

Control contribution for wear bearing recurrence process

K. WIERZCHOLSKI^{1*} and A. MISZCZAK²

¹ Institute of Technology and Education, Koszalin University of Technology, 2 Śniadeckich St., 75-453 Koszalin, Poland

² Gdynia Maritime University, 81-87 Morska St., 81-225 Gdynia, Poland

Abstract. This paper presents the methods of control problem solutions using recurrence equations implementation and UOS transformation for the bearing wear estimation during the finite and infinite time units of an operation process. If we have two wear value increase processes then very important is information which process is divergent more slowly. On the other hand, in comparison with two convergent processes we must decide which process is convergent more quickly. The wear process is determined mostly by the summation factor method. Such a method is applied for the solutions of recurrence equations with variable coefficients.

Key words: recurrences equations, wear bearing control process.

1. Preliminaries

The problems referring to the experimental and numerical wear of slide bearing exploitation questions require the knowledge of information about the features of the sequence of the existing wear process during the operation time [1–4]. The abovementioned information includes especially the kind and velocity of the wear value increases during the particular time units of the operation and very important are convergence and divergence phenomena of the wear value increases. If we have two wear value divergent processes, then very important is information which process is divergent more slowly. On the other hand, in comparison with two convergent processes, we must decide which process is convergent more quickly. This paper allows us to consider such a problem. If the wear value process in particular time units of the operating time is decreasing to some wear limit values, then it is possible to obtain the wear bearing stabilization in sufficient many time units of the operation. If the wear value process in particular time units of the operating time is increasing, then we have the phenomenon of wear bearing continuous increments in sufficient many time units of the operation. In both cases mentioned above, the sum of wear values after the considered time units of the operation, i.e. the cumulative function of wear values can be divergent. The most important for practical needs is the convergence or divergence of the cumulative values of bearing wear after the considered time units of the operation. To perform investigations of the abovementioned control problems, the wear value processes are described by non-homogeneous recurrent equations with a variable free term. The results obtained in this paper show that convergence or divergence of the wear processes depends on experimentally determined coefficients which occur in the considered wear recurrent equations. The coefficients depend on bearing materials, operation conditions, environmental external influences, and vibration influences. Therefore, the assumption and knowledge of materials, exploitation conditions and finally external influences

enable us to determine the control of wear during the operation time [3, 4]. The results obtained in this paper show how to anticipate wear of a slide bearing after many time units of the operation if bearing materials, exploitation and environmental conditions are known.

The presented paper is devoted both mathematical theory of solutions of summation or recurrence equations with variable coefficients and applications to HDD micro-bearing control wear process determination and its convergences in succeeding time units of operation.

While the authors have applied recurrence equations with variable coefficients for bearing material properties description, we can observe any relationship between the compositions, microstructure, processing or other physical characteristics of the tribo-components, that could add new information to identify or quantify the roots causes and control of wear of various bearing materials [5–8].

2. The object of the problem

We assume three sequences $\{g_n\}$, $\{f_n\}$, $\{h_n\}$ of three dimensional wear values g_n , f_n , h_n in mm^3 or μm^3 for particular time units $n = 1, 2, 3, \dots$ and for mentioned wear values we assign the respective wear processes described by the following three recurrent equations [1–8]:

$$g_{n+2} = P_{gn}g_{n+1} + Q_{gn}g_n, \quad P_{gn} \equiv \frac{b}{a} - \frac{c}{n+1}, \quad (1)$$

$$Q_{gn} \equiv \frac{b \cdot c}{a \cdot n}, \quad \frac{b}{a} > \frac{c}{n+1} \Rightarrow n > \frac{a \cdot c}{b} - 1,$$

$$f_{n+2} = P_{fn}f_{n+1} + Q_{fn}f_n + R_{fn},$$

$$P_{fn} \equiv 1 - \frac{b}{a \cdot n}, \quad (2)$$

$$Q_{fn} \equiv \frac{b}{a \cdot n}, \quad R_{fn} = \frac{d}{a \cdot n}, \quad n > \frac{b}{a},$$

$$n = 1, 2, 3, \dots$$

*e-mail: krzysztof.wierzcholski@wp.pl

$$\begin{aligned}
 h_{n+2} &= P_{hn}h_{n+1} + Q_{hn}h_n + R_{hn}, \\
 P_{hn} &= -\frac{n^2 + 1}{n + 1}, \\
 Q_{hn} &= \frac{2n^2}{n + 1}, \quad R_{hn} = \frac{d \cdot n}{n + 1}, \\
 n &= 1, 2, 3, \dots
 \end{aligned}
 \tag{3}$$

After experimental measurements it follows, that discrete wear values g_{n+2} , f_{n+2} , h_{n+2} of the sequences $\{g_n\}$, $\{f_n\}$, $\{h_n\}$ for $n = 1, 2, 3, \dots$ i.e. the wear values increases in mm^3 of HDD micro-bearing journal or sleeve equal to sums $(P_{gn}g_{n+1} + Q_{gn}g_n)$, $(P_{fn}f_{n+1} + Q_{fn}f_n + R_{fn})$, $(P_{hn}h_{n+1} + Q_{hn}h_n + R_{hn})$ of wear in two foregoing successive time units (may be months) where addends are multiplied by dimensionless variable stochastic wear coefficient P_{gn} and Q_{gn} or P_{fn} and Q_{fn} or P_{hn} and Q_{hn} . Variable coefficients P_{gn} , Q_{gn} or P_{fn} , Q_{fn} or P_{hn} , Q_{hn} as well free term R_{fn} , R_{hn} depend on the obtained in experimental way dimensional wear standard deviation of micro-bearing material, the journal angular velocity and the frequencies of vibrations. Variable n is numbered by natural numbers $1, 2, 3, \dots$. Dimensionless positive constants values $a, b, c > 0$ are independent of n in (1) but d is dimensional in (2), (3). Constants a, b are dimensionless in (2) too. Such experimental values depend on: bearing material properties, bearing operating conditions and so on. To confirm the positive wear values g_n, f_n for each n in first and second process (1), (2) we ought assume inequalities $n > (a \cdot c/b) - 1$ and $n > b/a$, $n = 1, 2, \dots$ respectively. To solve mentioned problem it is necessary to add the boundary conditions. Hence we assume that in two first time units (may be month for $n = 1$ and $n = 2$), the wear from experiments attains dimensional values $g_1 = f_1 = h_1 = W_1 [\mu\text{m}^3]$, and $g_2 = f_2 = h_2 = W_2 [\mu\text{m}^3]$ respectively.

The specific properties of wear values sequence described by the formula (3) denotes wear prognosis where after sufficient many time units $n \gg 1$ of exploitation process, observed wear value h_{n+2} in the next time units is exactly equal the wear value $d + n(2h_n - h_{n+1})$ in previous time units. The specific properties of wear values sequence described by the formula (2) denotes wear prognosis where after sufficient many time units $n \gg 1$ of exploitation process, observed wear value f_{n+2} in the next time units is exactly equal the wear value f_{n+1} in previous time units.

