crucial role in many real life applications (see, e.g., Lotov and Pospelova, 2008). In this paper, in addition to Pareto optimality, we consider the multicriteria problem of Integer Linear Programming (ILP) with the extreme optimality principle, i.e. with the set of solutions being individual optimizers of all criteria. This set is used to construct the payoff table, often serving for calculating the ideal point and estimating the nadir point of the Pareto optimal set (see, e.g., Ehrgott, 2005; Miettinen, 1999; Noghin, 2018; Steuer, 1986). We introduce a parameterized optimality principle which is implemented by means of partitioning the partial criteria set into non-empty subsets,

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inside which relations on the set of solutions are based on the Pareto dominance.

Notice that the idea of splitting and grouping criteria is frequently used in multicriteria decision making. For example, in Wilppu, Mäkelä and Nikulin (2017), a new parameterized achievement scalarizing function is using similar approach for treating partial criteria and found to be useful in modern applications (Montonen, Ranta and Mäkelä, 2019). In addition, the problem if ILP can be considered under game theoretic framework (see, e.g., Nikulin, 2009), where grouping criteria can be interpreted as forming coalition between players and then establishing optimality principles between coalitions.

A general approach to obtaining the formula of the stability radius for scalar combinatorial problem was suggested in Gordeev and Leontev (1996). A brief survey of some typical quantitative results and approaches to various multicriteria integer linear programming problems can be found in Emelichev et al. (2002).

In this paper, the lower and upper bounds for the radii of quasistability are obtained, and criteria are formulated as corollaries. This type of stability is interpreted as the existence of such perturbations in which new optimal solutions can appear but all the original optima should be preserved.

Why is it important to have information about quasistability radius bounds? First, if the radius of stability is not equal to zero, not only does it determine the solution of the original problem, but also to a series of problems with parameters located in the vicinity of the radius equal to a radius of quasistability. Second, for a number of particular cases one can build an algorithm for finding radii that uses and continues the same procedures that were involved in the problem, which actually means that the radius could be potentially calculated along with the optimal solution of the problem.

The paper is organized as follows. In Section 2, we formulate parametric optimality and introduce basic concepts along with the notation. Section 3 contains some auxiliary statements about norms and two lemmas used later for the proof of the main result. In Section 4, we formulate and prove the main result regarding the lower and upper bounds for the quasistability radius. Section 5 lists most important corollaries as well as presents a comment on how the result fares against the earlier known results. Section 6 describes quantitatively quasistability by means of formulating necessary and sufficient conditions. Concluding remarks appear in Section 7.

2. Problem formulation and basic definitions

Consider a multicriteria integer linear programming problem (ILP) in the following formulation. Let $C = [c_{ij}] \in \mathbf{R}^{m\times n}$ be a matrix, whose rows are denoted by $C_i = (c_{i1}, c_{i2}, ..., c_{in}) \in \mathbf{R}^n$, $i \in N_m = \{1, 2, ..., m\}$, $m \geq 1$. Let $x = (x_1, x_2, ..., x_n)^T \in X \subset \mathbf{Z}^n$, $n \geq 2$, the number of elements of the set X being finite and greater than one. On the set of (admissible) solutions X, we define a vector linear criterion

$$Cx = (C_1x, C_2x, ..., C_mx)^T \to \min_{x\in X}. \tag{1}$$

In the space $\mathbf{R}^k$ of arbitrary dimension $k \in \mathbf{N}$ we introduce a binary relation that generates the Pareto optimality principle, Pareto (1909),

$$y \succ y' \;\Leftrightarrow\; y \geq y' \;\&\; y \neq y',$$

where $y = (y_1, y_2, ..., y_k)^T \in \mathbf{R}^k$, $y' = (y'_1, y'_2, ..., y'_k)^T \in \mathbf{R}^k$.

The symbol $\overline{\succ}$, as usual, denotes the negation of the relation $\succ$.

Let $\emptyset \neq I \subseteq N_m, |I| = v$, and let C_I denote the submatrix of the matrix $C \in \mathbf{R}^{m\times n}$ consisting of rows of this matrix with the numbers of the subset I, i.e.

$$C_I = (C_{i_1}, C_{i_2}, ..., C_{i_v})^T,\; I = \{i_1, i_2, ..., i_v\},$$
$$1 \leq i_1 < i_2 < ... < i_v \leq m,\; C_I \in \mathbf{R}^{v\times n}.$$

Let $s \in N_m$, and let $N_m = \bigcup\limits_{k\in N_s} I_k$ be a partition of the set N_m into s nonempty sets, i.e. $I_k \neq \emptyset$, $k \in N_s$, and $i \neq j \Rightarrow I_i \cap I_j = \emptyset$. For this partition, we introduce a set of $(I_1, I_2, ..., I_s)$-efficient solutions according to the formula:

$$\begin{aligned} &G^m(C, I_1, I_2, ..., I_s) = \big\{x \in X \;: \\ &\quad \exists k \in N_s \;\forall x' \in X \;\big(C_{I_k}x \,\overline{\succ}\, C_{I_k}x'\big)\big\}. \end{aligned} \tag{2}$$

Sometimes for brevity we denote this set by $G^m(C)$.

