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NOVEL INTEGRAL TRANSFORM TREATING SOME Ψ-FRACTIONAL DERIVATIVE EQUATIONS

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Abstract: The paper deals with a new integral transformation method called Ψ-Elzaki transform (PETM) in order to analyze some Ψ-fractional differential equations. The proposed method uses a modification of the Elzaki transform that is well adapted to deal with Ψ-fractional operators. To solve the nonlinear Ψ-fractional differential equations, we combine the PETM by an iterative method to overcome this nonlinearity. To validate the accuracy and efficiency of this approach, we compare the results of the discussed numerical examples with the exact solutions.

Key words: Ψ-Integral transform, Ψ-Hilfer derivative, Ψ-Caputo derivative, Ψ-Hilfer fractional derivative equation.

1. INTRODUCTION

In the literature, we find a several applications of the fractional derivatives such as studying the undernutrition problems in pregnant women and predicting their intricacies, and investigating the behaviour of neural networks under challenging circumstances by modelling of Hindmarsh-Rose neuron (HRN) in the biomedical field [1, 2]. We also find the modelling of wind turbine system and their chaotic permanent magnet synchronous generator in the physical field [3], and describing the chemical reactions and activation energy by the artificial neural networks (ANN) in the chemical field [4]. The fractional derivatives can also present models for many applications in stochastic analysis [5], and can be used to extend the physical results obtained in [6−10].

Fractional derivatives and fractional integral operators have been well developed in the scientific literature, especially in recent years. Much of the results obtained are compatible with the needs of physical and biological problems. For example, we find that the mathematical models studied in biology which intervene in many epidemiological phenomena can lead to fractional systems as in [11], which relies on the stochastic perturbation technique to simulate the propagation evolution of the Lassa fever. Moreover, researchers have never lost sight of the fundamental need to ensure better models that describe certain problems more realistically (see for example [12]). Therefore, new methods and definitions of fractional sums/differences are developed for discrete fractional calculus (DFC) [13−15], and new methods and definitions of fractional derivatives are developed to give an even better description of some dynamical problems [16] and some chaotic systems [17]. Integral inequalities were also studied by Rashid et al., which are based on fractional calculus [18].

The classical derivatives of fractional order have been known by certain approaches such as Riemann-Liouville, Caputo, GrAnwald-Letnikov...etc. Despite the diversity of these notions, researchers have continued to develop some generalized notions that place these fractional derivatives into a more general concept. This is perhaps the main purpose of the birth of the Ψ-fractional derivative. This concept of the fractional derivative with respect to another function has existed since 1964 when Érdlyi started the discussion in [19] of this generalized derivative. Next, Olser in 1970 gave a precise definition of the Ψ-fractional derivative [20]. Since then, many scholars have developed this notion, including examples from Almeida [21, 22], Sousa, and Oliveira [23]. But the interest of these notions was not purely mathematical. Conversely, recent works show more and more the efficiency of certain models based on the type of Ψ-fractional differential equation. For example, the modeling taking into account the relaxation and/or the law of deformation of certain bodies is based on this type of generalized fractional equations Yang [24]. Concerning the study of Ψ-fractional differential equation by the integral transform method, we can cite the works of Jarad and Abdeljawad that proposed a Ψ-Laplace transformation to treat the type of the equations [25]. Using Elzaki transform Singh et al. in [26] has developed the Hilfer-Prabhakar fractional derivative to study the free electron laser equation. Motivated by these physical applications, we propose in this article to treat some equations comprising fractional derivatives respecting another function. However, the Ψ-fractional derivative problems are not easy to solve. Given their obvious interest, we propose a new technique called Ψ-Elzaki transform based on the Elzaki transform (ET) developed by Elzaki [27], which can enrich the research work on approximate solutions. Furthermore, we will be extending our technique to nonlinear problems, proposing a combination of the Ψ-Elzaki transform and an iterative procedure to overcome the nonlinearity.

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2. PRELIMINARY

In this section, we recall certain definitions and results. For more details concerning these definitions and the proofs of the properties, we can consult for example [3, 21, 28, 29].

2.1. Definitions

Let u an integrable function on $[\delta_1, \delta_1] \to \mathbb{R}, (\alpha, p) \in \mathbb{R} \times \mathbb{N}$ and $\Psi \in C^p([\delta_1, \delta_1])$ such that $\Psi'(t) > 0$. We have the following definitions:

– We define the right Ψ-Riemann-Liouville fractional integral of order $\alpha > 0$ as follow:

$$I_{\delta_1^+}^{\alpha,\Psi} u(t) = \frac{1}{\Gamma(\alpha)} \int_{\delta_1}^{t} \Psi'(s) u(s) \big(\Psi(t) - \Psi(s)\big)^{\alpha-1} ds. \quad (1)$$

– For $\alpha > 0$, $n-1 < \alpha \le n$, the left and right Ψ-Reimann-Liouville fractional derivative is given as:

$$D_{\delta^\pm}^{n-\alpha,\Psi} u(t) = \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^n I_{\delta^\pm}^{\alpha,\Psi} u(t).$$

– For $n-1 < \alpha \le n$, the left and right Ψ-Caputo fractional derivative of order α is defined as:

$${}^{C}D_{\delta^\pm}^{\alpha,\Psi} u(t) = I_{\delta^\pm}^{\alpha,\Psi} u_\Psi^{[n]}(t), \text{ with } u_\Psi^{[n]}(t) = \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^n u(t).$$

A more general definition that covers Ψ-Riemann-Liouville and Ψ-Caputo fractional derivatives, this is the following fractional derivative of Ψ-Hilfer.

Definition 1 ([30]): Let u be an integrable function defined on $[\delta_1, \delta_2]$, and $\Psi \in C^1([\delta_1, \delta_2], \mathbb{R})$ be an increasing function such that $\Psi'(t) \neq 0$, for all $t \in [\delta_1, \delta_2]$. The Ψ-Hilfer fractional derivative right-sided of order $\mu > 0$, $n = \lceil \mu \rceil$ and of type $0 \le \alpha \le 1$ is defined by:

$${}^{H}D_{\delta_1^+}^{\mu,\alpha,\Psi} u(t) = I_{\delta_1^+}^{\alpha(n-\mu),\Psi} \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^n I_{\delta_1^+}^{(1-\alpha)(n-\mu),\Psi} u(t).$$

In the rest, we denote $\Lambda_{\delta_1^+}$, $\Lambda_{\delta_1^-}$ respectively by Λ_{δ_1}, Λ_{δ_2} where Λ denotes the fractional integral or derivative operators defined in this paper. For these definitions, we have the following properties.