The distinguishing mark of the wear values in sequence (1) denotes wear prognosis where after sufficient many time units $n \gg 1$ of exploitation process, obtained wear value g_{n+2} in the next time units is equal to the wear value g_{n+1} in previous time units multiplied by the experimental determined constant coefficient b/a which can be controlled or modulated. After sufficient many time units for $n \gg 1$ or in the case for $c \rightarrow 0$, we have limits $P_{gn} \rightarrow b/a$, $Q_{gn} \rightarrow 0$ and mentioned exploitation process (1) tends to the following recurrent relation [9, 10]:

$$g_{n+2} = \frac{b}{a}g_{n+1}, \quad \text{for } n \gg 1, \quad \text{or } c \rightarrow 0. \tag{4}$$

In this singular case for additional assumption $0 < b/a < 1$ then we have phenomenon of wear values decreasing to zero after sufficient many time units of journal bearing operation. It is only necessity condition of convergence of wear cumulative value after sufficient many time units of operation. Nevertheless the sufficient condition is satisfied and imposing the boundary condition, the wear cumulative value after $N \gg 1$ and infinite many time units of exploitation for $N \rightarrow \infty$ has the following form [9–11]:

$$\begin{aligned}
 G_N &= \sum_{n=1}^N g_n = W_{1g} + W_{2g} \frac{1 - (b/a)^N}{1 - (b/a)}, \\
 G_\infty &= W_{1g} + W_{2g} \frac{a}{a - b}.
 \end{aligned}
 \tag{5}$$

3. A new variable method for wear cumulative process determination

Now we show the method to be sure on the new variable introduction which enables us to reduce the order of the considered recurrence equation for wear cumulative process [5, 7]. Wear process Eqs. (1), (2) can be described in the following form [12, 13]:

$$\text{for (1):} \quad a \cdot y_{n+1} - b \cdot y_n = 0, \tag{6a}_1$$

$$y_n \equiv g_{n+1} + \frac{c}{n}g_n, \tag{6a}_2$$

for $n = 1, 2, 3, \dots$

$$\text{for (2):} \quad a \cdot n \cdot y_{n+1} + b \cdot y_n = d, \tag{6b}_1$$

$$y_n \equiv f_{n+1} - f_n \tag{6b}_2$$

for $n = 1, 2, 3, \dots$

The homogeneous and non-homogeneous first order recurrent Eqs. (6a)₁, (6b)₁:

$$\text{for (6a)}_1: \quad a \cdot y_{n+1} - b \cdot y_n = 0 \Rightarrow y_{n+1} - \frac{b}{a}y_n = 0, \tag{7a}$$

for (6b)₁:

$$a \cdot n \cdot y_{n+1} + by_n = d \Rightarrow y_{n+1} + \frac{b}{a \cdot n}y_n = \frac{d}{a \cdot n}, \tag{7b}$$

have the general solutions in the form: $y_n = B_n(C_2)$ for $y_1 = C_2$, $n = 2, 3, 4, \dots$; with the arbitrary constant C_2 . Such solution y_n are introduced into recurrent Eqs. (6a)₂, (6b)₂ and hence we obtain the following, linear, non-homogeneous first order wear process recurrence equation [14]:

$$\text{for (6a)}_2: \quad g_{n+1} + \frac{c}{n}g_n = C_2 \left(\frac{b}{a}\right)^{n-1}, \tag{8a}$$

$$\begin{aligned}
 \text{for (6b)}_2: \quad f_{n+1} - f_n &= \frac{C_2}{(n-1)!} \left(-\frac{b}{a}\right)^{n-1} \\
 &+ \frac{d}{a} \sum_{m=1}^{n-1} \frac{(m-1)!}{(n-1)!} \left(-\frac{b}{a}\right)^{n-m-1}, \tag{8b}
 \end{aligned}$$

where $g_1 = C_1$, $f_1 = C_1$ (arbitrary summation constant), $n = 2, 3, 4, \dots$. We impose boundary conditions $g_1 = W_{1g}$

and $g_2 = W_{2g}$; $f_1 = W_{1f}$ and $f_2 = W_{2f}$ in places $n = 1$ and $n = 2$, on the general solution (8a), (8b).

At first we determine wear process $\{g_n\}$.

Equation (7a) has the general solution in the following form [13, 14]:

$$y_n = C_2(-1)^{n-1} \prod_{j=1}^{n-1} \left(-\frac{b}{a}\right) = C_2 \left(\frac{b}{a}\right)^{n-1} \equiv B_n(C_2) \quad (9)$$

for $n = 2, 3, 4, \dots$ where $C_2 = y_1$ denotes the arbitrary constant.

Solution (9) in presented form y_n is introduced into the formula (6a)₂ and hence we obtain the following implication and wear process solution [15]:

$$\begin{aligned} & \left(g_{n+1} + \frac{c}{n}g_n = y_n\right) \Rightarrow \\ g_n &= \frac{(-c)^{n-1}}{(n-1)!} \left\{ C_1 + C_2 \sum_{k=1}^{n-1} (-1)^k k! \left(\frac{b}{a \cdot c}\right)^{k-1} \right\}, \quad (10) \end{aligned}$$

where $n = 2, 3, \dots$ and $g_1 = C_1$.

On the general solution (10), we impose wear process boundary conditions $g_1 = W_{1g}$ and $g_2 = W_{2g}$ in first $n = 1$ and in second time units $n = 2$. Hence we obtain:

$$C_2 = W_2 + cW_{1g}, \quad C_1 = g_1 = W_{1g}. \quad (11)$$

We put constants (11) into implication result (10). After summation and terms ordering in solution (10) the wear value process solution has the final form:

$$g_n = W_{1g}g_{1n}(\delta, c, n) + W_{2g}g_{2n}(\delta, c, n), \quad (12)$$

where

$$\begin{aligned} g_{1n}(\delta, c, n) &\equiv \frac{(-c)^{n-1}}{(n-1)!} \left\{ 1 + \sum_{k=1}^{n-1} (-1)^k k! \left(\frac{\delta}{c}\right)^{k-1} \right\}, \\ g_{2n}(\delta, c, n) &\equiv \frac{(-c)^{n-2}}{(n-1)!} \sum_{k=1}^{n-1} (-1)^{k-1} k! \left(\frac{\delta}{c}\right)^{k-1}, \quad (13) \end{aligned}$$

whereas $n = 2, 3, 4, \dots$; $k = 1, 2, \dots, n - 1$; $\delta \equiv b/a$, and $g_{11} = 1$; $g_{12} = 0$; $g_{21} = 0$; $g_{22} = 1$. The wear values process during the particular time units for $c \rightarrow 0$, $0 < \delta < 1$ is convergent to zero ($g_\infty \rightarrow 0$). We have [16]:

$$\begin{aligned} g_\infty &= W_{1g}g_{1\infty} (|\delta| < 1, c \rightarrow 0, n \rightarrow \infty) \\ &+ W_{2g}g_{2\infty} (|\delta| < 1, c \rightarrow 0, n \rightarrow \infty), \quad (14) \end{aligned}$$

where

$$\begin{aligned} g_{1\infty} &\equiv \lim_{n \rightarrow \infty} \lim_{c \rightarrow 0} \frac{(-c)^{n-1}}{(n-1)!} \left\{ 1 + \sum_{k=1}^{n-1} (-1)^k k! \left(\frac{\delta}{c}\right)^{k-1} \right\} \\ &= \lim_{n \rightarrow \infty} \lim_{c \rightarrow 0} \left\{ \frac{(-c)^{n-1}}{(n-1)!} + \frac{1}{(n-1)!} \sum_{k=1}^{n-1} (-1)^{n+k-1} k! \cdot c^{n-k} \right\} = 0, \\ g_{2\infty} &\equiv \lim_{n \rightarrow \infty} \lim_{c \rightarrow 0} \frac{(-c)^{n-2}}{(n-1)!} \sum_{k=1}^{n-1} (-1)^{k-1} k! \left(\frac{\delta}{c}\right)^{k-1} \Big|_{|\delta| < 1} \\ &= \lim_{n \rightarrow \infty} \lim_{c \rightarrow 0} \left\{ \frac{1}{(n-1)!} \sum_{k=1}^{n-1} (-1)^{n+k-3} \cdot k! \cdot \delta^{k-1} \cdot c^{n-k-1} \right\} \Big|_{|\delta| < 1} = 0, \quad (15) \end{aligned}$$

where $n = 2, 3, 4, \dots$, $k = 1, 2, \dots, n - 1$, $\delta \equiv b/a$.

The sum of wear values i.e. cumulative wear values after considered N and infinite time units i.e. for $N \rightarrow \infty$ has the form [17, 18]:

$$\begin{aligned} G_N &= \sum_{n=1}^N g_n = \\ &W_{1g}G_{1N}(\delta, c, N) + W_{2g}G_{2N}(\delta, c, N), \quad (16) \end{aligned}$$

$$\begin{aligned} G_\infty &= \sum_{n=1}^\infty g_n \\ &= W_{1g}G_{1\infty}(\delta, c, \infty) + W_{2g}G_{2\infty}(\delta, c, \infty), \end{aligned}$$

$$\begin{aligned} G_{1N}(\delta, c, N) &\equiv \sum_{n=1}^N g_{1n}(\delta, c, n), \\ G_{2N}(\delta, c, N) &\equiv \sum_{n=1}^N g_{2n}(\delta, c, n). \quad (17) \end{aligned}$$

For $c > 0$ and $\delta > 1$ cumulative sums of wear G_N , G_{N+1} , G_{N+2}, \dots always increase after successive N , $N + 1$, $N + 2, \dots$ time units hence cumulative wear process in divergent. For $c \rightarrow 0$ and $0 < \delta < 1$ wear process is convergent and then cumulative wear sums have finally the form (5).

Now we determine wear process $\{f_n\}$.

Equation (7b) has the general solution in the following form:

$$\begin{aligned} y_n &= \frac{C_2}{(n-1)!} (-1)^{n-1} \left(\frac{b}{a}\right)^{n-1} + \frac{d}{a} f_{3n}, \\ f_{3n} &\equiv \sum_{m=1}^{n-1} \frac{(m-1)!}{(n-1)!} \left(-\frac{b}{a}\right)^{n-m-1} \quad \text{for } n = 2, 3, 4, \dots, \quad (18) \end{aligned}$$

where $C_2 = y_1$ denotes the arbitrary constant.

Solution (18) in presented form y_n is introduced into the formula (6b)₂ and hence we obtain the following implication and wear process solution:

$$\begin{aligned} f_{n+1} - f_n &= y_n \\ \Rightarrow f_n &= C_1 + \sum_{k=1}^{n-1} \left[\frac{C_2}{(k-1)!} \left(-\frac{b}{a}\right)^{k-1} + \frac{d}{a} f_{3k} \right], \quad (19) \end{aligned}$$

whereas $f_1 = C_1$, C_2 - arbitrary summation constant, f_{3n} - free term function $f_{31} = f_{32} = 0$.

To determine the particular solution of recurrence Eq. (2) we impose boundary conditions $f_1 = W_{1f}$ and $f_2 = W_{2f}$ in places $n = 1$ and $n = 2$ on the solution (19), and we obtain:

$$C_2 = W_{2f} - W_{1f} = f_2 - f_1 = y_1, \quad C_1 = W_{1f}. \quad (20)$$

We put constant C_2 and C_1 from Eq. (20) into Eq. (19), then the particular solution of recurrence Eq. (2) has the final particular wear value solution in following form [19]:

$$f_n = W_{1f}f_{1n}(\delta, n) + W_{2f}f_{2n}(\delta, n) + \Delta \cdot f_{3n}, \quad (21)$$

where

$$\begin{aligned}
 f_{1n}(\delta, n) &\equiv 1 - \sum_{k=1}^{n-1} \left[\frac{(-\delta)^{k-1}}{(k-1)!} \right], \\
 f_{2n}(\delta, n) &\equiv \sum_{k=1}^{n-1} \left[\frac{(-\delta)^{k-1}}{(k-1)!} \right] \quad n = 2, 3, \dots, \\
 f_{3n}(\delta, n) &\equiv + \sum_{k=1}^{n-1} \left[\sum_{m=1}^{k-1} \frac{(m-1)!}{(k-1)!} (-\delta)^{k-m-1} \right], \\
 &\quad n = 3, 4, \dots,
 \end{aligned}
 \tag{22}$$

whereas $\delta \equiv b/a$, $\Delta \equiv d/a$ and $f_{11} = 1$, $f_{21} = 0$, $f_{31} = 0$, $f_{32} = 0$, $m = 1, 2, \dots, k-1$; $k = 1, 2, \dots, n-1$.