Obviously, any N_m-efficient solution $x \in G^m(C, N_m)$ $(s = 1)$ is Pareto optimal, i.e. is an efficient solution to problem (1). Therefore, the set $G^m(C, N_m)$ is the Pareto set (Pareto, 1909):

$$P^m(C) = \big\{x \in X \;:\; \forall x' \in X \;\big(Cx \,\overline{\succ}\, Cx'\big)\big\}.$$

We also use the following set

$$X(x, C) = \big\{x' \in X \;:\; Cx \succ Cx'\big\},$$

which is a set of solutions $x' \in X$ such that x' dominates x in Pareto sense in problem (1). Therefore,

$$P^m(C) = \big\{x \in X \ : \ X(x, C) = \emptyset\big\}.$$

In the other extreme case, when $s = m$, $G^m(C, \{1\}, \{2\}, ..., \{m\})$ is a set of extreme solutions (Miettinen, 1999; Sholomov, 1989; Yudin, 1989). This set is denoted by $E^m(C)$. Thereby, we have

$$\begin{aligned} E^m(C) &= \big\{x \in X \ : \ \exists k \in N_m \ \forall x' \in X \ (C_k x \,\overline{\succ}\, C_k x')\big\} = \\ &= \big\{x \in X \ : \ \exists k \in N_m \ \forall x' \in X \ \big(C_k x \le C_k x'\big)\big\}. \end{aligned}$$

It is easy to see that the set is composed of the solutions that are the best for at least one criterion.

So, in this context, the parameterization of the optimality principle refers to the introduction of such a characteristic of the binary preference relation that allows us to connect the well-known choice functions, parameterizing them from the Pareto to the extreme.

Denoted by $Z^m(C, I_1, I_2, \dots, I_s)$, the multicriteria ILP problem consists in finding the set $G^m(C, I_1, I_2, \dots, I_s)$. Sometimes, for the sake of brevity, we use the notation $Z^m(C)$ for this problem.

It is easy to see that the set $P^1(C) = E^1(C)$ is the set of optimal solutions to the scalar (single-criterion) problem $Z^1(C, N_1)$, where $C \in \mathbf{R}^n$.

For any nonempty subset $I \subseteq N_m$ we introduce the notation

$$P(C_I) = \big\{x \in X : \ \forall x' \in X \ (C_I x \,\overline{\succ}\, C_I x')\big\},$$

$$X(x, C_I) = \big\{x' \in X : \ C_i x \succ C_i x')\big\},$$

i.e.

$$P(C_I) = \big\{x \in X : \ X(x, C_I) = \emptyset\big\}.$$

Then, by virtue of (2), we obtain

$$G^m(C, I_1, I_2, \dots, I_s) = \big\{x \in X : \ \exists k \in N_s \ \big(x \in P(C_{I_k})\big)\big\}. \tag{3}$$

Therefore, we have

$$G^m(C, I_1, I_2, \dots, I_s) = \bigcup_{k \in N_s} P(C_{I_k}), \ N_m = \bigcup_{k \in N_s} I_k.$$

It is obvious that all the sets given here are nonempty for any matrix $C \in \mathbf{R}^{m \times n}$.

The perturbation of the elements of the matrix C is imposed by adding matrices C' from $\mathbf{R}^{m\times n}$ to it. Thus, the perturbed problem $Z^m(C+C')$ has the form

$$(C+C')x \to \min_{x\in X},$$

and the set of its $(I_1, I_2, ..., I_s)$-efficient solutions is $G^m(C+C', I_1, I_2, \ldots, I_s)$.