2.2. Properties

If $u(t) = \big(\Psi(t) - \Psi(\delta_1)\big)^\eta$ where $\eta > n$ and $\alpha > 0$ then,

$${}^{C}D_{\delta_1}^{\alpha,\Psi} u(t) = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+1)} \big(\Psi(t) - \Psi(\delta_1)\big)^{\eta-\alpha}. \quad (2)$$

$$I_{\delta_1}^{\alpha,\Psi} u(t) = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)} \big(\Psi(t) - \Psi(\delta_1)\big)^{\eta+\alpha}. \quad (3)$$

$${}^{C}D_{\delta_1}^{\alpha,\Psi} I_{\delta_1}^{\alpha,\Psi} u(t) = u(t). \quad (4)$$

$$I_{\delta_1}^{n,\Psi} u_\Psi^{[n]}(t) = u(t) - \sum_{k=0}^{n-1} \frac{u_\Psi^{[n]}(\delta_1)}{k!} \big(\Psi(t) - \Psi(\delta_1)\big)^k. \quad (5)$$

$$I_{\delta_1}^{\alpha,\Psi}\, {}^{C}D_{\delta_1}^{\alpha,\Psi} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u_\Psi^{[n]}(\delta_1)}{k!} \big(\Psi(t) - \Psi(\delta_1)\big)^k. \quad (6)$$

Proof. For to prove the formula (2), we have

$$u_\Psi^{[n]}(t) = \frac{\Gamma(\eta+1)}{\Gamma(\eta-n+1)} u(t)^{\frac{\eta-n}{\eta}}.$$

Using the definition of the Ψ-Caputo fractional derivative, we can write

$${}^{C}D_{\delta_1}^{\alpha,\Psi} u(t) = \frac{\Gamma(\eta+1)}{\Gamma(\eta-n+1)} I_{\delta_1}^{n-\alpha,\Psi} u(t)^{\frac{\eta-n}{\eta}} =$$

$$\frac{\Gamma(\eta+1)}{\Gamma(n-\alpha)\Gamma(\eta-n+1)} \int_{\delta_1}^{t} \Psi'(t) \big(\Psi(t) - \Psi(s)\big)^{n-\alpha-1} u(s)^{\frac{\eta-n}{\eta}} ds =$$

$$\Omega(t) \int_{\delta_1}^{t} \frac{u'(s)}{\eta} \left(\frac{(u(s))^{1-\eta}}{u(t)}\right)^{\frac{1}{\eta}} \left(1 - \left(\frac{u(s)}{u(t)}\right)^{\frac{1}{\eta}}\right)^{n-\alpha-1} \left(\frac{u(s)}{u(t)}\right)^{\frac{\eta-n}{\eta}} ds,$$

where $\Omega(t) = \frac{\Gamma(\eta+1) u(t)^{\frac{\eta-\alpha}{\eta}}}{\Gamma(n-\alpha)\Gamma(\eta-n+1)}$.

With the change of variables $v(s) = \left(\frac{u(s)}{u(t)}\right)^{\frac{1}{\eta}}$, we obtain

$${}^{C}D_{\delta_1}^{\alpha,\Psi} u(t) = \Omega(t) \int_0^1 (1-v)^{n-\alpha-1} v^{\eta-n} dv = \Omega(t) B(n-\alpha, \eta-n+1),$$

where B is the Beta function that satisfies $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ which ends the proof of the formula (2).

In the formula proof (2), we have already implied the formula proof (3) where we have

$$\frac{\Gamma(\eta+1)}{\Gamma(\eta-n+1)} I_{\delta_1}^{n-\alpha,\Psi} u(t)^{\frac{\eta-n}{\eta}} = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+1)} u(t)^{\frac{\eta-\alpha}{\eta}}.$$

Then

$$I_{\delta_1}^{n-\alpha,\Psi} \big(\Psi(t) - \Psi(\delta_1)\big)^{\eta-\alpha} = \frac{\Gamma(\eta-n+1)}{\Gamma(\eta-\alpha+1)} \big(\Psi(t) - \Psi(\delta_1)\big)^{\eta-\alpha}.$$

With simple changes of variables for $n - \alpha$ and $\eta - n$, we complete the formula proof (3).

Formula (4) is a direct consequence of formulas (2) and (3) and, formulas (5) and (6) have been well proven in [29].

3. NOVEL Ψ-INTEGRAL TRANSFORM METHOD

This section discusses a type of method that has been and still is useful for solving certain differential equations, namely integral transforms. This method is useful because it can turn a complicated problem into a simpler one. In addition to their clarity, integral transformations have also helped us to develop some formulas in fractional calculus. We can cite for example Laplace transforms, Sumudu transform, Jafari transform, Elzaki transform, etc. [31–33].

3.1. Elzaki transform

Considering the following Elzaki transform:

Definition 2 Let a function u with respect to t, let $s_1, s_2 > 0$, it exists $M > 0$ such that $|u(t)| < M \exp\left(\frac{|t|}{s_j}\right)$, for $t \in (-1)^j \times [0, \infty)$. Elzaki transform is defined by:

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$$U(s) = \mathcal{E}[u(t)] = \mathcal{E}(s) = s\int_0^\infty u(t)\, exp\left(-\frac{t}{s}\right) dt, \quad (7)$$

where s is a complex number, $t > 0$ and $|s| \in [s_1, s_2]$.

