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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS OF A SEMILINEAR ELLIPTIC SYSTEM IN A BOUNDED DOMAIN

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Abstract. Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a smooth boundary $\partial\Omega$. We discuss in this paper the existence and the asymptotic behavior of positive solutions of the following semilinear elliptic system

$$-\Delta u = a_1(x)u^\alpha v^r \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0,$$
$$-\Delta v = a_2(x)v^\beta u^s \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0.$$

Here $r, s \in \mathbb{R}$, $\alpha, \beta < 1$ such that $\gamma := (1-\alpha)(1-\beta) - rs > 0$ and the functions a_i $(i = 1, 2)$ are nonnegative and satisfy some appropriate conditions with reference to Karamata regular variation theory.

Keywords: semilinear elliptic system, asymptotic behavior, Karamata class, sub-super solution.

Mathematics Subject Classification: 31B25, 34B15, 34B18, 34B27.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a smooth boundary $\partial\Omega$. This paper deals with the study of the following semilinear elliptic system

$$\begin{cases} -\Delta u = a_1(x)u^\alpha v^r, \ u > 0 & \text{in } \Omega, \\ -\Delta v = a_2(x)v^\beta u^s, \ v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $r, s \in \mathbb{R}$ and $\alpha, \beta < 1$ such that

$$\gamma := (1-\alpha)(1-\beta) - rs > 0. \tag{1.2}$$

The weight functions a_1, a_2 are nonnegative on Ω and satisfy a suitable condition with reference to a functional class $\mathcal{K}$ called the Karamata class and defined on $(0, \eta]$, ($\eta > \operatorname{diam}(\Omega)$) by

$$\mathcal{K} := \left\{ t \mapsto L(t) := c \exp\Big(\int_t^\eta \frac{z(s)}{s} ds \Big); \quad c > 0 \quad \text{and} \quad z \in C([0,\eta]), \;\; z(0) = 0 \right\}.$$

The Karamata regular variation theory is an innovation in the study of differential nonlinear equations where we not only discuss the existence of solutions vanishing on $\partial\Omega$ but also the blow-up boundary solutions and give an asymptotic behavior of such solutions (see, e.g. [2–5, 13–16, 18, 19] and references therein).

By a classical solution of (1.1), we understand a pair (u, v) such that $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ and satisfy (1.1) point wise.

Existence of such solutions and their behavior have been extensively investigated in the literature with various methods (see, e.g. [1, 6–13, 17, 21] and references therein). Indeed, systems of type (1.1) with $a_1 = a_2 = 1$ and $r, s, \alpha, \beta \geq 0$ have received much attention in the last decade. Namely, Maniwa proved in [17] the existence and the uniqueness of a positive solution for the following more general Dirichlet problem

$$\begin{cases} -\Delta u_i = \prod_{j=1}^N u_j^{p_{ij}}, & u_i > 0 \text{ in } \Omega, \\ u_{i|\partial\Omega} = 0, & i = 1, 2, \ldots, N, \end{cases}$$

where $p_{ij} \geq 0$ and the matrix $P = (p_{ij})$ satisfies the following assumption

$$I - P \text{ is a nonsingular irreducible M-matrix,} \tag{1.3}$$

where I is the identity matrix (see [17, Definition 1-2]).

Especially, for the case $N = 2$, putting

$$P := \begin{pmatrix} \alpha & r \\ s & \beta \end{pmatrix},$$

condition (1.3) is satisfied if and only if (1.2) is satisfied with $r, s > 0$ and $\alpha, \beta < 1$.

Most recently, there has been some interest in systems of type (1.1) where the authors considered $a_1 = a_2 = 1$ on Ω, but where the exponents r, s, α, β are not necessarily positive. In [12], Ghergu showed the following.

Theorem 1.1 ([12])**.** *Let $\alpha, \beta \leq 0, r, s < 0$ satisfying* (1.2) *and one of the following conditions:*

(i) $-r \min\{1, \frac{2+s}{1-\beta}\} \leq 1 + \alpha, -s < 2,$

(ii) $-s \min\{1, \frac{2+r}{1-\alpha}\} \leq 1 + \beta, -r < 2,$

(iii) $\alpha, \beta \leq -1$ *and* $r, s > -2$.