In the space of solutions $\mathbf{R}^n$, we define an arbitrary Hölder's norm l_p, $p \in [1,\infty]$, i.e. by the norm of the vector $a = (a_1, a_2, ..., a_n)^T \in \mathbf{R}^n$ we mean the number

$$\|a\|_p = \begin{cases} \left(\sum\limits_{j\in N_n} |a_j|^p \right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\left\{ |a_j| \ : \ j \in N_n \right\} & \text{if } p = \infty. \end{cases}$$

In the space of criteria $\mathbf{R}^m$, we define an arbitrary Hölder's norm l_q, $q \in [1,\infty]$. By the norm of the matrix $C \in \mathbf{R}^{m\times n}$ with the rows $C_i,\ i \in N_m$, we mean the norm of a vector whose components are the norms of the rows of the matrix. By that, we have

$$\|C\|_{pq} = \big\| (\|C_1\|_p, \|C_2\|_p, \ldots, \|C_m\|_p) \big\|_q.$$

For an arbitrary number $\varepsilon > 0$, we define the set of perturbing matrices

$$\Omega(\varepsilon) = \left\{ C' \in \mathbf{R}^{m\times n} \ : \ \|C'\|_{pq} < \varepsilon \right\}.$$

Following Emelichev et al. (2002), Emelichev and Nikulin (2019), Emelichev and Podkopaev (1998, 2001, 2010), the *quasistability radius* of the ILP problem $Z^m(C, I_1, I_2, \ldots, I_s)$, $m \in \mathbf{N}$ (called T_4-stability radius in the terminology of Emelichev et al., 2014 and Sergienko and Shilo, 2003) is the number

$$\rho = \rho^m_{pq}(C, I_1, I_2, \ldots, I_s) = \begin{cases} \sup \Xi & \text{if } \Xi \ne \emptyset, \\ 0 & \text{if } \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \left\{ \varepsilon > 0 \ : \ \forall C' \in \Omega(\varepsilon) \quad \big(G^m(C) \subseteq G^m(C+C') \big) \right\}.$$

Thus, the quasistability radius of the problem $Z^m(C)$ determines the limit level of perturbations of the elements of the matrix C that preserves optimality of all the solutions of the set $G^m(C)$ of the original problem, but new extreme solutions are allowed to arise in the perturbed problem.

The quasistability radius of the problem $Z^m(C)$ can also be determined using the well-known (see, for example, Emelichev et al., 2014; Sergienko and Shilo, 2003) concept of the stability kernel of the problem. Indeed, it is easy to see that

$$\rho = \sup \left\{ \varepsilon > 0 \ : \ Ker^m(C, \varepsilon) = G^m(C) \right\},$$

where

$$Ker^m(C,\varepsilon) = \big\{x \in G^m(C) \,:\, \forall C' \in \Omega(\varepsilon) \;\; (x \in G^m(C+C'))\big\}.$$

The last set is called the *kernel* of ε-stability of the problem, and the set

$$Ker^m(C) = Ker^m(C, I_1, I_2, \dots, I_s) = \big\{x \in G^m(C, I_1, I_2, \dots, I_s) :$$

$$\exists \varepsilon > 0 \; \forall C' \in \Omega(\varepsilon) \; \big(x \in G^m(C + C', I_1, I_2, \dots, I_s)\big)\big\}$$

is called the stability kernel of the problem $Z^m(C, I_1, I_2, \dots, I_s)$. Thus, the kernel of the stability of a problem is the set of all solutions that are stable with respect to small perturbations of the parameters of the problem.

3. Auxiliary statements and lemmas

In the solution space $\mathbf{R}^n$, along with the norm $l_p,\ p \in [1,\infty]$, we will use the conjugate norm l_{p^*}, where the numbers p and p^* are connected, as usual, by the equality

$$\frac{1}{p} + \frac{1}{p^*} = 1,$$

assuming $p^* = 1$ if $p = \infty$, and $p^* = \infty$ if $p = 1$. Therefore, we further suppose that the range of the variation of the numbers p and p^* is the closed interval $[1,\infty]$, and the numbers themselves are connected by the above conditions.

So, it is easy to see that for any $a = (a_1, a_2, \dots, a_n)^T \in \mathbf{R}^n$ with

$$|a_j| = \alpha,\ j \in N_n,$$

the following equality holds

$$\|a\|_p = \alpha n^{1/p} \tag{4}$$

for any $p \in [1,\infty]$.

Further, we use the well-known Hölder's inequality

$$|a^T b| \le \|a\|_p \|b\|_{p^*} \tag{5}$$

that is true for any two vectors $a = (a_1, a_2, \dots, a_n)^T \in \mathbf{R}^n$ and $b = (b_1, b_2, \dots, b_n)^T \in \mathbf{R}^n$.

It is also well-known (see, e.g., Hardy, Littlewood and Polya, 1988) that Hölder's inequality becomes an equality for $1 < p < \infty$ if and only if

a) one of a or b is a zero vector;

b) the two vectors obtained from non-zero vectors a and b by raising their components' absolute values to the powers of p and p^*, respectively, are linearly dependent (proportional), and the sign of $(a_i b_i)$ is independent of i.

