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Monte Carlo method for reliability of parallel system with dependent failures of component

Keywords

reliability, stochastic process, Weibull distribution, dependency

Abstract

A problem of parallel system reliability with dependent failures of components is presented in the paper. It is assumed that lifetimes of components are independent random variable having Weibull distribution. We take under consideration a parallel (in reliability meaning) system consisting of *n* independent at the beginning of work and identical components. We assume that a load of the working system affects on the reliability of its components and the load of the system is distributed on all working components. Therefore, a failure rate of each component is changeable during run of the system and depends on a number of working elements at this point in time. As a model of the system failures we construct a stochastic process which value at the moment *t* denotes the number of working components. Generally it is neither Markov nor semi-Markov process. To assess the reliability characteristics of the system we simulate this stochastic process using the Monte-Carlo method and we calculate values of nonparametric kernel density and reliability functions estimators.

1. Introduction

Reliability of most real systems depend on the load of its component. Problem of parallel system reliability with depend failures of components exponentially distributed was considered in [1]. In [2] a multistate approach to the reliability analysis of series systems with dependent components according to the local load sharing rule is proposed. In case of Weibull distribution this problem was studied in [3], [4]. In this paper an extension of that problem is discussed. We take under consideration a parallel (in reliability meaning) system consisting of *n* independent at the beginning and identical components. We assume that a load of the working system affects the reliability of its components and the load of the system is distributed on all working components. From these assumptions it follows that a failure rate of each component is changeable during running of the system and depends on number of working elements at this point in time. As a model of the system failures we construct a stochastic process the value of which at the moment *t* denotes the number of working components. Generally it is neither Markov nor semi-Markov process. To assess the reliability characteristics of the system we simulate the stochastic process using Monte-Carlo method and we calculate values of the nonparametric kernel density and reliability functions estimators.

2. Assumptions

We suppose that the reliability of a parallel system consisting of *n* components depend on their load. We assume that a load *L* of the system is distributed on all working components. It means that a failure rate of each component (element) is changeable during work of the system and depends on the number of working elements at this point in time. We assume that times to failure of all starting the work components are identical independent random variables having Weibull distribution with parameters α and λ. Hence, the failure rate of each element is given by the formula

$$\lambda(t) = \lambda\,\alpha\,t^{\alpha-1}, \quad t \geq 0, \quad \lambda > 0, \quad \alpha > 0, \tag{1}$$

and the reliability function of each element is given by

$$R(t) = \exp[-\lambda\,t^{\alpha}],\ i = 1,...,n, \\ t \geq 0,\ \lambda > 0, \alpha > 0. \tag{2}$$

We assume that a scale parameter of an element failure rate is dependent on its load. It is assumed that each component of the system with i working and n-i failing components has the failure rate with the scale parameters given by the function

$$\lambda(i) = \frac{n}{n-i+1}\lambda, \quad i = 1,2,...,n$$

We also assume that a shape parameter of each component depends on a number of the working components and it is determined by three kinds of functions

$$\alpha(i), \quad i = 1,..., n: \tag{4}$$

$$\alpha(i) = \alpha \cdot \frac{n}{n-i+1}, \; i = 1,...,n, \tag{5}$$

$$\alpha(i) = \alpha[1 + (i-1) \cdot r], \;\; r \geq 0, \, i = 1,..., n, \tag{6}$$

$$\alpha(i) = \alpha \cdot q^{i-1}, \;\; q \geq 1, \;\; i = 1,..., n. \tag{7}$$

The failure rate of component changes during operation of the system and depends on the number of working elements at the moment

$$\lambda(t, i) = \lambda(i)\alpha(i)\, t^{\alpha(i)-1}, \quad t \geq 0 \tag{8}$$

We are going to investigate the Monte Carlo model for three kinds of functions describing shape parameter $\alpha(i), i = 1,..., n$. These functions are given by (5), (6) and (7).