Then the following system

$$\begin{cases} -\Delta u = u^\alpha v^r, \;\; u > 0 & \text{in } \Omega, \\ -\Delta v = v^\beta u^s, \;\; v > 0 & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

has at least one solution.

Later, inspired by the above result, Zhang studied in [21] system (1.4) for a different range of exponents to those in [12]. Indeed, he proved that for $\alpha, \beta, r, s < 0$ satisfying some appropriate conditions, problem (1.4) has at least one classical solution (u, v) satisfying,

$$m\delta(x) \le u(x) \le M\delta(x)^{\frac{2(1+\beta-r)}{\gamma}},$$
$$m\delta(x) \le v(x) \le M\delta(x)^{\frac{2(1+\alpha-s)}{\gamma}},$$

where m, M are positive constants and $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. Zhang proved also in [21] an exact boundary behavior and a uniqueness result to system (1.4).

We study in this paper system (1.1) in a more general situation that treat the cases $a_1, a_2 \neq 1$ with no restriction on the sign of the exponents. Our approach relies on the asymptotic behavior of solutions to the following singular elliptic problem

$$\begin{cases} -\Delta u = a(x)u^{\alpha}, \ \ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\alpha < 1$ and a satisfies

(H_0) $a \in C^{\mu}_{loc}(\Omega), 0 < \mu < 1$, satisfying for each $x \in \Omega$

$$a(x) \approx \delta(x)^{-\lambda} L(\delta(x)),$$

where $\lambda \le 2$ and $L \in \mathcal{K}$ such that $\int_0^{\eta} t^{1-\lambda} L(t)dt < \infty$.

Here and throughout, the notation $f(x) \approx g(x)$, $x \in S$, for f and g nonnegative functions defined on a set S, means that there exists $c > 0$ such that $\frac{1}{c}g(x) \le f(x) \le cg(x)$ for each $x \in S$.

In [14], Mâagli showed the following.

Theorem 1.2 ([14])**.** *Let $\alpha < 1$ and assume (H_0). Then problem (1.5) has a unique classical solution u satisfying for $x \in \Omega$,*

$$u(x) \approx \delta(x)^{\min(1, \frac{2-\lambda}{1-\alpha})} \tilde{L}(\delta(x)),$$

where $\tilde{L}$ is the function defined on $(0, \eta]$ by

$$\tilde{L}(t) := \begin{cases} \left(\int_0^t \frac{L(s)}{s} ds \right)^{\frac{1}{1-\alpha}} & \text{if } \lambda = 2, \\ (L(t))^{\frac{1}{1-\alpha}} & \text{if } 1 + \alpha < \lambda < 2, \\ \left(\int_t^{\eta} \frac{L(s)}{s} ds \right)^{\frac{1}{1-\alpha}} & \text{if } \lambda = 1 + \alpha, \\ 1 & \text{if } \lambda < 1 + \alpha. \end{cases}$$

Our approach relies closely on the result of [14] stated in Theorem 1.2 above. So, by applying the Karamata regular variation theory, we improve and extend the estimates established in [12, 21]. The proof of the existence in this paper is based on the sub-supersolution method.

2. MAIN RESULTS

Let us introduce our condition.

(H) For $i = 1, 2$, the function a_i is in $C^{\mu}_{loc}(\Omega)$, $0 < \mu < 1$ satisfying for each $x \in \Omega$

$$a_i(x) \approx \delta(x)^{-\lambda_i} L_i(\delta(x)),$$

where $\lambda_i \in \mathbb{R}$ and $L_i \in \mathcal{K}$.

We also introduce the quantities

$$\sigma_1 = \min\left\{1, \frac{2-\lambda_2}{1-\beta}\right\}, \quad \sigma_2 = \min\left\{1, \frac{2+s-\lambda_2}{1-\beta}\right\},$$

$$\sigma_3 = \frac{(2-\lambda_1)(1-\beta) + r(2-\lambda_2)}{\gamma} \quad \text{and} \quad \sigma_4 = \frac{(2-\lambda_2)(1-\alpha) + s(2-\lambda_1)}{\gamma}.$$

The above values of σ_i $(i = 1, \dots, 4)$ are related to the boundary behavior of solutions to problem (1.1), as it will be explained in our main results stated in the following theorems.