When $p = 1$, the inequality (5) transforms into the following inequality:

$$|\sum_{i \in N_n} a_i b_i| \leq \max_{i \in N_n} |b_i| \sum_{i \in N_n} |a_i|.$$

The last inequality holds as equality if, for example, b is the zero vector or if $a_j \neq 0$ for some j such that $|b_j| = \|b\|_\infty \neq 0$, and $a_i = 0$ for all $i \in N_n \backslash \{j\}$.

When $p = \infty$, the inequality (5) transforms into the following inequality:

$$|\sum_{i \in N_n} a_i b_i| \leq \max_{i \in N_n} |a_i| \sum_{i \in N_n} |b_i|.$$

This last inequality holds as equality if, for example, b is the zero vector or if $a_i = \sigma \operatorname{sign}(b_i)$ for all $i \in N_n$ and $\sigma > 0$.

So, we have just proven the first Lemma.

LEMMA 1 *For any $p \in [1, \infty]$ the following formula holds*

$$\forall b \in \mathbf{R}^n \ \forall \sigma > 0 \ \exists a \in \mathbf{R}^n$$

$$\big(|a^T b| = \sigma \|b\|_{p*} \ \& \ \|a\|_p = \sigma\big).$$

Hereinafter, a^+ is a projection of a vector $a = (a_1, a_2, \dots, a_k) \in \mathbf{R}^k$ on a positive orthant, i.e.

$$a^+ = [a]^+ = (a_1^+, a_2^+, \dots, a_k^+),$$

where $+$ implies the positive cut of vector a, i.e.

$$a_i^+ = [a_i]^+ = \max\{0, a_i\}.$$

LEMMA 2 *Given $x, x^0 \in X$, $x^0 \neq x$, $\emptyset \neq I \subseteq N_m$, $v = |I|$, $C_I \in \mathbf{R}^{v \times n}$ with rows C_i, $i \in I$, and a vector η with $\eta_i > 0$ such that for any $i \in I$*

$$[C_i(x - x^0)]^+ < \eta_i \|x - x^0\|_{p^*}, \tag{6}$$

then for any $\varepsilon > \|\eta\|_q$ there exists a perturbing matrix $C_I^0 \in \mathbf{R}^{v \times n}$ with rows C_i^0, $i \in I$ such that

$$x \in X(x^0, C_I + C_I^0), \ \|C_i^0\|_p = \eta_i, \ i \in I, \ \|C_I^0\|_{pq} < \varepsilon.$$

Proof Let us choose $\varepsilon > \varphi$. According to (5), for any $C' \in \mathbf{R}^{v\times n}$ with rows $C'_i,\ i \in I$, the following inequalities hold

$$C'_i(x - x^0) \leq \|C'_i\|_p \|x - x^0\|_{p^*},\ i \in I.$$

Therefore, for every $i \in I$, according to Lemma 1, there exists a vector row C^0_i with the norm $\|C^0_i\|_p = \eta_i$ such that

$$C^0_i(x - x^0) = -\eta_i \|x - x^0\|_{p^*}.$$

From the above, taking into consideration (6), we deduce inequalities below

$$(C_i + C^0_i)(x - x^0) = [C_i(x - x^0)]^+ - \eta_i \|x - x^0\|_{p^*} < 0,\ i \in I.$$

This implies $x \in X(x^0, C_I + C^0_I)$, and $\|C^0_I\|_{pq} = \|\eta\|_q < \varepsilon$. □

4. Bounds on quasistability radius

For the multicriteria ILP problem $Z^m(C, I_1, I_2, \ldots, I_s)$, $m \in \mathbf{N}$, for any $p, q \in [1, \infty]$ and $s \in N_m$ we define

$$\varphi = \varphi^m_{pq}(C, I_1, I_2, \ldots, I_s) = \min_{x' \in G^m(C)} \max_{k \in N_s} \min_{x \in X\setminus\{x'\}} \frac{\|[C_{I_k}(x - x')]^+\|_q}{\|x - x'\|_{p^*}}.$$

It is obvious that $\varphi \geq 0$.

Theorem 1 *For any $m \in \mathbf{N}$, $p, q \in [1, \infty]$ and $s \in N_m$, the quasistability radius of the multicriteria ILP problem $Z^m(C, I_1, I_2, \ldots, I_s)$ has the following lower and upper bounds:*

$$\varphi \leq \rho^m_{pq}(C, I_1, I_2, \ldots, I_s) \leq s^{\frac{1}{q}} \varphi.$$

Proof First, we prove the inequality $\rho \geq \varphi$. For $\varphi = 0$, this inequality is obvious. Let $\varphi > 0$. Then, according to the definition of the number φ, we have

$$\begin{aligned}&\forall x' \in G^m(C, I_1, I_2, \ldots, I_s) \quad \exists k \in N_s \quad \forall x \in X\setminus\{x'\}\\ &\Big(0 < \varphi\|x - x'\|_{p^*} \leq \|[C_{I_k}(x - x')]^+\|_q\Big).\end{aligned} \tag{7}$$