By using the properties of this proposed integral transform, we can have an efficient tool to deal with linear fractional equations. Moreover, we can reach nonlinear cases by combining ET with other techniques. For this, let us recall the following formulas dealing with the integral and the classical fractional derivative by ET.

$$\mathcal{E}[{}^{RL}\mathrm{I}_0^\alpha u] = s^\alpha U(s). \quad (8)$$

$$\mathcal{E}[{}^{RL}D_0^\alpha u] = \frac{U(s)}{s^\alpha} - \sum_{k=0}^{n-1} s^{2-n+k}\, {}^{RL}\mathrm{I}_0^{n-k-\alpha} u(0). \quad (9)$$

$$\mathcal{E}[{}^{C}D_0^\alpha u] = \frac{U(s)}{s^\alpha} - \sum_{k=0}^{n-1} s^{2-\alpha+k} u^{[k]}(0). \quad (10)$$

3.2. Generalized Elzaki transform

The aim of this study is to show the relevance of this novel transform integral and its efficiency in solving certain Ψ-fractional equations. We will consider the novel following definition of generalized integral transform:

Definition 3 Let a function u on $[0,\infty)$ with respect to t, and Ψ an increasing function such that $\Psi(0) = 0$. Let $s_1, s_2 > 0$, if it exists $M > 0$ such that $|u(t)| < M\exp\left(\frac{|t|}{s_j}\right)$, for $t \in (-1)^j \times [0,\infty)$. Then, the Elzaki transform of u with respect to Ψ is defined by:

$$U_\Psi(s) = \mathcal{E}_\Psi[u(t)](s) = s\int_0^\infty \Psi'(t) u(t)\, exp\left(-\frac{\Psi(t)}{s}\right) dt, \quad (11)$$

where s is a complex number, $t > 0$ and $|s| \in [s_1, s_2]$.

Jarad and Abdeljawad in [25] have defined the Ψ-Laplace transform. As we have Duality's ownership between ET and LT, we can easily check duality between the Ψ-ELzaki and Ψ-Laplace transforms. The duality relation is given by the following relations:

$$\mathcal{E}_\Psi[u(t)](s) = s\mathcal{L}_\Psi[u(t)]\left(\frac{1}{s}\right), \quad (12)$$

$$\mathcal{L}_\Psi[u(t)](s) = s\mathcal{E}_\Psi[u(t)]\left(\frac{1}{s}\right). \quad (13)$$

Using this duality between the two transformations, we can make certain results developed in [25, 34] for the Ψ-Laplace transform into Ψ-Elzaki transform.

3.3. Convergence of PET and properties

Theorem 1: Let Ψ an increasing function with $\Psi(0) = 0 \in \mathbb{R}$, and u is continuous on $[0, +\infty[$ and a Ψ-exponentially function bounded order $\frac{1}{d} > 0$ (i.e. $\|u\|_\infty \le M\exp\left(\frac{\Psi(t)}{d}\right)$; with M a positive constant). Then, the Ψ-Elzaki transform of u exists for $s > d$.

Proof: We have

$$|\mathcal{E}_\Psi[u(t)]| = \left|s\int_0^\infty \Psi'(t)u(t)\, exp\left(-\frac{\Psi(t)}{s}\right) dt\right|$$
$$\le s\int_0^\infty \Psi'(t)\, exp\left(-\frac{\Psi(t)}{s}\right)\|u\|_\infty dt$$
$$\le sM\int_0^\infty \Psi'(t)\, exp\left(-\frac{\Psi(t)}{s} + \frac{\Psi(t)}{d}\right) dt.$$

Using the fact that $s > d$, so the primitive limit is cancels for infinity, we get

$$|\mathcal{E}_\Psi[u(t)]| \le \frac{s^2 Md}{s-d}.$$

Then the PET is convergent.

We can also define the inverse of PET by:

$$\mathcal{E}_\Psi^{-1}[U(s)](t) = \frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty} U\left(\frac{1}{s}\right) exp\left(-\frac{\Psi(t)}{s}\right) s\, dt. \quad (14)$$

Remark 1: We get the immediate result: If

$$\mathcal{E}_\Psi[u(t)](s) = U(s), \text{ then } \mathcal{E}_\Psi\left[u\big(\Psi(t)\big)\right](s) = U(s). \quad (15)$$

i.e. we have $U_\Psi(s) = U(s)$.

Using Remark. 1 and the duality propriety of ET and LT, and the results proved in [25] we can easily get the following proprieties:

Properties:

$$\mathcal{E}_\Psi\left[exp\big(\lambda\Psi(t)\big)\right] = \frac{s^2}{1-\lambda s}. \quad (16)$$

$$\mathcal{E}_\Psi\left[\big(\Psi(t)\big)^\beta\right] = \Gamma(\beta+1)s^{\beta+2}. \quad (17)$$

3.4. PET and Ψ-Fractional operators

In this section, we will discuss some results of PET that can be useful for solving fractional equations of the type Ψ-Caputo. But these results can also be extended to Ψ-Hilfer or Ψ-Riemann-Liouville Fractional equations type and other equations based on Ψ-fractional differential operators.

Proposition 1: If $\alpha \in]n-1, n]$, $n \in \mathbb{N}$ then

$$\mathcal{E}_\Psi\left[{}^{RL}\mathrm{I}_0^{\alpha,\Psi} u\right] = s^\alpha U_\Psi(s). \quad (18)$$

$$\mathcal{E}_\Psi\left[{}^{RL}D_0^{\alpha,\Psi} u\right] = \frac{U_\Psi(s)}{s^\alpha} - \sum_{k=0}^{n-1} s^{2-n+k}\, {}^{RL}\mathrm{I}_0^{n-k-\alpha,\Psi} u(0). \quad (19)$$

$$\mathcal{E}_\Psi\left[{}^{C}D_0^{\alpha,\Psi} u\right] = \frac{U_\Psi(s)}{s^\alpha} - \sum_{k=0}^{n-1} s^{2-\alpha+k}\, {}^{RL}D_0^{k,\Psi} u(0). \quad (20)$$

$$\mathcal{E}_\Psi\left[{}^{H}D_0^{\mu,\alpha,\Psi} u\right] = \frac{U_\Psi(s)}{s^\mu} - \sum_{k=0}^{n-1} s^{2-n(1-\alpha)+k}\, \mathrm{I}_0^{(1-\alpha)(n-\mu)-k,\Psi} u(0). \quad (21)$$