3. Monte Carlo Model

Let $\{X(t): \; t \geq 0\}$ be a stochastic process in which the state at the moment t denotes the number of working components. A set $S = \{0,1,\ldots,n\}$ is the state space of the process. This process is defined by the formula

$$X(t) = n - i \;\text{ for }\; t \in [\tau_i, \tau_{i+1}), \, i = 1,2,..., n \tag{9}$$

where
$0 = \tau_0 < \tau_1 < \tau_2 < ... < \tau_n < \tau_{n+1} = \infty$

denote moments of the system state changes. A trajectory of that process is shown in *Figure 1*.
From (9) it follows that simulation of the process trajectory is equivalent to the simulation of the realization of the system state changes moments. Notice that

$$\tau_1 = T^{(1)} = \min\{T_1^{(1)}, T_2^{(1)},..., T_n^{(1)}\}, \tag{10}$$

where $T_1^{(1)}, T_2^{(1)},..., T_n^{(1)}$ are independent random variables having identical Weibull distribution with parameters $\alpha(1)$ and λ. Hence

$$F_1^{(1)}(t) = P(T_1^{(1)} \leq t) = 1 - \exp[-n\lambda t^{\alpha(1)}], \tag{11}$$
$$i = 1,2,\ldots, n, \; t \geq 0, \, \lambda > 0, \, \alpha > 0.$$

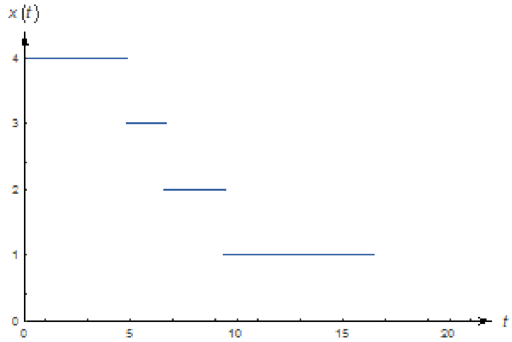

Figure 1. Trajectory of the stochastic process $\{X(t): t \geq 0\}$

To obtain a generator of the random variable value t_1 we have to solve an equation

$$u_1 = 1 - \exp[-n\lambda t_1^{\alpha(1)}], \, t_1 > 0, \, u_1 \in (0, 1) \tag{12}$$

with respect to t_1, where the number u_1 is a realization of a random variable U_1 uniformly distributed on the interval (0, 1).
A solution of the equation is

$$t_1 = [-\frac{1}{n\lambda}\ln(1-u_1)]^{\frac{1}{\alpha}}, \, u_1 \in (0, 1). \tag{13}$$

This number is the realization of the random variable $T^{(1)}$.
Let $T_j^{(2)}, \, j = 1,2,..., n-1$ be time to failure of the component j for the system with n-1 working components and let $T_{j,t_1}^{(2)}, \, j = 1,2,..., n-1$ denotes time to failure of the component j since the moment t_1. Notice that

$$P(T_{j,t_1}^{(2)} > t) = P(T_j^{(2)} - t_1 > t \,|\, T_i^{(2)} > t_1) =$$
$$= \frac{P(T_j^{(2)} > t + t_1)}{P(T_j^{(2)} > t_1)} = \frac{\exp[-\lambda(2)(t+t_1)^{\alpha(2)}]}{\exp[-\lambda(2) t_1^{\alpha(2)}]}.$$

The second failure of the component takes place after the time t_2, which is a value of the random variable

$$T_{t_1}^{(2)} = \min\{T_{1,t_1}^{(2)}, T_{2,t_1}^{(2)},..., T_{n-1,t_1}^{(2)}\} \tag{14}$$

which has a distribution

$$
\begin{aligned}
&P(T_{t_1}^{(2)} > t) = \\
&= P(T_{1,t_1}^{(2)} > t)P(T_{2,t_1}^{(2)} > t)...P(T_{n-1,t_1}^{(2)} > t) = \\
&= \frac{\exp[-(n-1)\lambda(2)(t+t_1)^{\alpha(2)}]}{\exp[-(n-1)\lambda(2)t_1^{\alpha(2)}]} = \\
&= \frac{\exp[-(n-1)\lambda(2)(t+t_1)^{\alpha(2)}]}{\exp[-(n-1)\lambda(2)t_1^{\alpha(2)}]} = \\
&= \exp[-(n-1)\lambda(2)(t+t_1)^{\alpha(2)} - (n-1)\lambda(2)t_1^{\alpha(2)})].
\end{aligned}
$$