Further, for any perturbing matrix $C' \in \Omega(\varphi)$, we deduce

$$(C_{I_k} + C'_{I_k})x' \,\overline{\succ}\, (C_{I_k} + C'_{I_k})x,\ x \in X\setminus\{x'\}. \tag{8}$$

Assume the opposite, i.e. assume that there exist a matrix $C^0 \in \Omega(\varphi)$ and solutions $x^0 \neq x'$ such that

$$(C_{I_k} + C^0_{I_k})x' \ \succ\ (C_{I_k} + C^0_{I_k})x^0.$$

Then, for any index $i \in I_k$, we have

$$(C_i + C_i^0)(x^0 - x') \leq 0.$$

Therefore, due to (5), the following inequalities are valid

$$[C_i(x^0 - x')]^+ \leq \|C_i^0\|_p \|x^0 - x'\|_{p^*},\ i \in I_k. \tag{9}$$

Let

$$I_k = \{i_1, i_2, \dots, i_v\},\ 1 \leq i_1 \leq i_2 \leq \dots \leq i_v \leq m.$$

Then, due to monotonicity of l_q, i.e.

$$\forall y, y' \in \mathbf{R}_+^v\ \big(y \leq y' \Rightarrow \|y\|_q \leq \|y'\|_q\big),$$

inequalities (9) yield the following equalities:

$$\|[C_{I_k}(x^0 - x')]^+\|_q = \|[C_{i_1}(x^0 - x')]^+, [C_{i_2}(x^0 - x')]^+, \dots, [C_{i_v}(x^0 - x')]^+\|_q \leq$$
$$\big\|(\|C_{i_1}^0\|_p, \|C_{i_2}^0\|_p, \dots, \|C_{i_v}^0\|_p)\big\|_q \|x^0 - x'\|_{p^*} =$$
$$\|C_{I_k}^0\|_{pq} \|x^0 - x'\|_{p^*} \leq \|C^0\|_{pq} \|x^0 - x'\|_{p^*} < \varphi \|x^0 - x'\|_{p^*}.$$

The last contradicts (7). So, (8) is true, i.e. $x' \in P(C_{I_k} + C'_{I_k})$. Thus, according to (3), $x' \in G^m(C + C', I_1, I_2, \dots, I_s)$ if $C' \in \Omega(\varphi)$. Therefore, we conclude that

$$\forall C' \in \Omega(\varphi) \quad (G^m(C, I_1, I_2, \dots, I_s) \subset G^m(C + C', I_1, I_2, \dots, I_s)),$$

i.e. $\rho \geq \varphi$.

Next, we prove the inequality $\rho \leq s^{\frac{1}{q}}\varphi$. Let $\varepsilon \geq s^{\frac{1}{q}}\varphi$ and $\Theta > 1$ be such that

$$\frac{\varepsilon}{s^{\frac{1}{q}}} > \Theta\varphi > \varphi. \tag{10}$$

In accordance with the definition of the number φ we get the formula

$$\exists x^0 \in G^m(C, I_1, I_2, \dots, I_s) \quad \forall k \in N_s \quad \exists \hat{x} \in X \backslash \{x^0\}$$
$$\left(\|[C_{I_k}(\hat{x} - x^0)]^+\|_q \leq \varphi \|\hat{x} - x^0\|_{p^*}\right).$$

Then, for any index $k \in N_s$ there exists a vector η with $\eta_i > 0,\ i \in I_k$ such that

$$\Theta[C_i(\hat{x} - x^0)]^+ \leq \eta_i \|\hat{x} - x^0\|_{p^*},\ i \in I_k,\ \|\eta\|_q = \Theta\varphi.$$

Therefore, according to Lemma 2, for every index $k \in N_s$ there exists a perturbing matrix $C_{I_k}^0 \in \mathbf{R}^{v \times n},\ v = |I_k|$ with rows $C_i^0, i \in I_k$ such that

$$\hat{x} \in X(x^0, C_{I_k} + C_{I_k}^0),\ \|C_i^0\|_p = \eta_i,\ i \in I_k,$$

$$\|C^0_{I_k}\|_{pq} = \|\eta\|_q = \Theta\varphi < \varepsilon.$$

Therefore, using (4) and (10), we have

$$\|C^0\|_{pq} = s^{\frac{1}{q}}\Theta\varphi < \varepsilon.$$

Thus, we get $x^0 \notin P(C_{I_k} + C^0_{I_k})$ for any $k \in N_s$. So, due to (3), x^0 is not extreme in $Z^m(C + C^0, I_1, I_2, \ldots, I_s)$ if $C^0 \in \Omega(\varepsilon)$. It implies that for any $\varepsilon > s^{\frac{1}{q}}\varphi$, the inequality $\rho < \varepsilon$ holds. Hence, $\rho \leq s^{\frac{1}{q}}\varphi$. □