During the infinity time units ($n \rightarrow \infty$) the wear attains the value [16, 19]:

$$\begin{aligned}
 f_{\infty} &= W_{1f}f_{1\infty}(\delta, \infty) + W_{2f}f_{2\infty}(\delta, \infty) + \Delta \cdot f_{3\infty}(\delta, \infty), \\
 f_{1\infty}(\delta, \infty) &= 1 - e^{-\delta}, \quad f_{2\infty}(\delta, \infty) = e^{-\delta}, \\
 f_{3\infty}(\delta, \infty) &= + \sum_{k=2}^{\infty} \left[\sum_{m=1}^{k-1} \frac{(m-1)!}{(k-1)!} (-\delta)^{k-m-1} \right], \\
 &\quad 1 - e^{-\delta} > 0.
 \end{aligned}
 \tag{23}$$

The sum of wear values i.e. cumulative wear values after considered N and infinite time units i.e. for $N \rightarrow \infty$ has the form:

$$\begin{aligned}
 F_N &= \sum_{n=1}^N f_n = W_{1f}F_{1N}(\delta, N) \\
 &\quad + W_{2f}F_{2N}(\delta, N) + \Delta \cdot F_{3N}, \\
 F_{\infty} &= \sum_{n=1}^{\infty} f_n = W_{1f}F_{1\infty}(\delta, \infty) \\
 &\quad + W_{2f}F_{2\infty}(\delta, \infty) + \Delta \cdot F_{3\infty}, \\
 F_{1N}(\delta, N) &\equiv \sum_{n=1}^N f_{1n}(\delta, n), \\
 F_{2N}(\delta, N) &\equiv \sum_{n=1}^N f_{2n}(\delta, n), \\
 F_{3N}(\delta, N) &\equiv \sum_{n=1}^N f_{3n}(\delta, n), \\
 F_{1\infty}(\delta, \infty) &\equiv \sum_{n=1}^{\infty} f_{1n}(\delta, n), \\
 F_{2\infty}(\delta, \infty) &\equiv \sum_{n=1}^{\infty} f_{2n}(\delta, n), \\
 F_{3\infty}(\delta, \infty) &\equiv \sum_{n=1}^{\infty} f_{3n}(\delta, n).
 \end{aligned}
 \tag{25}$$

The sums of wear $F_N, F_{N+1}, F_{N+2}, \dots$ always increase after successive $N, N+1, N+2, \dots$ time units. Hence the considered wear process is divergent.

The particular sums of wear values after successive time units $N = 1, 3, \dots$ have the following form [19]:

$$\begin{aligned}
 F_1 &= W_{1f} + W_{2f} \cdot 0, \quad F_2 = W_{1f} + W_{2f}, \\
 F_N &\equiv W_{1f} \sum_{n=1}^N \left\{ 1 - \sum_{k=1}^{n-1} \left[\frac{(-\delta)^{k-1}}{(k-1)!} \right] \right\} \\
 &\quad + W_{2f} \sum_{n=1}^N \left\{ \sum_{k=1}^{n-1} \left[\frac{(-\delta)^{k-1}}{(k-1)!} \right] \right\} \\
 &\quad + \Delta \cdot \sum_{n=1}^N \left\{ \sum_{k=1}^{n-1} \left[\sum_{m=1}^{k-1} \frac{(m-1)!}{(k-1)!} (-\delta)^{k-m-1} \right] \right\}, \\
 &\quad N = 2, 3, 4, \dots
 \end{aligned}
 \tag{27}$$

$$\begin{aligned}
 \dots\dots\dots \\
 F_{\infty} &\equiv W_{1f} \sum_{n=1}^{\infty} \left\{ 1 - \sum_{k=1}^{n-1} \left[\frac{(-\delta)^{k-1}}{(k-1)!} \right] \right\} \\
 &\quad + W_{2f} \sum_{n=1}^{\infty} \left\{ \sum_{k=1}^{n-1} \left[\frac{(-\delta)^{k-1}}{(k-1)!} \right] \right\} \\
 &\quad + \Delta \cdot \sum_{n=1}^{\infty} \left\{ \sum_{k=1}^{n-1} \left[\sum_{m=1}^{k-1} \frac{(m-1)!}{(k-1)!} (-\delta)^{k-m-1} \right] \right\} \rightarrow \infty.
 \end{aligned}$$

To prove that the sums of wear values after successive time units $N = 1, 2, 3, \dots$ have always the positive values, we write the wear values (22) for $\Delta = 0$ in following form [20]:

$$\begin{aligned}
 f_3 &= W_{1f} + (W_{2f} - W_{1f})(1 - \delta), \\
 f_4 &= W_{1f} + (W_{2f} - W_{1f}) \left[1 + \frac{1}{1!}(-\delta) + \frac{1}{2!}(-\delta)^2 \right], \\
 \dots\dots\dots \\
 f_n &= W_{1f} + (W_{2f} - W_{1f}) \\
 &\quad \cdot \left[1 + \frac{1}{1!}(-\delta) + \frac{1}{2!}(-\delta)^2 + \dots + \frac{1}{(n-1)!}(-\delta)^{n-1} \right],
 \end{aligned}
 \tag{28}$$

and during the infinity time units the wear attain the value:

$$\begin{aligned}
 f_{\infty} &= W_{1f} + (W_{2f} - W_{1f}) \exp(-\delta) \\
 &= W_{1f}(1 - e^{-\delta}) + W_{2f}e^{-\delta}, \quad 1 - e^{-\delta} > 0.
 \end{aligned}
 \tag{29}$$

whereas $W_{2f} > W_{1f}$.

4. Summation factor tools for wear process determination

Let l.h.s. of the following second order non-homogeneous recurrence Eq. (3) describing wear process $\{h_n\}$ for $n = 1, 2, \dots$ and with free term R_{hn} :

$$h_{n+2} - P_{hn}h_{n+1} - Q_{hn}h_n = R_{hn}, \tag{30}$$

is the total sum or the total differential. In this case we can choose such coefficients L_n, M_n that the following equality is true [19, 20]:

$$S_{\varepsilon\rho}^1(L_n h_{n+1} + M_n h_n) = h_{n+2} - P_{hn} h_{n+1} - Q_{hn} h_n = R_{hn}. \quad (31)$$

Unified Operator of Summation (UOS) of first rank imposed on the complex number (...) is defined in the following form [19, 20]:

$$S_{\varepsilon\rho}^1(\dots) \equiv S_0^1(\dots) + \varepsilon\rho(\dots), \quad (32)$$

whereas Unitary Translation Operator (UTO) gives:

$$S_0^1(h_n) \equiv h_{n+1}, \quad S_0^0(h_n) \equiv h_n. \quad (33)$$

We denote: ρ – complex number of the complex variable, $\varepsilon\rho$ – basis of the unified operator of summation S in particular case is equal $\varepsilon\rho = +1$ or $\varepsilon\rho = -1$, 1 – the first rank as an upper index of the operator of the unified summation, -1 – the first rank as an upper index of the reciprocal operator of the unified summation.