Proof: For to prove the formula (18):

Using the duality between ET and LT, we can write

$$\mathcal{E}_\Psi\left[{}^{RL}\mathrm{I}_0^{\alpha,\Psi} u\right](s) = s\mathcal{L}_\Psi\left[{}^{RL}\mathrm{I}_0^{\alpha,\Psi} u\right]\left(\frac{1}{s}\right).$$

With Laplace transform, the Ψ-Riemann-Liouville integral is given by

$$\mathcal{L}_\Psi\left[{}^{RL}\mathrm{I}_0^{\alpha,\Psi} u\right](s) = s^{-\alpha}\mathcal{L}_\Psi[u](s).$$

Using both previous formulas, we get

$$\mathcal{E}_\Psi\left[{}^{RL}\mathrm{I}_0^{\alpha,\Psi} u\right](s) = \left(\frac{1}{s}\right)^{-\alpha} s\mathcal{L}_\Psi[u]\left(\frac{1}{s}\right) = s^\alpha U_\Psi(s).$$

The remainders of the results can be easily demonstrated in the same way.

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3.5. PET and some special functions

Special functions play an important role in the definitions of certain fractional derivatives (such as fractional derivatives involving the kernel of the Mittag-Leffler function (ML) [24]) or even in the resolution of certain fractional differential equations. For this, it is useful to present some results concerning these functions.

Definition 4: We consider the following generalized version of the Mittag-Leffler function defined in [24], recognized as the Prabhakar function

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{\gamma_k}{k!\Gamma(\alpha k+\beta)}z^k, \tag{22}$$

where $\gamma_k=\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$; $(\alpha,\beta,\gamma,z)\in\mathbb{C}^4$, with $Re(\alpha)>0$.

In fractional differences, the authors use the Kernel of the discrete Mittag-Leffler function, see [14, 15] for more details.

By PET, we have the following results:

Proposition 2 Assume that $\mathrm{Re}(\alpha)>0$ and $|\mathrm{a}\,\mathrm{s}^{\beta}|<1$. Applying the PET, we get for ML function with one parameter

$$\mathcal{E}_{\Psi}\left[E_{\beta}\left(a(\Psi(t))^{\beta}\right)\right]=\frac{s^2}{1-as^{\beta}}. \tag{23}$$

For two parameter ML function we have

$$\mathcal{E}_{\Psi}\left[(\Psi(t))^{\beta-1}E_{\beta,\beta}\left(a(\Psi(t))^{\beta}\right)\right]=\frac{s^{\beta+1}}{1-as^{\beta}}. \tag{24}$$

For Prabhakar function, we get

$$\mathcal{E}_{\Psi}\left[(\Psi(t))^{\alpha-1}E^{\gamma}_{\beta,\alpha}\left(a(\Psi(t))^{\beta}\right)\right]=\mathcal{E}\left[t^{\alpha-1}E^{\gamma}_{\beta,\alpha}\left(at^{\beta}\right)\right]=\frac{s^{\alpha+1}}{\left(1-as^{\beta}\right)^{\gamma}}. \tag{25}$$

We have

$$^{C}D_0^{\eta,\Psi}\left(\Psi(t)\right)^{\alpha-1}E^{\gamma}_{\beta,\alpha}\left(a(\Psi(t))^{\beta}\right)=\left(\Psi(t)\right)^{\beta-\eta-1}E^{\gamma}_{\beta,\alpha-\eta}\left(a(\Psi(t))^{\beta}\right) \tag{26}$$

The proof of these properties is immediate if Remark.1 is taken into account with the properties of ET developed in the scientific literature.

4. SOLUTION FOR Ψ-FRACTIONAL DERIVATIVE EQUATIONS

Although the concept of Ψ-fractional operator has been discussed for decades, the notion of Ψ-fractional derivative equation has only been discussed frequently in recent years. For example, the random differential equations via the fractional derivative Ψ-Hilfer [37], Ψ-fractional differential equations [29], and for modeling taking into account the relaxation of certain bodies [24]. Guided by these motivations, we propose to discuss the solution to the following types of equations:

$$\begin{cases} ^{C}D_{\delta_1}^{\alpha,\Psi}u(t)=g\,(t,u), \qquad t\in[\delta_1,\delta_2], \qquad n=\lceil\alpha\rceil,\\ u^{(k)}(\delta_1)=u_k, u^{(n-1)}(\delta_2)=u_{n-1}, \ \ k=0,1,\cdots,n-1.\end{cases} \tag{27}$$

Applying the PET on the (27), and using (20), we get

$$\mathcal{E}_{\Psi}[u]=s^{\alpha}\mathcal{E}_{\Psi}[g\,(t,u)]+\sum_{k=0}^{n-1}s^{2+k}\,D_0^{k,\Psi}\,u_k. \tag{28}$$

By the inverse of PET, we obtain

$$u(t)=\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[g\,(t,u)]\}+\sum_{k=0}^{n-1}\frac{\left(\Psi(t)-\Psi(0)\right)^k}{k!}D_0^{k,\Psi}\,u_k. \tag{29}$$

To overcome the nonlinear term of this equation, one can use a direct iterative method or the one described for example by [33].

Assume that have the following equation:

$$u=N(u)+f, \tag{30}$$

with N a non linear operator, and f a given function. We can solving (30) by the following iterative method

$$\begin{cases}u_0=f,\\ u_{p+1}=\mathcal{N}(u_p).\end{cases} \tag{31}$$

With $\mathcal{N}$ is approximate of N. Then, the solution is

$$\mathrm{u}=\sum_{\mathrm{k}=0}^{\infty}\mathrm{u}_{\mathrm{k}}.$$

In the situation of problem (27), we can pose

$$u_0=\sum_{k=0}^{n-1}\frac{t^k}{k!}D_0^{k,\Psi}\,u(0). \tag{32}$$

$$N(u)=\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[g\,(t,u)]\}. \tag{33}$$

Remark 2 If N is linear the iterative procedure in (31) is as follows: $\mathrm{u}_{\mathrm{p}+1}=\mathrm{N}(\mathrm{u}_{\mathrm{p}})$.