We obtain the realization of the random variable $T_{t_1}^{(2)}$ from solving the equation

$$
\begin{aligned}
&u_2 = 1 - \exp[-(n-1)\lambda(2)(t+t_1)^{\alpha(2)} \\
&-(n-1)\lambda(2)t_1^{\alpha(2)})]
\end{aligned}
$$

with respect to t_2. A number u_2 denotes a realization of the random variable U_2 with uniform distribution on (0, 1). Hence

$$
\begin{aligned}
&t_2 = [-\frac{1}{(n-1)\lambda(2)}\ln(1-u_2) + t_1^{\alpha(2)}]^{\frac{1}{\alpha(2)}} - t_1, \\
&u_2 \in (0,1).
\end{aligned}
$$

A number

$$
\begin{aligned}
&\boldsymbol{\tau}_2 = t_1 + t_2 = \\
&\left[-\frac{1}{(n-1)\,\lambda(2)}\ln(1-u_2) + t_1^{\alpha(2)}\right]^{\frac{1}{\alpha(2)}}, \quad u_2 \in (0,1)
\end{aligned}
$$

is a realization of the random variable $\boldsymbol{\tau}_2$ which denotes an instant of the second failure of the system component. Similarly, the *i*-th failure of the system component takes place after time t_i, which is a value of the random variable

$$
T_{t_{i-1}}^{(i)} = \min\{T_{1,t_{i-1}}^{(i)}, T_{2,t_{i-1}}^{(i)}, ..., T_{n-i+1,t_{i-1}}^{(i)}\}
$$

which has a distribution

$$
\begin{aligned}
&P(T_{t_{i-1}}^{(i)} > t) = \\
&P(T_{1,t_{i-1}}^{(2)} > t)P(T_{2,t_{i-1}}^{(2)} > t)...P(T_{n-i+1,t_{i-1}}^{(2)} > t) = \\
&= \frac{\exp[-(n-i+1)\cdot\lambda(i)\cdot(t+t_1)^{\alpha(i)}]}{\exp[-(n-i+1)\cdot\lambda(i)\cdot t_{i-1}^{\alpha(i)}]} = \\
&= \exp[-(n-i+1)\cdot\lambda(i)\cdot(t+t_{i-1})^{\alpha(i)} \\
&-(n-i+1)\cdot\lambda(i)\cdot t_{i-1}^{\alpha(i)})].
\end{aligned} \qquad (18)
$$

We obtain the realization of the random variable $T_{t_{i-1}}^{(i)}$ by solution of the equation (15)

$$
\begin{aligned}
&u_i = 1 - \exp[-(n-i+1)\cdot\lambda(i)\cdot(t_i + t_{i-1})^{\alpha(i)} \\
&-(n-i+1)\cdot\lambda(i)\cdot t_{i-1}^{\alpha(i)})]
\end{aligned} \qquad (19)
$$

with respect to t_i. A number u_i denotes a realization of the random variable U_i with uniform distribution on (0, 1). Hence we obtain a generator of the state changes instants of the process $\{X(t):\ t \ge 0\}$:

$$
\begin{aligned}
&\tau_i = t_1 + t_2 + ... + t_i = \\
&= \left[-\frac{1}{(n-i+1)\,\lambda(i)}\ln(1-u_i) + t_{i-1}^{\alpha(i)}\right]^{\frac{1}{\alpha(i)}}, \\
&u_i \in (0,1), \quad i = 1,2,...,n.
\end{aligned} \qquad (20)
$$

(16)

The lifetime of the system is equal to the random variable $\boldsymbol{\tau}_n$.

For the scale parameters defined by (3):

$$
\lambda(i) = \frac{n}{n-i+1}\lambda, \quad i = 1,2,...,n
$$

we have

$$
\tau_i = \left[-\frac{1}{n\,\lambda}\ln(1-u_i) + t_{i-1}^{\alpha(i)}\right]^{\frac{1}{\alpha(i)}}, i = 1,2,...,n. \qquad (21)
$$

That formula allows to construct both generator of the failure process and the lifetime of the system as well.