5. Corollaries

Theorem 1, proven in the previous section, extends the earlier known bounds for the cases with Pareto and extreme optimality to the more general case with parametric optimality, allowing for a greater flexibility for a decision maker regarding expressing his or her preferences along with taking into consideration possible input parameter uncertainty. Particularly, from Theorem 1, we have the following two known results (see Emelichev and Nikulin, 2019; Emelichev and Kuzmin, 2013).

Corollary 1 *For any $m \in \mathbf{N}$ and $p, q \in [1, \infty]$ for the quasistability radius $\rho^m_{pq}(C, N_m)$ ($s = 1$) of the multicriteria ILP $Z^m(C, N_m)$ consisting in finding the Pareto set $P^m(C)$, the following formula is true:*

$$\rho^m_{pq}(C, N_m) = \min_{x' \in P^m(C)} \quad \min_{x \in X \setminus \{x'\}} \frac{\|[C(x - x')]^+\|_q}{\|x - x'\|_{p^*}}.$$

Corollary 2 *For any $m \in \mathbf{N}$ and $p, q \in [1, \infty]$ for the quasistability radius $\rho^m_{pq}(C, \{1\}, \{2\}, \ldots, \{m\})$ ($s = m$) of the multicriteria ILP $Z^m(C, 1, 2, \ldots, m)$ consisting in finding the extreme set $E^m(C)$, the following lower and upper bounds are true:*

$$\psi^m_p \leq \rho^m_{pq}(C, \{1\}, \{2\}, \ldots, \{m\}) \leq m^{\frac{1}{q}}\psi^m_p,$$

where

$$\psi^m_p = \min_{x' \in E^m(C)} \quad \max_{i \in N_m} \quad \min_{x \in X \setminus \{x'\}} \frac{[C_i(x - x')]^+}{\|x - x'\|_{p^*}}.$$

Corollary 1 proves that the lower bound $\varphi^m_{pq}(C, I_1, I_2, \ldots, I_s)$, specified in Theorem 1 is attainable if $s = 1$. The attainability is also evident if $q = \infty$ and $m = 1$. Corollary 1 also illustrates the fact that the bounds for the quasistability radius can be transformed into the equation formula for some special cases. It might also be possible that the equation formula takes place in more general situations, not only in the Pareto case. In such situation, the analytical expression for the quasistability radius should have a bit more general form that can turn into the lower and upper bounds, specified in

Theorem 1 and Corollary 2, respectively.

The following example shows that the upper bound $m^{\frac{1}{q}}\psi_p^m$ specified in corollary 2 is also attainable.

EXAMPLE 1 *Let* $p = \infty,\ q \in [1,\infty],\ X = \{x^1, x^2, \dots, x^n\} \subset \mathbf{E}^n = \{0,1\}^n$, *where* $n \geq 3$, *and every solution* $x^j,\ j \in N_n$, *be a unit vector, i.e. a column of identity matrix of size* $n \times n$. *Let matrix* $C = [c_{ij}] \in \mathbf{R}^{m\times n}$ *with rows* $C_i \in \mathbf{R}^n$, $i \in N_m,\ m = n-1$ *be constructed as follows*

$$C = \begin{pmatrix} 0 & M & \dots & M & -2\alpha \\ M & 0 & \dots & M & -2\alpha \\ \dots & \dots & \dots & \dots & \dots \\ M & M & \dots & 0 & -2\alpha \end{pmatrix},$$

where $M \gg \alpha > 0$, *and* M *is a number large enough. Then we have*

$$Cx^1 = (0, M, \dots, M, M)^T \in \mathbf{R}^m,$$
$$Cx^2 = (M, 0, \dots, M, M)^T \in \mathbf{R}^m,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$Cx^{n-1} = (M, M, \dots, M, 0)^T \in \mathbf{R}^m,$$
$$Cx^n = (-2\alpha, -2\alpha, \dots, -2\alpha, -2\alpha)^T \in \mathbf{R}^m.$$

Thus, $x^n \in E^m(C)$, $x^j \notin E^m(C)$, $j \in N_m$. *Moreover, the following equality is evident*

$$\psi_\infty^m = \max_{i\in N_m} \quad \min_{j\in N_m} \quad \frac{C_i(x^j - x^n)}{2} = \alpha. \tag{11}$$