Imposing reciprocal first order operator UOS on the both sides of Eq. (31) we obtain [19]:

$$L_n h_{n+1} + M_n h_n = S_{\varepsilon\rho}^{-1}(R_{hn}) + C(-\varepsilon\rho)^n. \quad (34)$$

The symbol $S_{\varepsilon\rho}^{-1}$ is the reciprocal unified operator of summation UOS with the basis $\varepsilon\rho$ and C -arbitrary summation constant. If the equality (30) not valid i.e. total sum not exists, we can find the summation factor U_n for Eq. (30). After multiplication of both sides of the Eq. (30) by this factor, we can always find such coefficients L_n^* , M_n^* that following equality is true [20]:

$$S_{\varepsilon\rho}^1(L_n^* h_{n+1} + M_n^* h_n) = U_n h_{n+2} - U_n P_{hn} h_{n+1} - U_n Q_{hn} h_n = U_n R_{hn}. \quad (35)$$

Imposing the reciprocal first order operator UOS on the both sides of Eq. (35) we obtain [20]:

$$L_n^* f_{n+1} + M_n^* f_n = S_{\varepsilon\rho}^{-1}(U_n R_n) + C(-\varepsilon\rho)^n. \quad (36)$$

Reciprocal UOS is always univocal because [20]:

$$S_{\varepsilon\rho}^{-1}[C(-\varepsilon\rho)^n] \equiv 0. \quad (37)$$

Formulae (34) and (36) present the first order non-homogeneous recurrent equation with variable coefficients.

5. Wear process resolved by means of summation factor

Now we are going to the determination of the wear process sequence $\{h_n\}$ for $n = 1, 2, 3, \dots$ i.e. the wear solution from the Eq. (3) presented in the following form [20]:

$$\left(1 + \frac{1}{n}\right) h_{n+2} + \left(n + \frac{1}{n}\right) h_{n+1} - 2n \cdot h_n = d \quad (38)$$

$$\Leftrightarrow h_{n+2} = -\frac{n^2 + 1}{n + 1} h_{n+1} + \frac{2n^2}{n + 1} h_n + \frac{d \cdot n}{n + 1},$$

where $d \neq 0$ denotes an arbitrary coefficient independent of n . We take into account wear solution for known boundary conditions i.e. wear values $h_1 = W_{1h}$, $h_2 = W_{2h}$ in two successive time units $n = 1$ and $n = 2$. In abovementioned problem L.h.s. of recurrence Eq. (38) does not present the total sum.

Symbol $U_n \equiv n$ is the summation factor, because multiplying by n both sides of Eq. (38) we obtain:

$$(n + 1) h_{n+2} + (n^2 + 1) h_{n+1} - 2n^2 h_n = d \cdot n \quad (39)$$

and by virtue of UOS transformation [20] the following equality is true:

$$S_{-2}^1(n \cdot h_{n+1} + n^2 h_n) = (n + 1) h_{n+2} + (n + 1)^2 h_{n+1} - 2n \cdot h_{n+1} - 2n^2 h_n \quad (40)$$

$$= (n + 1) h_{n+2} + (n^2 + 1) h_{n+1} - 2n^2 h_n.$$

Hence Eq. (39) can be written in the following form:

$$S_{-2}^1(n \cdot h_{n+1} + n^2 h_n) = d \cdot n. \quad (41)$$

When a reciprocal UOS operator is imposed on the both sides of Eq. (41), we obtain the following expression [20]:

$$n \cdot h_{n+1} + n^2 h_n = S_{-2}^{-1}(d \cdot n). \quad (42)$$

By virtue of properties of reciprocal UOS operator, we obtain [20]:

$$S_{-2}^{-1}(d \cdot n) = d \cdot n \cdot J - d \cdot J^2 + C_3 \cdot [-(-2)]^n, \quad (43)$$

$$J \equiv \frac{1}{1 + \varepsilon\rho} = \frac{1}{1 - 2} = -1.$$

Symbol C_3 denotes the first summation constant. Equation (42) has the form:

$$n \cdot h_{n+1} + n^2 \cdot h_n = -d \cdot n - d + C_3 \cdot 2^n. \quad (44)$$

Dividing both sides of Eq. (44) by n , we obtain:

$$h_{n+1} + n \cdot h_n = -d \left(1 + \frac{1}{n}\right) + C_3 \cdot \frac{2^n}{n}. \quad (45)$$

The general solution of Eq. (45) has after transformation the following form [19–23]:

$$h_n = (-1)^{n-1} \cdot (n - 1)! \left\{ C_4 + \sum_{k=1}^{n-1} \left[(-1)^k \cdot \frac{C_3 \cdot 2^k - d \left(1 + \frac{1}{k}\right)}{k!} \right] \right\}. \quad (46)$$

$n = 2, 3, \dots$ where $h_1 = C_4$. On solution (46) we impose boundary condition $h_1 = W_{1h}$ for $n = 1$ and $h_2 = W_{2h}$ in place $n = 2$, hence we obtain the following summation constants:

$$C_4 = W_{1h}, \quad (47)$$

$$C_3 = d + 0.5(W_{1h} + W_{2h}).$$

In presented formula (46) we can now show a linear combination of two linear independent particular solutions of the homogeneous equation plus the particular solution of non-homogeneous recurrence accordingly with Eq. (38). Above-mentioned linear combination of particular solutions is written in following form [20]:

$$\begin{aligned}
 h_n &= W_{1h} h_{1n} + W_{2h} h_{2n} + h_{3n}, \quad n = 1, 2, 3, \dots, \\
 h_{1n} &= (-1)^{n-1} (n-1)! \left[1 + \sum_{k=1}^{n-1} (-1)^k \frac{2^{k-1}}{kk!} \right], \\
 &\quad n = 2, 3, \dots, \\
 h_{2n} &= (-1)^{n-1} (n-1)! \sum_{k=1}^{n-1} (-1)^k \frac{2^{k-1}}{kk!}, \quad (48) \\
 &\quad n = 2, 3, \dots, \\
 h_{3n} &= d(-1)^{n-1} (n-1)! \sum_{k=1}^{n-1} \frac{(-1)^k}{kk!} (1+k-2^k), \\
 &\quad n = 2, 3, \dots, k = 1, 2, \dots, n-1, \\
 &\quad h_{11} = 0, \quad h_{21} = 0, \quad h_{31} = 0.
 \end{aligned}$$

6. The general numerical implementations

The numerical calculations are performed using Mathcad 15 Professional Program.

To illustrate the cumulative dimensionless wear values G_{1N} , G_{2N} after $N = 10, 15, 20, 25, 30$ time units presented by sums (16), (17) taking into account results (13) we perform the numerical calculations.

The sum of dimensionless wear values computation results for functions G_{1N} , G_{2N} after $N = 10, 15, 20, 25, 30$ time units and for dimensionless ratio $0.3 < \delta < 1$ and $0.1 \leq c < 1$ are presented in Figs. 1–3.