4. Generator of the failure process

To generate the realizations of that random variable we construct the computer program in MATHEMATICA system using Monte Carlo Method. (17) This program allows to compute moments of the system component failure, instant of the system failure and enables to calculate a mean time to failure and standard deviation of the system. Moreover that program plots figures of a value of the density lifetime estimator and corresponding reliability function of the system. To achieve this we apply the empirical nonparametric estimator with the Gaussian kernel. Recall that the real function $K(u)$ which takes the non-negative values and satisfies the condition

$$\int_{-\infty}^{\infty} K(u) = 1$$

is called a statistical kernel.
A function

$$\hat{f}(x) = \frac{1}{m \cdot h} \sum_{k=1}^{m} K\left(\frac{x - \tau_n(k)}{h}\right)$$

is said to be a value of the non-parametric kernel estimator of a density function $f(x)$. A number h is called a bandwidth or a smoothing parameter or a window. The choice of the kernel and the choice of the bandwidth is significant problem of density estimation. One of well known proposal is a Gaussian kernel

$$K(u) = \frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}}, \quad u \in (-\infty, \infty)$$

A number $h = 1.06\ s^{-0.2}$ where s is the empirical standard deviation is a practical estimate of the bandwidth in this case. The empirical standard deviation is determined by the rule

$$s = \sqrt{\frac{1}{m} \sum_{k=1}^{m} (\tau_n(k) - \bar{\tau}_n(k))^2} .$$

Generator of the failure process reliability characteristics

Main part of the program for the scale parameter $\lambda(i) = \frac{n}{n-i+1}\lambda$ ***and the shape parameter*** $\alpha(i) = \alpha + (i-1) \cdot r$.

```
Print["Number of the system elements
      n=",n=4]
Print["Size of the sample m=",m=200]
Print["Shape parameter of the Weibull
      distribution α=",b=1.2]
Print["Parameter =", r=0.2]
Print["Scale parameter of the Weibull
      distribution λ=",λ=0.0005 ]
Print["Number of the system elements
       n=",n=4]
Print["Size of the sample m=",m=200]
Print["Shape parameter of the Weibull
      distribution α=",b=1.2]
Print["Parameter =", r=0.2]
Print["Scale parameter of the Weibull
       distribution λ=",λ=0.0005 ]
Do[For[i=1,i<=n,i++, u[0]=0;
Print["u[",i,"]=",u[i]=Random[Real,{0,1
                                     }]];
a[0]=0;
Print["a[",i,"]=",
a[i]=b+b(i-1)r];//N;
y[0]=0;
y[i]=(-Log[1-u[i]])/n λ +(y[i-1])
Print["y[",i,"]=", y[i] ]];
Print["τn[",k,"]=",
       w[k]=y[n]], {k,1,m}];
Print["Mean time to failure = ",
       ys=(Sum[w[k],{k,1,m}])/m//N];
Print["Second moment=",
      dm=(Sum[(w[k])2,{k,1,m}])/m //N];
Print["Standard deviation=", s=(dm-
      ys2)0.5//N];
Print["Window h=", h=1.06 *s*
       1/m0.2//N];
f[x_]:=1/(2Pi)0.5 Exp[-(x2/2)]
g[x_]:=1/(m h) Sum[ f[(x-w[i])/h]//N,
                    {i,1,m}]
Plot[g[x],{x,0,400}]
Plot[d[z]=1-NIntegrate[g[x], {x,0,z}],
     {z,0,400}]
```

5. Results of simulation

For the sample size of m = 100 we have computed parameters of system reliability like the mean time to failure and the standard deviation. For

$$\alpha = 1.2, \quad \lambda = 0.0005, \quad r = 0.2, \quad q = 1.21$$
$$b = 1.2 \quad , n = 4.$$

the results are shown in *Table 1*.

Table 1. Mean time to failure and standard deviation of the system.

	$\lambda(i),\ i=1,\dots,n$ / $\alpha(i),\ i=1,\dots,n$	$\lambda(i) = \frac{n}{n-i+1}\lambda$
1	$\alpha(i) = \alpha \cdot \frac{n}{n-i+1}$	$E(\tau_n) = 186{,}74$ $D(\tau_n) = 122.45$
2	$\alpha(i) = \alpha \cdot [1 + (i-1) \cdot r]$	$E(\tau_n) = 224.60$ $D(\tau_n) = 118{,}48$
3	$\alpha(i) = \alpha \cdot q^{i-1}$	$E(\tau_n) = 180{,}99$ $D(\tau_n) = 99.10$

For

$$\alpha = 1.2, \quad \lambda = 0.001 \quad r = 0.2, \quad q = 1.21$$
$$b = 1.2, \quad n = 4.$$

the results are shown in *Table 2*.