Let $C' = [c'_{ij}] \in \Omega_{\infty q}(\alpha m^{\frac{1}{q}})$ *be an arbitrary perturbing matrix with rows* $C'_1, C'_2, \dots, C'_m$, *i.e.* $C' \in \mathbf{R}^{m\times n}$, $\|C'\|_{\infty q} < \alpha m^{\frac{1}{q}}$. *Proving by contradiction, it is easy to show that there exists an index* $k \in N_m$ *with* $\|C'_k\|_\infty < \alpha$. *Therefore,* $|c'_{kj}| < \alpha$ *for any* $j \in N_n$. *So, we deduce that*

$$(C_k + C'_k)(x^k - x^n) = 2\alpha + c'_{kk} - c'_{kn} \geq 2\alpha - |c'_{kk}| - |c'_{kn}| > 0,$$

and hence for any index $i \in N_m \backslash \{k\}$ *we obtain*

$$(C_i + C'_i)(x^k - x^n) = C_i(x^k - x^n) + C'_i(x^k - x^n) = M + 2\alpha + c'_{ik} - c'_{in} > 0.$$

As a result, we conclude that $x^n \in E^m(C + C')$, *and for any perturbing matrix* $C' \in \Omega_{\infty q}(\alpha m^{\frac{1}{q}})$ *the following inequality holds*

$$\rho_{\infty,q}^m(C, \{1\}, \{2\}, \dots, \{m\}) \geq m^{\frac{1}{q}}\alpha.$$

Taking into account Corollary 2 and using (11), we get the equality:

$$\rho_{\infty,q}^m(C, \{1\}, \{2\}, \dots, \{m\}) = m^{\frac{1}{q}}\psi_\infty^m.$$

6. Quasistability criteria

We call a multicriteria ILP problem $Z^m(C)$, $m \geq 1$, quasistable (to perturbations of the elements of the matrix C) if there exists a number $\varepsilon > 0$ such that

$$\forall C' \in \Omega(\varepsilon) \quad \big(G^m(C) \subseteq G^m(C + C')\big).$$

It is obvious that the property of quasistability is a discrete analogue of the lower semicontinuity property (according to Hausdorff) at the point $C \in \mathbf{R}^{m \times n}$ of the optimal mapping

$$G^m(C) \, : \, \mathbf{R}^{m \times n} \to 2^X,$$

i.e. of the point-to-set mapping, which associates with each set of problem parameters (each matrix C) the set of $(I_1, I_2, \ldots, I_s)$-efficient solutions $G^m(C, I_1, I_2, \ldots, I_s)$.

In order to formulate the necessary and sufficient conditions of quasistability, we introduce some notation. In the space $\mathbf{R}^k$ of arbitrary dimension $k \in \mathbf{N}$ we introduce one more binary relation:

$$y \vdash y' \;\Leftrightarrow\; y_i \geq y'_i, \; i \in N_k,$$

where $y = (y_1, y_2, \ldots, y_k)^T \in \mathbf{R}^k$, $y' = (y'_1, y'_2, \ldots, y'_k)^T \in \mathbf{R}^k$.

Now, we define a set of strictly extreme solutions to the problem $Z^m(C)$ according to the formula:

$$SG^m(C) = SG^m(C, I_1, I_2, \ldots, I_s) =$$
$$= \{x \in X \, : \, \exists k \in N_s \; \forall x' \in X \backslash \{x\} \quad \big(C_{I_k} x \;\overline{\vdash}\; C_{I_k} x'\big)\}.$$

It is obvious that $SG^m(C) \subseteq G^m(C)$ for any matrix $C \in \mathbf{R}^{m \times n}$ and any partition $(I_1, I_2, \ldots, I_s)$.

From Theorem 1, we get the following corollary.

COROLLARY 3 *For any $m \in \mathbf{N}$, $p \in [1, \infty]$ and $s \in N_m$, for the multicriteria ILP problem $Z^m(C, I_1, I_2, \ldots, I_s)$ the following statements are equivalent:*
(i) the problem $Z^m(C, I_1, I_2, \ldots, I_s)$ is quasistable;
(ii) $G^m(C) = SG^m(C) = Ker^m(C)$;
(iii) $\varphi^m_{pq}(C, I_1, I_2, \ldots, I_s) > 0$.