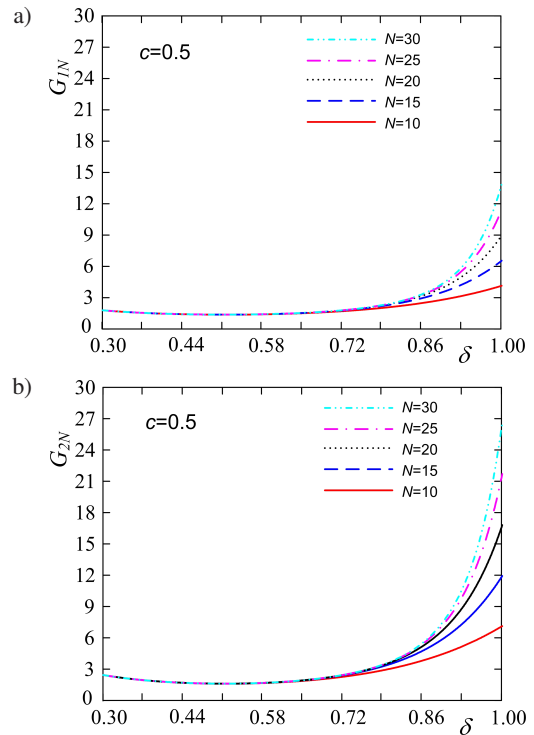


Fig. 2. The cumulative dimensionless wear values: a) for G_{1N} , b) for G_{2N} after $N = 10, 15, 20, 25, 30$ time units of operations versus dimensionless parameter ratio $\delta = b/a$ inside interval $(0.30, 1.0)$ presenting bearing material properties and operating conditions with $d = 0.50$

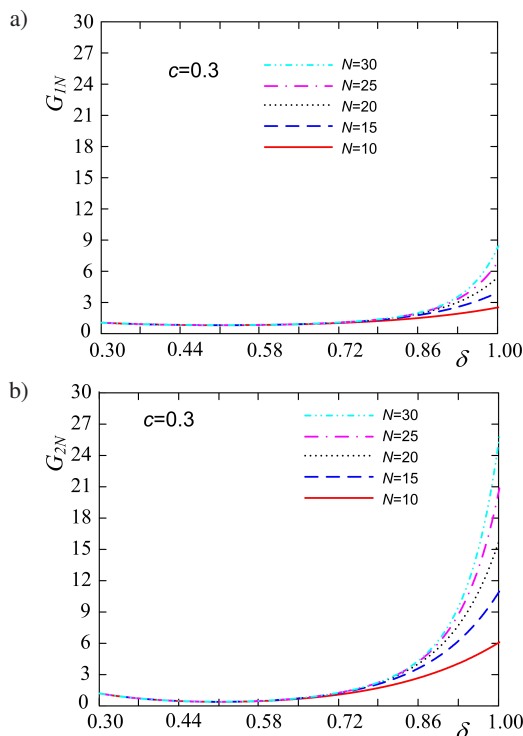


Fig. 1. The cumulative dimensionless wear values: a) for G_{1N} , b) for G_{2N} after $N = 10, 15, 20, 25, 30$ time units of operations versus dimensionless parameter ratio $\delta = b/a$ inside interval $(0.30, 1.0)$ presenting bearing material properties and operating conditions with $c = 0.30$

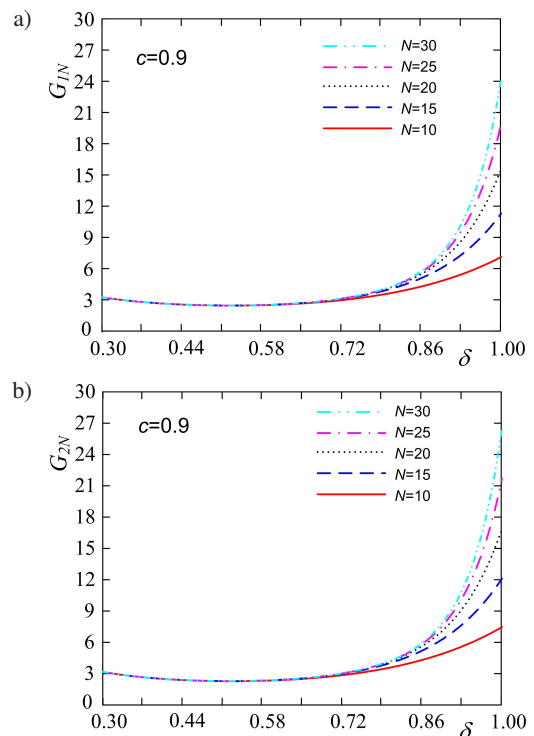


Fig. 3. The cumulative dimensionless wear values: a) for G_{1N} , b) for G_{2N} after $N = 10, 15, 20, 25, 30$ time units of operations versus dimensionless parameter ratio $\delta = b/a$ inside interval $(0.30, 1.0)$ presenting bearing material properties and operating conditions with $d = 0.90$

Sums of dimensionless wear values G_{1N} , G_{2N} , are convergent if time unit N increases. To obtain bearing dimensional wear values after N time units we must multiply the dimensionless values G_{1N} , G_{2N} , indicated in Fig. 1–3 by the in experimental way obtained dimensional wear values W_{1g} , W_{2g} in μm^3 respectively. The cumulative wear value after finite N or infinite time units of operation has finally the form (16).

By virtue of performed calculations we can see that the largest cumulative wear values increase after N time units of bearing operations are observed for the dimensionless ratio $\delta = b/a$ occurring in interval from 0.85 to 1.00 for G_{1N} , G_{2N} . Moreover the increases of the ratio δ imply consequently the cumulative wear increments presented by the values G_{1N} , G_{2N} . This fact is very easy to explain, because ratio δ increases for the bearing material especially hardness decreases.

Wear divergence process increases if increases dimensionless parameter c depending on standard deviation of vibration frequencies.

To illustrate the cumulative dimensionless wear values F_{1N} , F_{2N} , F_{3N} after $N = 10, 15, 20, 25, 30$ time units presented by sums (23), (24), (25) taking into account results (27) we perform the numerical calculations. The sum of dimensionless wear values computation results for functions F_{1N} , F_{2N} , F_{3N} after $N = 10, 15, 20, 25, 30$ time units and for dimensionless ratio $1 < \delta < 3$ are presented in Figs. 4–9.

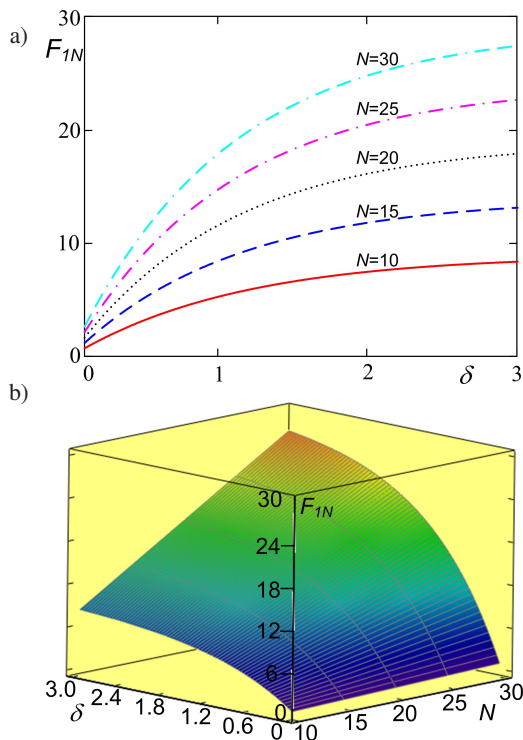


Fig. 4. The cumulative wear dimensionless values F_{1N} after N time units of operations versus dimensionless parameter ratio $\delta = b/a$ in interval (0,3) presenting bearing material properties and operating conditions: a) two dimensional chart of dependencies, b) three dimensional view