5. NUMERICAL EXAMPLES

Based on the Ψ-Elzaki transformation, we present some numerical examples to discuss the efficiency of the proposed method. The main objective is to compare the results obtained with the exact solution in each of the considered examples.

Example 1: We begin with an illustrative example that seeks the eigenfunction of the Ψ-Caputo fractional derivative operator. This is a relaxation model based on a fractional derivative with respect to another function. The model is given by:

$$\begin{cases} ^{C}D_0^{\alpha,\Psi}u(t)=\lambda u(t), \ for\ \ t\in[0,T], T>0, \alpha\in\,]0,1],\\ u(0)=1.\end{cases} \tag{34}$$

By the formula (26), we can find the exact solution to the problem

$$\mathrm{u(t)}=\mathrm{E}_{\alpha}(\lambda(\Psi(\mathrm{t})-\Psi(\mathrm{s}))^{\alpha}).$$

Applying the PET to the problem (34), we obtain

$$\mathrm{u(t)}=\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[\lambda\mathrm{u}]\}+\mathrm{u}(0).$$

In this case, the iterative procedure reduces to the following system

$$u_0=1,$$
$$u_{p+1}=\lambda\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[u_p]\}.$$

Then,

$$u_1=\lambda\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[u_0]\}=\lambda\mathcal{E}_{\Psi}^{-1}\{s^{\alpha+2}\}=\lambda\frac{\left(\Psi(t)-\Psi(0)\right)^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2=\lambda\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[u_1]\}=\frac{\lambda^2}{\Gamma(\alpha+1)}\mathcal{E}_{\Psi}^{-1}\{s^{2\alpha+2}\Gamma(\alpha+1)\}=$$

$\lambda^2 \frac{(\Psi(t)-\Psi(0))^{2\alpha}}{\Gamma(2\alpha+1)}$, and so on.

We can write

$$u_p = \lambda^p \frac{(\Psi(t)-\Psi(0))^{p\alpha}}{\Gamma(p\alpha+1)}.$$

The solution is given by

$$u = \sum_{k=0}^{\infty} \lambda^p \frac{(\Psi(t)-\Psi(0))^{p\alpha}}{\Gamma(p\alpha+1)} = E_\alpha\left(\lambda(\Psi(t)-\Psi(0))^\alpha\right).$$

Example 2: We choose an inhomogeneous relaxation model based on the Ψ-fractional derivative. This model has been solved numerically in [38] having only the Caputo fractional derivative.

$$\begin{cases} {}^{C}D_0^{\alpha,\Psi} u(t) + u(t) = \frac{(\Psi(t)-\Psi(0))^{4-\alpha}}{\Gamma(5-\alpha)}, \ t \geq 0, \alpha \in]0,1]. \\ u(0) = 0. \end{cases} \quad (35)$$

Applying (29), we obtain

$$u(t) = -\mathcal{E}_\Psi^{-1}\{s^\alpha \mathcal{E}[u]\} + \mathcal{E}_\Psi^{-1}\left\{s^\alpha \mathcal{E}\left[\frac{(\Psi(t)-\Psi(0))^{4-\alpha}}{\Gamma(5-\alpha)}\right]\right\}.$$

Then

$$\begin{cases} u_0 = \mathcal{E}_\Psi^{-1}\left\{s^\alpha \mathcal{E}\left[\frac{(\Psi(t)-\Psi(0))^{4-\alpha}}{\Gamma(5-\alpha)}\right]\right\}, \\ u_{p+1} = -\mathcal{E}_\Psi^{-1}\{s^\alpha \mathcal{E}[u_p]\}, \quad p \geq 0. \end{cases}$$

Putting

$$u_0 = \mathcal{E}_\Psi^{-1}\left\{s^\alpha \mathcal{E}\left[\frac{(\Psi(t)-\Psi(0))^{4-\alpha}}{\Gamma(5-\alpha)}\right]\right\} = \frac{(\Psi(t)-\Psi(0))^{4}}{\Gamma(5)}.$$

$$u_1 = -\mathcal{E}_\Psi^{-1}\left\{s^\alpha \mathcal{E}\left[\frac{(\Psi(t)-\Psi(0))^{4}}{\Gamma(5)}\right]\right\} = -\frac{(\Psi(t)-\Psi(0))^{\alpha+4}}{\Gamma(\alpha+5)}.$$

$$u_2 = -\mathcal{E}_\Psi^{-1}\left\{s^\alpha \mathcal{E}\left[-\frac{(\Psi(t)-\Psi(0))^{\alpha+4}}{\Gamma(\alpha+5)}\right]\right\} = \frac{(\Psi(t)-\Psi(0))^{2\alpha+4}}{\Gamma(2\alpha+5)}$$

$$u_3 = -\frac{(\Psi(t)-\Psi(0))^{3\alpha+4}}{\Gamma(3\alpha+5)}.$$

Then

$$u(t) = (\Psi(t)-\Psi(0))^4\left\{\frac{1}{\Gamma(5)} - \frac{(\Psi(t)-\Psi(0))^{\alpha}}{\Gamma(\alpha+5)} + \frac{(\Psi(t)-\Psi(0))^{2\alpha}}{\Gamma(2\alpha+5)} - \frac{(\Psi(t)-\Psi(0))^{3\alpha}}{\Gamma(3\alpha+5)} + \cdots\right\}$$

$$= (\Psi(t)-\Psi(0))^4 E_{\alpha,5}\left[-(\Psi(t)-\Psi(0))^\alpha\right].$$

For the first performance test we set $\alpha = 0.2$ and $\Psi = t^2$. This involves comparing the exact solution with a numerical solution estimated after 3 iterations of PET method. We notice in Fig.1 that the numerical solution is almost identical to the exact solution.

Fig.2 shows the variation of the solution according to the values of the fractional derivative α. Knowing that in this case, we took $\Psi = t^2$.

The main objective of defining Ψ-fractional operators is to obtain a generalized notion that gathers some fractional derivatives in a single definition. In Fig.3, we present the solutions according to different values of the functions Ψ.