In the case when $s = 1$, the set $SG^m(C, N_m)$ turns into the well-known Smale set (Smale, 1974), i.e. into the set of strictly efficient solutions of the problem $Z^m(C, N_m)$:

$$Sm^m(C) = \big\{x \in X \, : \, \forall x' \in X \backslash \{x\} \quad \big(Cx \;\overline{\vdash}\; Cx'\big)\big\}.$$

Therefore, Corollary 3 implies the following well-known result (see Emelichev et al., 2002, 2009, 2014; Emelichev and Podkopaev, 1998, 2001; Sergienko and Shilo, 2003):

COROLLARY 4 *For any* $m \in \mathbf{N}$, *and* $p \in [1, \infty]$ *for the multicriteria ILP problem* $Z^m(C, N_m)$, *consisting in finding the Pareto set* $P^m(C)$, *the following statements are equivalent:*
(i) the problem $Z^m(C, N_m)$ *is quasistable;*
(ii) $P^m(C) = Sm^m(C) = Ker^m(C, N_m)$;
(iii) $\varphi^m_{pq}(C, N_m) > 0$.

From this, in particular, we obtain the following corollary:

COROLLARY 5 *The single criterion (scalar) ILP problem* $Z^1(C)$, $C \in \mathbf{R}^n$, *consisting in finding optimal solutions, is quasistable if and only if it has a unique optimal solution.*

In the case when $s = m$, the set

$$SG^m(C) = SG^m(C, \{1\}, \{2\}, \ldots, \{m\})$$

turns into the set of strictly extreme solutions of the problem $Z^m(C, \{1\}, \{2\}, \ldots, \{m\})$:

$$SE^m(C) = \{x \in X \,:\, \exists k \in N_m \ \forall x' \in X \backslash \{x\} \ (C_k x < C_k x')\}.$$

Thus, from Corollary 3, while taking into consideration (11) and Corollary 2, we derive the following corollary:

COROLLARY 6 *For any* $m \in \mathbf{N}$ *and* $p \in [1, \infty]$, *for the multicriteria ILP problem* $Z^m(C, \{1\}, \{2\}, \ldots, \{m\})$, *consisting in finding the set of extreme solutions* $E^m(C)$, *the following statements are equivalent:*
(i) the problem $Z^m(C, \{1\}, \{2\}, \ldots, \{m\})$ *is quasistable;*
(ii) $E^m(C) = SE^m(C) = Ker^m(C, \{1\}, \{2\}, \ldots, \{m\})$;
(iii) $\psi^m_p > 0$.

7. Conclusion

As a result of the parametric analysis performed in this paper, the lower and upper bounds on the quasistability radius were obtained for the multicriteria ILP problem with parametric optimality in the case, in which the criterion and solution spaces are endowed with various Hölder's norms l_p, $1 \leq p \leq \infty$, and l_q, $1 \leq q \leq \infty$, respectively. Parametrization was done on the basis of partitioning of the partial criteria set into non-empty subsets such that Pareto optimality principle is used within each subset and extreme optimality principle is used between the subsets.

The generality of our approach made it possible to obtain the achievable lower and upper bounds only for the quasistability radius. As stated in Theorem 1, the bounds turn into the equation formula for cases of $s = 1$ and $q = \infty$. The existence of the lower and upper bounds naturally leads us to the proof that the bounds could not be improved, since, as it was shown in Example 1 and Corollary 1, so the quasistability radius can be equal either to lower or upper bound for some classes of ILP problems. Theorem 1 made it also possible to formulate the quasistability criteria (see Corollaries 3 – 6).

One of the biggest challenges in this field is to construct efficient algorithms to calculate the analytical expressions of the bounds. To the best of our knowledge, there are not so many results known in that area, and, moreover, some of those results, which have been already known, put more questions than answers.

For example, in Kuzmin (2015), the formulas along with the lower and upper exact bounds of stability radii were obtained for solutions of the multiobjective maximum cut problem as well as for the various types of stability of the problem under assumption that Hölder's metrics are given on the spaces of a perturbing parameter. In Kuzmin (2015), it was also shown that the problem of finding the radii of every type of stability is intractable unless $P = NP$. As it was specifically mentioned in Nikulin, Karelkina and Mäkelä (2013), calculating exact values of stability radii is an extremely difficult task in general, and so one concentrates either on finding easy computable classes of problems or on developing general metaheuristic approaches.

An example of such metaheuristic approach can be found in Karelkina, Nikulin and Mäkelä (2011), where non-dominated sorting genetic algorithm based approach is proposed for calculating stability radius of an optimal solution to the single criterion shortest path problem. The key idea of the method is defractionalization of the objective by means of transforming a nonlinear single objective problem into biobjective problem with linear objectives. Such transformation is performed locally (within the genetic population), which makes the problem of finding the approximation of the Pareto frontier in biobjective case realistic when compared to the case, in which such linearization would have been made globally, that is, with respect to the original set of feasible solutions. This approach may become beneficial in comparison with other possible methods, such as, for example, applying genetic algorithm directly to the single objective problem with nonlinear (fractional) function, which would require some efficient nonlinear optimization tool to deal with. Extension of this idea to the case of several objectives could be an interesting avenue for future research in this and related areas.

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