Theorem 2.1. *Assume (H) and that the exponents $\alpha, \beta < 1$ and $r, s \in \mathbb{R}$ satisfy (1.2). Suppose that*

$$\lambda_1 - r\sigma_1 = 2, \quad \lambda_2 < 2, \quad \lambda_2 \neq 1 + \beta$$

and L_1, L_2 satisfy

(a) $\int_0^\eta \frac{L_1(t)(L_2(t))^{\frac{r}{1-\beta}}}{t} dt < \infty$ *if* $1 + \beta < \lambda_2 < 2$,
(b) $\int_0^\eta \frac{L_1(t)}{t} dt < \infty$ *if* $\lambda_2 < 1 + \beta$.

Then system (1.1) has a classical solution (u, v) satisfying

$$u(x) \approx \tilde{L}_1(\delta(x)) \quad \textit{and} \quad v(x) \approx \delta(x)^{\sigma_1} \tilde{L}_2(\delta(x)),$$

where $\tilde{L}_1, \tilde{L}_2 \in \mathcal{K}$.

Theorem 2.2. *Assume (H) and that the exponents $\alpha, \beta < 1$ and $r, s \in \mathbb{R}$ satisfy (1.2). Suppose that*

$$\lambda_1 - r\sigma_2 \leq 1 + \alpha, \lambda_2 < 2 + s \quad \textit{and} \quad (\lambda_1, \lambda_2) \neq (1 + \alpha + r, 1 + \beta + s).$$

Then system (1.1) has a classical solution (u, v) satisfying

$$u(x) \approx \delta(x)\tilde{L}_1(\delta(x)) \quad \textit{and} \quad v(x) \approx \delta(x)^{\sigma_2} \tilde{L}_2(\delta(x)),$$

where $\tilde{L}_1, \tilde{L}_2 \in \mathcal{K}$.

Theorem 2.3. *Assume (H) and that the exponents* $\alpha, \beta < 1$ *and* $r, s \in \mathbb{R}$ *satisfy* (1.2). *Suppose that*

$$1+\beta < \lambda_2 - s\sigma_3 < 2 \quad \textit{and} \quad 1+\alpha < \lambda_1 - r\sigma_4 < 2,$$

or equivalently

$$0 < \sigma_3 < 1 \quad \textit{and} \quad 0 < \sigma_4 < 1.$$

Then system (1.1) *has a classical solution* (u, v) *satisfying*

$$u(x) \approx \delta(x)^{\sigma_3} \tilde{L}_1(\delta(x)) \quad \textit{and} \quad v(x) \approx \delta(x)^{\sigma_4} \tilde{L}_2(\delta(x)),$$

where $\tilde{L}_1, \tilde{L}_2 \in \mathcal{K}$.

Remark 2.4. (i) Since system (1.1) is invariant under the transform

$$(u, v, \alpha, \beta, r, s) \longrightarrow (v, u, \beta, \alpha, s, r),$$

then we have also the existence of a classical solution (u, v) for system (1.1) that behave like $\delta(x)^{\sigma} \tilde{L}(\delta(x))$ where $\sigma \in [0, 1]$ depends on $\alpha, \beta, r, s, \lambda_1, \lambda_2$ and $\tilde{L} \in \mathcal{K}$, if one of the following conditions holds:

(a) $\lambda_2 - s \min\{1, \frac{2-\lambda_1}{1-\alpha}\} = 2, \lambda_1 < 2$ and $\lambda_1 \neq 1 + \alpha$,
(b) $\lambda_2 - s \min\{1, \frac{2+r-\lambda_1}{1-\alpha}\} \leq 1 + \beta$, $\lambda_1 < 2 + r$ and $(\lambda_1, \lambda_2) \neq (1 + \alpha + r, 1 + \beta + s)$.

(ii) Theorems 2.1, 2.2 and 2.3 extend results in a previous work [13] due to Giacomoni, Hernandez and Sauvy in the semi linear case, which involve a smaller class of nonlinearities.