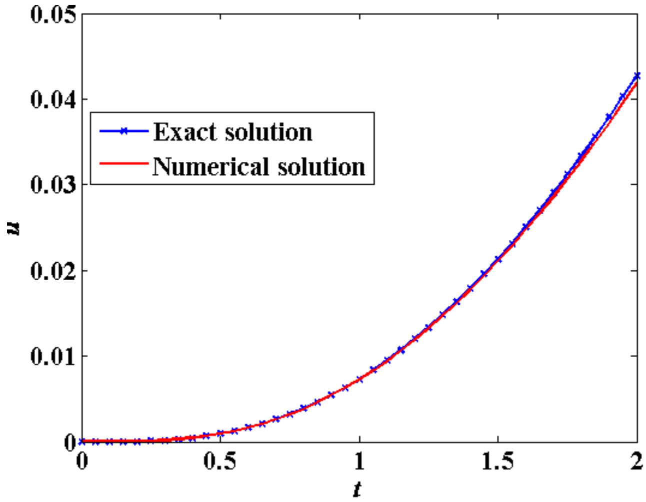

Fig. 1. Comparison of the exact solution and the numerical solution after 3 iterations of PET method

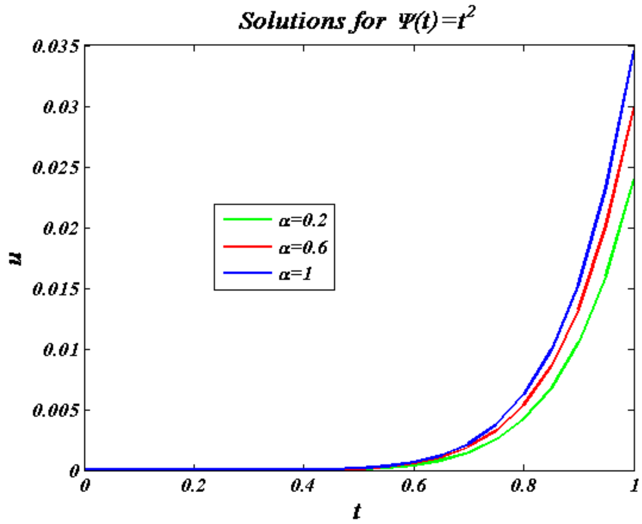

Fig. 2. Solutions for different values of α and for $\Psi(t) = t^2$

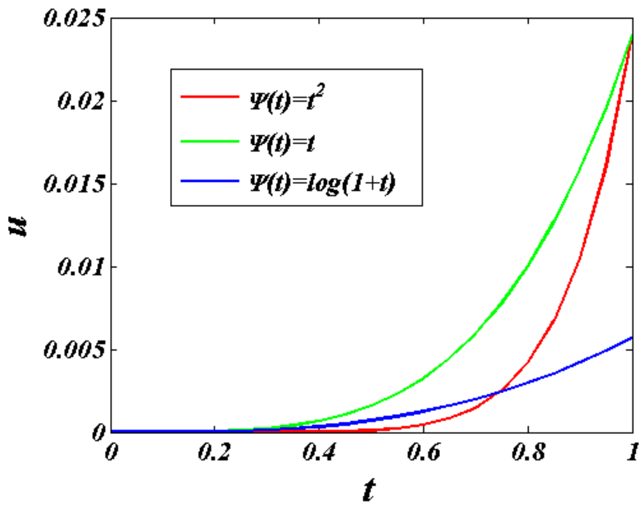

Fig. 2. Solutions for different values of Ψ

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Example 3: Considering the following inhomogeneous Ψ-fractional derivative equation:

$$\begin{cases} {}^{C}D_{0}^{\alpha,\Psi} u(t) - \lambda u(t) = C, \ t \geq 0, and\ 0 < \alpha \leq 1. \\ u(0) = 0. \end{cases} \quad (36)$$

Applying (29), we obtain

$$u(t) = \lambda \mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[u]\} + C\mathcal{E}_{\Psi}^{-1}\{s^{\alpha+2}\}.$$

Then

$$\begin{cases} u_0 = C\mathcal{E}_{\Psi}^{-1}\{s^{\alpha+2}\} = C\frac{(\Psi(t)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)}, \\ u_{p+1} = \lambda \mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[u_p]\}, \quad p \geq 0. \end{cases}$$

Then

$$u_1 = \lambda \mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[u_0]\} = \lambda C \mathcal{E}_{\Psi}^{-1}\left\{s^{\alpha}\mathcal{E}\left[\frac{(\Psi(t)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right]\right\}$$

$$= \lambda C\frac{(\Psi(t)-\Psi(0))^{2\alpha}}{\Gamma(2\alpha+1)}.$$

$$u_2 = \lambda \mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[u_1]\} = \lambda^2 C\frac{(\Psi(t)-\Psi(0))^{3\alpha}}{\Gamma(3\alpha+1)}.$$

And

$$u_3 = \lambda^3 C\frac{(\Psi(t)-\Psi(0))^{4\alpha}}{\Gamma(4\alpha+1)}.$$

Then

$$u(t) = \frac{C}{\lambda}\left\{\lambda\frac{(\Psi(t)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)} + \lambda^2 C\frac{(\Psi(t)-\Psi(0))^{2\alpha}}{\Gamma(2\alpha+1)} + \lambda^3 C\frac{(\Psi(t)-\Psi(0))^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right\},$$

$$= \frac{C}{\lambda}\left(E_{\alpha}\left(\lambda(\Psi(t)-\Psi(0))^{\alpha}\right) - 1\right).$$

Example 4: With Ψ-Caputo fractional derivative, we consider the non-linear fractional boundary value problem

$$\begin{cases} {}^{C}D_{0}^{\alpha,\Psi} u(t) + u^2(t) = 1, \qquad t \geq 0\ and\ 0 < \alpha \leq 1. \\ u(0) = 0. \end{cases} \quad (37)$$

Applying (29), we obtain

$$u(t) = -\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[u^2]\} + \mathcal{E}_{\Psi}^{-1}\{s^{\alpha+2}\}$$

$$= -\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[u^2]\} + \frac{(\Psi(t)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)}.$$