Table 2. Mean time to failure and standard deviation of the system.

	$\lambda(i),\ i=1,\dots,n$ / $\alpha(i),\ i=1,\dots,n$	$\lambda(i) = \dfrac{n}{n-i+1}\lambda$
1	$\alpha(i) = \alpha \cdot \dfrac{n}{n-i+1}$	$E(\tau_n) =$ 101.69 $D(\tau_n) =$ 75.36
2	$\alpha(i) = \alpha \cdot [1 + (i-1)r]$	$E(\tau_n) =$ 148.64 $D(\tau_n) =$ 77.65
3	$\alpha(i) = \alpha \cdot q^{i-1}$	$E(\tau_n) =$ 120,52 $D(\tau_n) = 68.22$

Figures 2 and *3* show the density function estimator and the reliability function estimator of the system lifetime for the data presented in *Table 2* (2).

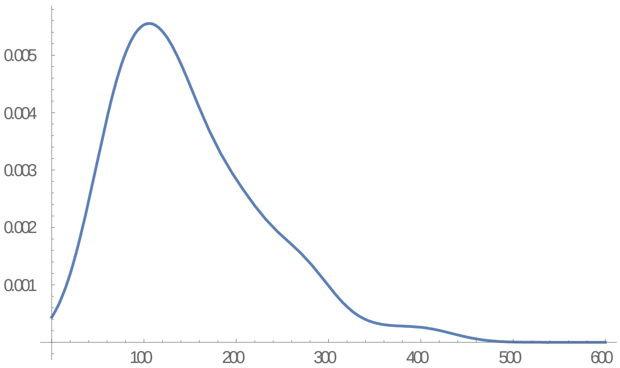

Figure 2. A value of the density function estimator of the system lifetime for the data presented in *Table 2* (2)

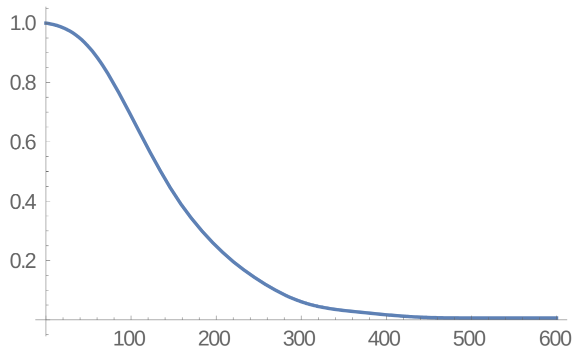

Figure 3. A value of the reliability function estimator of the system lifetime for the data presented in *Table 1* (2)

5. Simulation of parallel system with independent components

For comparison, we will discuss the reliability of the system with independent components. Now we suppose that times to failure of the parallel system components are represented by independent random variables having Weibull distribution. The reliability function of the system is

$$R(x) = 1 - \prod_{i=1}^{n} \left[1 - \exp(\lambda_i x^{\alpha_i})\right]$$

Using Monte Carlo method we investigate the process of the failure of the parallel system consisting of independent components the lifetime of which are represented by random variables having Weibull distribution with parameters

$$(\lambda_i \ \alpha_i), \quad i = 1,2,3,4 .$$

We assume that the scale parameter and the shape parameters are

$$\lambda_i = \lambda(i) = \frac{n}{n-i+1}\lambda$$
$$\alpha_i = \alpha(i) = \alpha \cdot [1 + (i-1) \cdot r]$$

where the rest of parameters are the same as in 2 (2).