To simplify our statements, we give some notations to be used later. We refer to $C(\bar{\Omega})$ as the collection of all continuous functions in $\bar{\Omega}$ and $C_0(\Omega)$ the subclass of $C(\bar{\Omega})$ consisting of functions which vanish continuously on $\partial\Omega$. We denote by $G_\Omega(x, y)$ the Green function of the Laplace operator $-\Delta$ in Ω, with Dirichlet boundary conditions. The potential kernel V is defined on $B^+(\Omega)$ by

$$Vf(x) = \int_\Omega G_\Omega(x, y) f(y) dy, \quad x \in \Omega.$$

We recall that if $f \in C^\mu_{loc}(\Omega), 0 < \mu < 1$, then $Vf \in C^{2,\mu}_{loc}(\Omega) \cap C_0(\Omega)$ and satisfies

$$-\Delta Vf = f \quad \text{in} \quad \Omega.$$

The letter c denotes a generic positive constant which may vary from line to line.

The rest of the paper is as follows. In Section 3, we give an existence result based on the sub-supersolution method that is a key tool to prove the existence of solutions in Theorems 2.1, 2.2 and 2.3. In Section 4, we recall some already known properties of functions in $\mathcal{K}$ and we give the proof of our main results. Section 5 deals with an example that illustrates the asymptotic behavior of solutions in Theorems 2.1, 2.2, 2.3 and that involves further some limiting cases which we reach in this example.

3. TECHNICAL CONDITION TO EXISTENCE RESULT

In this section, we adopt a sub-supersolution method. We consider the more general system

$$\begin{cases} -\Delta u = h_1(x,u,v), & u>0 \quad \text{in } \Omega, \\ -\Delta v = h_2(x,u,v), & v>0 \quad \text{in } \Omega, \\ u=v=0 & \quad\ \ \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Definition 3.1. A pair of positive functions $(u,v) \in C^2(\Omega)\cap C(\bar{\Omega})$ is called a supersolution to problem (3.1) if

$$\begin{cases} -\Delta u \geq h_1(x,u,v) & \text{in } \Omega, \\ -\Delta v \geq h_2(x,u,v) & \text{in } \Omega, \\ u=v=0 & \text{on } \partial\Omega. \end{cases}$$

If the above inequalities are reversed, (u,v) is called a subsolution to problem (3.1).

Lemma 3.2 ([21, Lemma 3.1])**.** *Let $h_i : \Omega\times(0,\infty)\times(0,\infty)\longrightarrow\mathbb{R}$ be a continuous function for $(i=1,2)$. Suppose that problem (3.1) has a supersolution $(\bar{u},\bar{v})$ and a subsolution $(\underline{u},\underline{v})$ such that $\underline{u}\leq\bar{u}$ and $\underline{v}\leq\bar{v}$ on $\bar{\Omega}$, then system (3.1) has a positive solution $(u,v)\in C^2(\Omega)\cap C(\bar{\Omega})$ such that $\underline{u}\leq u\leq\bar{u}$ and $\underline{v}\leq v\leq\bar{v}$ in $\bar{\Omega}$.*

Proposition 3.3. *Assume (H) and that the exponents $\alpha,\beta<1$ and $r,s\in\mathbb{R}$ satisfy (1.2). Suppose that there exist nonnegative functions θ and φ in $C_0(\Omega)$ satisfying for each $x\in\Omega$,*

$$V(a_1\theta^{\alpha}\varphi^{r})(x)\approx\theta(x) \quad \text{and} \quad V(a_2\theta^{s}\varphi^{\beta})(x)\approx\varphi(x). \tag{3.2}$$

Then system (1.1) has a classical solution (u,v) satisfying for each $x\in\Omega$,

$$u(x)\approx\theta(x) \quad \text{and} \quad v(x)\approx\varphi(x).$$

Proof. Let $c>1$ and θ,φ be nonnegative functions in $C_0(\Omega)$ satisfying for each $x\in\Omega$,

$$\frac{1}{c}\theta(x)\leq V(a_1\theta^{\alpha}\varphi^{r})(x)\leq c\theta(x)$$

and

$$\frac{1}{c}\varphi(x)\leq V(a_2\theta^{s}\varphi^{\beta})(x)\leq c\varphi(x).$$

We point that in view of (1.2), there exist $\nu_1,\nu_2>0$ such that

$$\begin{cases} (1-\alpha)\,\nu_1 - r\nu_2 > 0, \\ -s\nu_1 + (1-\beta)\,\nu_2 > 0. \end{cases}$$