Using for this example the Adomian decomposition method

$$\begin{cases} u_0 = \frac{(\Psi(t)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)}, \\ u_{p+1} = -\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[A_p]\}, \quad p \geq 0. \end{cases}$$

With $\mathrm{A_p}$ is p-th order of Adomian polynomial of the term u^2. Putting

$$u_0 = \frac{(\Psi(t)-\Psi(0))^{\alpha}}{\Gamma(\alpha+1)}.$$

$$u_1 = -\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[A_0]\} = -\mathcal{E}_{\Psi}^{-1}\left\{s^{\alpha}\mathcal{E}\left[\frac{(\Psi(t)-\Psi(0))^{2\alpha}}{\Gamma^2(\alpha+1)}\right]\right\}$$

$$= -\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\mathcal{E}_{\Psi}^{-1}\{s^{3\alpha+2}\}$$

$$= -\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}(\Psi(t)-\Psi(0))^{3\alpha}.$$

$$u_2 = -\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[A_1]\} = -\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[2u_0u_1]\}$$

$$= \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}(\Psi(t)-\Psi(0))^{5\alpha}.$$

And

$$u_3 = -\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[A_2]\} = -\mathcal{E}_{\Psi}^{-1}\{s^{\alpha}\mathcal{E}[u_0u_2 + u_1^2]\}$$

$$= -\frac{4\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)}{\Gamma^4(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)}(\Psi(t)-\Psi(0))^{7\alpha}$$

$$-\frac{\Gamma^2(2\alpha+1)\Gamma(6\alpha+1)}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(7\alpha+1)}(\Psi(t)-\Psi(0))^{7\alpha}.$$

Note that at $\alpha = 1$ the solution of (37) is given by

$$u(t) = (\Psi(t)-\Psi(0)) - \frac{(\Psi(t)-\Psi(0))^3}{3} + \frac{2(\Psi(t)-\Psi(0))^5}{15} - \frac{17(\Psi(t)-\Psi(0))^7}{315} + \cdots = \tanh(\Psi(t)-\Psi(0)).$$

6. CONCLUSIONS

This paper discusses and develops a generalized integral transform PET to find exact or approximate solutions to the linear or nonlinear Ψ-fractional differential equations. Contemplating the preliminary, several existing details of fractional calculus can be identified in the literature. The Ψ-fractional differential equations thus proposed in numerical tests were processed by the PETM combined with an iterative method, and this combination was very suitable for dealing with the fractional derivative involving a function Ψ. We conclude the effectiveness and success of this method for solving certain type of linear or nonlinear Ψ-fractional differential equations. Therefore, the proposed method in this paper and its numerical results can stimulates to work for applying on other operators fractional calculus.

REFERENCES

1. Chu Y-M, Rashid S, Karim S, Khalid A, Elagan S-K. Deterministic-stochastic analysis of fractional differential equations malnutrition model with random perturbations and crossover effects. Sci Rep. 2023;13(1):14824.
https://doi: 10.1038/s41598-023-41861-4
2. Al-Qurashi M, Asif Q. U-A, Chu Y-M, Rashid S, Elagan SK. Complexity analysis and discrete fractional difference implementation of the Hindmarsh–Rose neuron system. Results in Physics. 2023;51 106627:2211-3797. https://doi.org/10.1016/j.rinp.2023.106627
3. Alsharidi AK, Rashid S, Elagan SK. Short-memory discrete fractional difference equation wind turbine model and its inferential control of a chaotic permanent magnet synchronous transformer in time-scale analysis. AIMS Mathematics. 2023;8(8):19097-19120.
https://doi.10.3934/math.2023975
4. Kanan M, Ullah H, Raja M-A. Z, Fiza M, Ullah H, Shoaib M., Akgül A, Asad J. Intelligent computing paradigm for second-grade fluid in a rotating frame in a fractal porous medium. Fractals. 2023;31(08): 2340175. https://doi.org/10.1142/S0218348X23401758