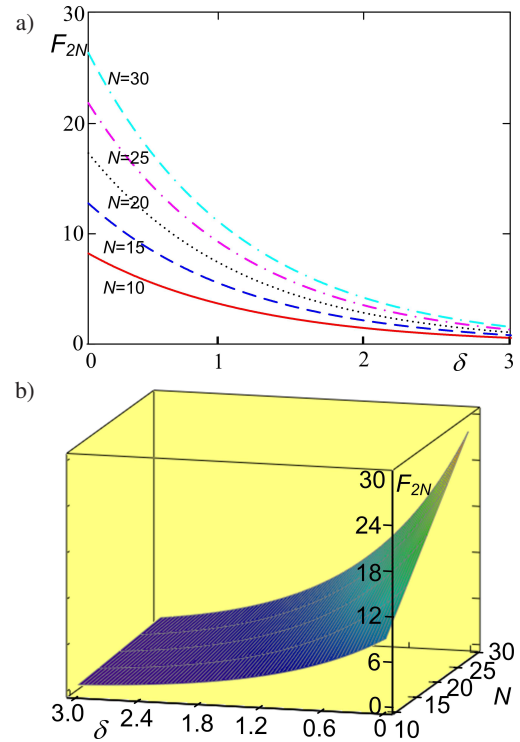


Fig. 5. The cumulative wear dimensionless values F_{2N} after N time units of operations versus dimensionless parameter ratio $\delta = b/a$ in interval (0,3) presenting bearing material properties and operating conditions: a) two dimensional chart of dependencies, b) three dimensional view

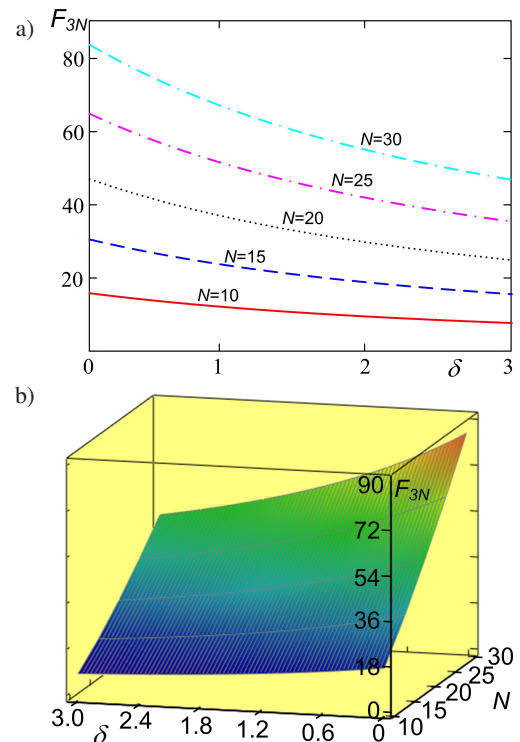


Fig. 6. The cumulative wear dimensionless values F_{3N} after N time units of operations versus dimensionless parameter ratio $\delta = b/a$ in interval (0,3) presenting bearing material properties and operating conditions: a) two dimensional chart of dependencies, b) three dimensional view

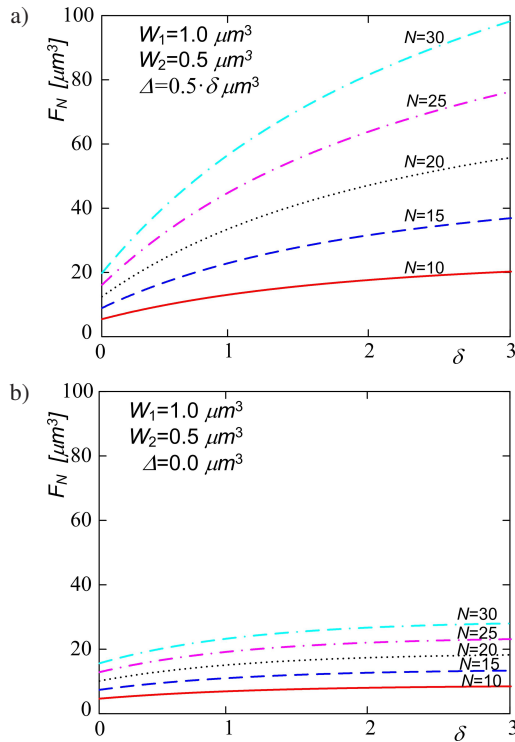


Fig. 7. The cumulative dimensional wear values: a) for interruption mode of bearing operation $\Delta > 0$, b) for continuous mode $\Delta = 0$, after N time units of operation versus dimensionless ratio $\delta = b/a$ presenting bearing material properties for $W_{1f} = 1.0 \mu\text{m}^3$, $W_{2f} = 0.5 \mu\text{m}^3$

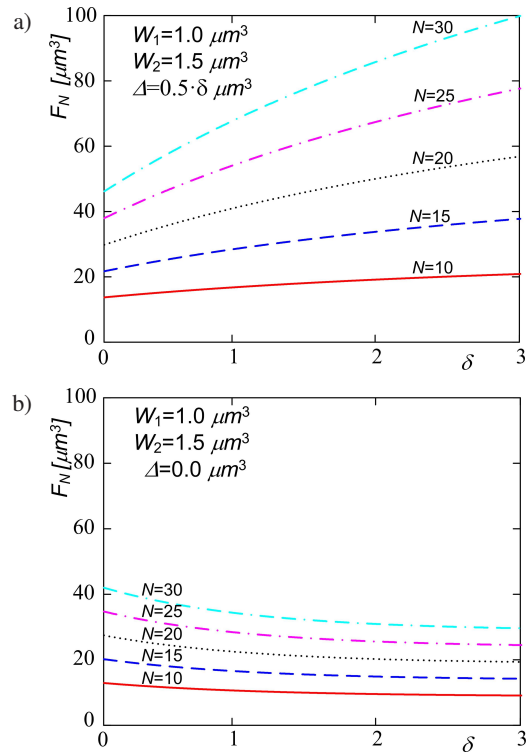


Fig. 9. The cumulative dimensional wear values: a) for interruption mode of bearing operation $\Delta > 0$, b) for continuous mode $\Delta = 0$, after N time units of operation versus dimensionless ratio $\delta = b/a$ presenting bearing material properties and operating conditions for $W_1 = 1.0 \mu\text{m}^3$, $W_2 = 1.5 \mu\text{m}^3$