Generator of failures of parallel system with independent components

Main part of the program for the data presented in *Table 2* (2):

```
Print["Number of system elements n=",
       n=4]
Print["Sample size m=",m=100]
Print["Parameters of Weibull
       distribution"]
Print["α₁=",a[1]=1.2]
Print["λ₁=",λ[1]=0.0005 ]
Print["α₂=",a[2]=1.44]
Print["λ₂=",λ[2]=0.00066 ]
Print["α₃=",a[3]=1.68]
Print["λ₃=",λ[3]=0.001 ]
Print["α₄=",a[4]=1.92]
Print["λ₄=",λ[4]=0.002 ]
Do[ For[i=1,i<=n,i++,
u[i]=Random[Real,{0,1}];
y[i]=(-Log[1-u[i]]/λ[i])^α[i]//N;];
Print["Tmax[",k,"]=",w[k]=Max[y[1],y[2],
       y[3],y[4]]], {k,1,m}]
Print["Mean ",
       ys=(Sum[w[k],{k,1,m}])/m//N];
Print["Second moment=",
       dm=(Sum[(w[k])²,{k,1,m}])/m//]
Print["Standard deviation=",
```

```
s=(dm-ys²)⁰·⁵//N
```

We obtain the mean time to failure and the standard deviation:

$$E(\tau_n) = \ 555.82, \quad D(\tau_n) = \ 407.96$$

as a result of the above mentioned part of the computer program. *Figures 4* and *5* show the density function estimator and the reliability function estimator of the system lifetime for the data presented in *Table 2* (*2*).

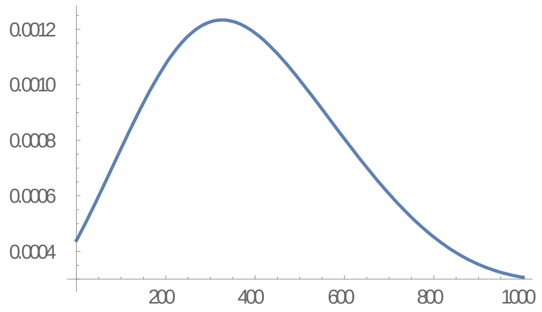

Figure 4. A value of the density function estimator for the parallel system with independent components

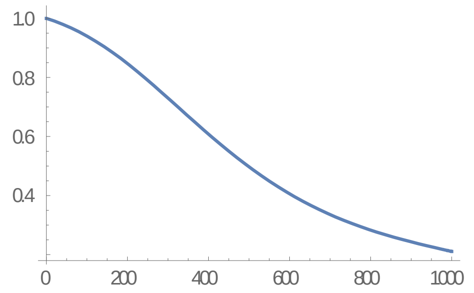

Figure 5. A value of the reliability function estimator for the parallel system with independent components

6. Conclusions

The model presented above deal with the parallel (in reliability sense) system such that the failure rate of each system component is changeable during running of the system and depends on a number of working elements at this point of time. There is constructed the model of the system failure which is a stochastic process. The value of the process at the moment *t* denotes the number of working components. Generally it is neither Markov nor semi-Markov process. We obtain only for parameters $\alpha(i)$ = 1 and $\lambda(i) = \lambda$ a Markov process. The key point of the Monte-Carlo model is an algorithm of generating instants of system components failure under assumption that a failure rate of each component is changeable during running of the system and depends on a number of working elements at this point of time. Main equation is determined by (20).

For

$$\alpha = 1.2, \quad \lambda = 0.0005, \quad r = 0.2, \quad q = 1.21$$
$$b = 1.2, \quad n = 4$$

and all functions $\alpha_i = \alpha(i), \quad i = 1,2,...,n$ presented in 1,2 and 3 rows of *Table 1*, the differences between average times to failures are significant (186,74, 218.25, 180,99).

In the second example parameter λ is two times bigger than in the previous case.

For $\alpha = 1.2$, $\lambda = 0.001$, $r = 0.2$, $q = 1.21$, $n = 4$ the average lifetimes are less than in the previous case (109.68, 148.63, 120,53). For comparison we have investigated the reliability of the system with independent components assuming that times to failure of parallel system components are represented by independent random variables having Weibull distribution with such parameters the case that was shown in *Table 1* (2). The mean time to failure of the system is much greater in this case $E(\tau_n) = 555.82$. It means that by using the classical model of the reliability under the assumption of independence of the components, times to failures leads to very optimistic results in comparison with the corresponding model which takes into account the changeable failure rate during operation of the system.

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