6. Rashid S, Noorb MA, Noor K. I. Caputo fractional derivatives and inequalities via preinvex stochastic processes, Published by Faculty of Sciences and Mathematics. University of Nis. Serbia. Filomat. 2023;37(19):6569–6584. https://doi.org/10.2298/FIL2319569R
7. Li W, Farooq U, Waqas H, Alharthi AM, Fatima N, Hassan AM, Muhammad T, Akgül A. Numerical simulations of Darcy-forchheimer flow of radiative hybrid nanofluid with Lobatto-IIIa scheme configured by a stretching surface. Case Studies in Thermal Engineering. 2023;49:103364:214-157X. https://doi.org/10.1016/j.csite.2023.103364
8. Faridi WA, Abu Bakar M, Akgül A, Abd El-Rahman M, El Din SM. Exact fractional soliton solutions of thin-film ferroelectric matrial equation by analytical approaches. Alexandria Engineering Journal. 2023;78:483-497. https://doi.org/10.1016/j.aej.2023.07.049
9. Ashraf R, Hussain S, Ashraf F, Akgül A, El Din SM. The extended Fan's sub-equation method and its application to nonlinear Schrödinger equation with saturable nonlinearity. Results in Physics. 2023;52:106755 https://doi.org/10.1016/j.rinp.2023.106755
10. Khan SA, Yasmin S, Waqas H, Az-Zo'bi EA, Alhushaybari A, Akgül A, Hassan A. M, Imran M. Entropy optimized Ferro-copper/blood based nanofluid flow between double stretchable disks: Application to brain dynamic. Alexandria Engineering Journal. 2023;79:296-307. https://doi.org/10.1016/j.aej.2023.08.017
11. Faridi WA, Abu Bakar M, Myrzakulova Z, Myrzakulov R, Akgül A, El Din S. M. The formation of solitary wave solutions and their propagation for Kuralay equation. Results in Physics. 2023;52:106774. https://doi.org/10.1016/j.rinp.2023.106774
12. Rashid S, Karim S, Akgül A, Bariq A, Elagan SK. Novel insights for a nonlinear deterministic-stochastic class of fractional-order Lassa fever model with varying kernels. Sci Rep 2023;13:15320. https://doi.org/10.1038/s41598-023-42106-0
13. Zhou S-S, Rashid S, Set E, Garba Ahmad A, Hamed YS. On more general inequalities for weighted generalized proportional Hadamard fractional integral operator with applications. AIMS Mathematics. 2021;6(9):9154–9176. https://doi.org/1010.3934/math.2021532
14. Rashid S, Abouelmagd E. I, Sultana S, Chu Y-M. New developments in weighted n-fold type inequalities via discrete generalized $\widehat{\hbar}$-proportional fractional operators. Fractals. 2022; 30(02):2240056. https://doi.org/10.1142/S0218348X22400564
15. Rashid S, Abouelmagd E. I, Khalid A, Farooq FB, Chu Y-M. Some recent developments on dynamical $\widehat{\hbar}$-discrete fractional type inequalities in the frame of nonsingular and nonlocal kernels. Fractals. 2022; 30 (02):2240110. https://doi.org/10.1142/S0218348X22401107
16. Rashid S, Sultana S, Hammouch Z, Jarad F, Hamed YS. Novel aspects of discrete dynamical type inequalities within fractional operators having generalized $\widehat{\hbar}$-discrete Mittag-Leffler kernels and application. Chaos. Solitons & Fractals. 2021;151:111204. https://doi.org/10.1016/j.chaos.2021.111204
17. Atangana A, Baleanu D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. Therm. Sci. 2016;20(2):763--769. http://dx.doi.org/10.2298/TSCI160111018A
18. Chu Y-M, Rashid S, Asif Q. U-A, Abdalbagi M. On configuring new choatic behaviours for a variable fractional-order memristor-based circuit in terms of Mittag-Leffler kernel. Results in Physics. 2023;53: 106939. https://doi.org/10.1016/j.rinp.2023.106939
19. Rashid S, Khalid A, Bazighifan O, Oros G.I. New Modifications of Integral Inequalities via $\wp$-Convexity Pertaining to Fractional Calculus and Their Applications. Mathematics. 2021;9:1753. https://doi.org/10.3390/math9151753
20. Érdlyi A. An integral equation involving Legendre functions, J. Soc. Indust. Appl. Math. 1964;12(1):15-30. https://doi.org/10.1137/0112002
21. OSLR TJ. Leibniz rule for fractional derivatives and an application to infinite series. SIAM J. Appl. Math. 1970;18(3):658--674. https://doi.org/10.1137/0118059
22. Almeida R. A caputo fractional derivative of a function with respect to another function. Communications in Nonlinear Science and Numerical Simulation. 2017;44:460--481. https://doi.org/10.1016/j.cnsns.2016.09.006
23. Almeida R. Further properties of Osler's generalized fractional integrals and derivatives with respect to another function. Rocky Mountain J. Math. 2019;49(8):2459--2493. https://doi.org/10.1216/RMJ-2019-49-8-2459
24. Sousa JV da C, Oliveira EC de. On the Ψ-Hilfer Fractional Derivative. Commun. Nonlinear Sci. Numer. Simul. 2018;60:72-91. https://doi.org/10.1016/j.cnsns.2018.01.005
25. Yang X-J. General fractional derivatives: theory, methods and applications. CRC Press. New York 2019. https://doi.org/10.1201/9780429284083
26. Jarad F, Abdeljawad T. Generalized fractional derivatives and Laplace transform. Discrete Contin. Dyn. Syst. 2020;13(3):709–722. https://doi.org/10.3934/dcdss.2020039
27. Singh Y, Gill V, Kundu S, Kumar D. On the Elzaki transform and its application in fractional free electron laser equation. Acta Univ. Sapientiae Mathem. 2019;11(2):419--429. https://doi.org/10.2478/ausm-2019-0030
28. Elzaki TM. The New Integral Transform (Elzaki Transform) fundamental properties investigations and applications. GJPAM. 2011;7(1):57—64.
29. Almeida R, Malinowska AB, Odzijewicz T. An extension of the fractional gronwall inequality, in Conference on Non-Integer Order Calculus and Its Applications. Springer. 2018:20-28. https://doi.org/10.1007/978-3-030-17344-9_2
30. Ali A, Minamoto T. A new numerical technique for investigating boundary value problems with Ψ-Caputo fractional operator. Journal of Applied Analysis & Computation. 2023;13(1):275--297. https://doi.org/10.11948/20220062
31. Sousa JV da C, Oliveira E C de. On the Ψ-fractional integral and applications. Comp. Appl. Math. 2019;38(4). https://doi.org/10.1007/s40314-019-0774-z
32. Bulut H, Baskonus HM, Bin Muhammad Belgacem F. The Analytical Solutions of Some Fractional Ordinary Differential Equations By Sumudu Transform Method. Abs. Appl. Anal. 2013;2013(6):203875. https://doi.org/10.1155/2013/203875
33. Jafari H. A new general integral transform for solving integral equations. J Adv Res. 2021;32:133--138. https://doi.org/10.1016/j.jare.2020.08.016
34. Elzaki MT, Chamekh M. Solving nonlinear fractional differential equations using a new decomposition method. Universal Journal of Applied Mathematics & Computation. 2018;6:27-35.
35. Fahad HM, Ur Rehman M, Fernandez A. On Laplace transforms with respect to functions and their applications to fractional differential equations. Math. Methods Appl. Sci. 2021;1-20. https://doi.org/10.1002/mma.7772
36. Prabhakar TR. A singular integral equation with a generalized Mittag-Leffler function in the kernel. Yokohama Math. J. 1971;19:7-15.
37. Pang D, Jiang W, Niazi AUK. Fractional derivatives of the generalized Mittag-Leffler functions. Adv. Differ. Equ. 2018;2018:415. https://doi.org/10.1186/s13662-018-1855-9
38. Harikrishnan S, Shah K, Baleanu D, et al. Note on the solution of random differential equations via Ψ-Hilfer fractional derivative. Adv Differ Equ. 2018;2018:224. https://doi.org/10.1186/s13662-018-1678-8
39. Li C, Zeng FH. Numerical methods for fractional calculus. Chapman and Hall/CRC 2015. https://doi.org/10.1201/b18503

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