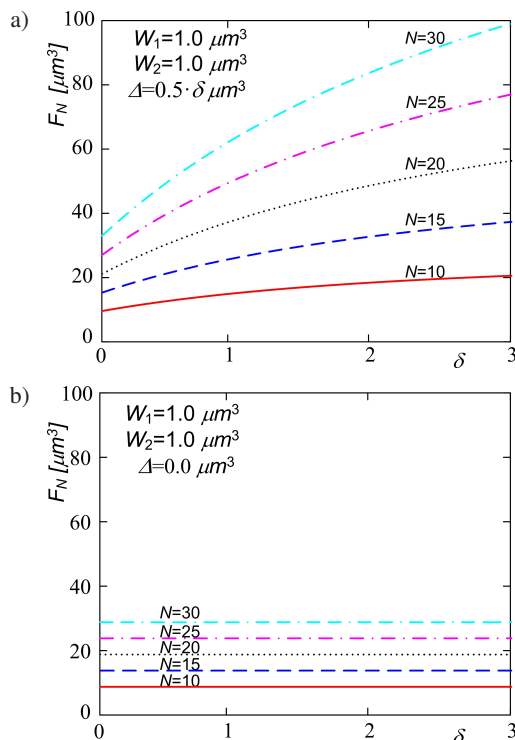


Fig. 8. The cumulative dimensional wear values: a) for interruption mode of bearing operation $\Delta > 0$, b) for continuous mode $\Delta = 0$, after N time units of operation versus dimensionless ratio $\delta = b/a$ presenting bearing material properties and operating conditions for $W_{1f} = 1.0 \mu\text{m}^3$, $W_{2f} = 1.0 \mu\text{m}^3$

It is evident that the values of sums F_{1N} , F_{2N} , F_{3N} in each case are increasing if the number of time units N increases. By virtue of performed calculations we can see that the largest cumulative wear values increase after N time units of bearing operations are observed for the dimensionless ratio $\delta = b/a$ occurring in interval from 2 to 3 for F_{1N} see Fig. 4 and from 0 to 1 for F_{2N} and F_{3N} see Figs. 5, 6. Therefore the influence of the δ ratio value on the final cumulative wear value, will be decide after calculations presenting in Figs. 7–9.

To obtain bearing wear dimensional values after N time units we must multiply the dimensionless values F_{1N} , F_{2N} , F_{3N} indicated in Figs. 4–6 by the in experimental way obtained dimensional wear values W_{1f} , W_{2f} , Δ in μm^3 respectively. Value $\Delta = d/a$ describes various operation conditions. The cumulative wear dimensional value after finite N or infinite time units of operation has finally the form (27).

To obtain the synchronous influences of the dimensionless $\delta = b/a$ ratio value presenting various properties of bearing materials and influences of bearing operation conditions presented by parameter dimensional Δ on the final cumulative wear value, we perform the additionally calculations in Figs. 7–9.

For example the final cumulative dimensional wear values after $N = 10, 15, 20, 25, 30$ time units of operation time and for selected dimensional wear values W_{1f} , W_{2f} in two first time units as well for selected dimensional values Δ in μm^3 are illustrated in Figs. 7–9.

7. Particular calculation example

In two first successive time units slide bearing journal or of artificial human hip joint attains wear values W_{1f} [μm^3], W_{2f} [μm^3]. Determine sum of the wear after ten time units if we know that in sliding nod we have following parameters obtained in experimental way: $\delta = 0.5$; $d = 0$, $\Delta = 0$. After measurements we obtain wear (during 100 days) after first time unit $W_{1f} = 0.1 \mu\text{m}^3$ and wear during the next time unit $W_{2f} = 0.2 \mu\text{m}^3$. Determine wear after 1000 days i.e. after $N = 10$ time units.

Solutions:

From charts presented in Fig. 4a,b and Fig. 5a,b for $\delta = 0.5$ and $N = 10$ we read $F_{1,10} = 3.2379$ and $F_{2,10} = 5.7621$. In other hand by virtue of the Eqs. (22), (24), (25) the total wear value after ten time units attain value:

$$\begin{aligned} F_{10} &= 0.1\mu\text{m}^3 \sum_{n=1}^{10} f_{1n}(\delta, n) + 0.2\mu\text{m}^3 \sum_{n=1}^{10} f_{2n}(\delta, n) \\ &= 0.1\mu\text{m}^3 \cdot 3.2379 + 0.2\mu\text{m}^3 \cdot 5.7621 \\ &= 1.47621 \mu\text{m}^3. \end{aligned} \quad (49)$$

Abovementioned example can be applies for wear determination during operation of slide journal bearing as well operation of human artificial hip joint (end prosthesis) [24].

8. Finally remarks

From calculations presented in Figs. 7–9 we show:

- The trends of the marked increases of final cumulative wear values of bearing surfaces for the hardness decreases of bearing material i.e. δ increases and after interruption mode increases of bearing exploitation (i.e. for positive $\Delta > 0$ increases);
- The tendencies of the very small changes of final cumulative wear values of bearing surfaces for the hardness decreases of bearing material i.e. δ increases and after continuous mode of bearing exploitation (and $\Delta = 0$);
- The inclinations of the increases of final cumulative wear values of bearing surfaces for the operation time units N increases;
- The tendencies of the increases of final cumulative wear values of bearing surfaces for the interrupted bearing operation mode increases (i.e. for $\Delta > 0$ increases).

From calculations presented in Figs. 1–3 we show:

- The trends of the increases of final cumulative wear values of bearing surfaces for the standard deviation increases of vibration frequencies (i.e. for c dimensionless increases).

9. Conclusions

1. In this paper the bearing wear prognosis was performed for the wear process where after sufficient many time units of the exploitation process, the wear value in the next time units is equal to the wear value in previous time unit multiplied by the various form of experimental determined functions depended on a bearing material, an operation mode,

vibration conditions where behavior of such functions has various limits obtained in an experimental way after sufficient many time units of an operation time.

2. The application of the presented theory is devoted to the analytical methods for wear slide bearing divergence and convergence course prognosis in particular time units of operating time and after sufficient many time units of bearing operation. Moreover, the kind and velocity of divergence and convergence process of the sum of wear i.e. cumulative values after arbitrary time units of bearing exploitation has been considered. In the case of convergence wear processes, the cumulative sums of wear has been obtained in analytical form.
3. The presented analytical tools for the mentioned thesis examination are constituted by the method of solution of the second order non homogeneous recurrence equations with variable coefficients with various limits tendencies after many time units of operation time which are determined from the measurements. This fact enables to control the physical characteristics of bearing materials, vibration influences, bearing operation materials that could add a new unknown influence on the bearing wear.
4. The recurrence equations determining the wear values of slide journal bearing are solved by means of the new methods namely UOS transformation, replacement of variables and summation factor presented by the authors in foregoing papers [20, 23]. The method of a summation factor is in this paper implemented by the reciprocal UOS (Unified Operator of Summation) tool derived by the author and applied for the analytical solutions of the recurrent non-homogeneous second order equation with variable coefficients and a variable free term.

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