Hence, put $c_1=m^{\nu_1}$ and $c_2=m^{\nu_2}$ for m large enough, we have $c_1,c_2>1$ and

$$\begin{cases} c^{|\alpha|+|r|}\leq c_1^{1-\alpha}c_2^{-r}, \\ c^{|s|+|\beta|}\leq c_1^{-s}c_2^{1-\beta}. \end{cases}$$

Put

$$\bar{u} = c_1 V(a_1\theta^\alpha\varphi^r), \quad \underline{u} = \frac{1}{c_1} V(a_1\theta^\alpha\varphi^r),$$

$$\bar{v} = c_2 V(a_2\theta^s\varphi^\beta) \quad \text{and} \quad \underline{v} = \frac{1}{c_2} V(a_2\theta^s\varphi^\beta).$$

Since θ, φ are nonnegative functions in $C_0(\Omega)$, then $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ are in $C^2(\Omega) \cap C_0(\Omega)$ and satisfy $\underline{u} \leq \bar{u}$ and $\underline{v} \leq \bar{v}$. Moreover, we have

$$\begin{aligned} -\Delta\bar{u} &= c_1 a_1 \theta^\alpha \varphi^r \\ &\geq c^{-|\alpha|-|r|} c_1 a_1 (V(a_1\theta^\alpha\varphi^r))^\alpha (V(a_2\theta^s\varphi^\beta))^r \\ &= c^{-|\alpha|-|r|} c_1^{1-\alpha} c_2^{-r} a_1 \bar{u}^\alpha \bar{v}^r \geq a_1 \bar{u}^\alpha \bar{v}^r \end{aligned}$$

and

$$\begin{aligned} -\Delta\underline{u} &= c_1^{-1} a_1 \theta^\alpha \varphi^r \\ &\leq c^{|\alpha|+|r|} c_1^{-1} a_1 (V(a_1\theta^\alpha\varphi^r))^\alpha (V(a_2\theta^s\varphi^\beta))^r \\ &= c^{|\alpha|+|r|} c_1^{\alpha-1} c_2^{r} a_1 \underline{u}^\alpha \underline{v}^r \leq a_1 \underline{u}^\alpha \underline{v}^r. \end{aligned}$$

Similarly, we have

$$\begin{aligned} -\Delta\bar{v} &= c_2 a_2 \theta^s \varphi^\beta \\ &\geq c^{-|\beta|-|s|} c_2 (V(a_1\theta^\alpha\varphi^r))^s (V(a_2\theta^s\varphi^\beta))^\beta \\ &= c^{-|\beta|-|s|} c_2^{1-\beta} c_1^{-s} a_2 \bar{u}^s \bar{v}^\beta \geq a_2 \bar{u}^s \bar{v}^\beta \end{aligned}$$

and

$$\begin{aligned} -\Delta\underline{v} &= c_2^{-1} a_2 \theta^s \varphi^\beta \\ &\leq c^{|\beta|+|s|} c_2^{-1} a_2 (V(a_1\theta^\alpha\varphi^r))^s (V(a_2\theta^s\varphi^\beta))^\beta \\ &= c^{|\beta|+|s|} c_2^{\beta-1} c_1^{s} a_2 \underline{u}^s \underline{v}^\beta \\ &\leq a_2 \underline{u}^s \underline{v}^\beta. \end{aligned}$$

Hence $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ are respectively a subsolution and a supersolution to system (1.1). Then, the result follows by using Lemma 3.2. □

4. EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS

4.1. THE KARAMATA CLASS $\mathcal{K}$

We recapitulate in this paragraph some properties of functions in the class $\mathcal{K}$ with reference to Karamata regular variation theory which are useful for our study.

Lemma 4.1 ([20]).

(i) *A function L is in $\mathcal{K}$ if and only if L is a positive function in $C^1\big((0,\eta]\big)$ such that*

$$\lim_{t\to 0}\frac{tL'(t)}{L(t)}=0.$$

(ii) *Let $L_1, L_2\in\mathcal{K}, p\in\mathbb{R}$ and $\varepsilon>0$, then we have*

$$L_1L_2\in\mathcal{K},\quad L_1^p\in\mathcal{K}$$

and

$$\lim_{t\longrightarrow 0}t^{\varepsilon}L_1(t)=0.$$

Lemma 4.2 ([3]). *Let $L\in\mathcal{K}$ be defined on $(0,\eta]$. Then, we have*

$$t\mapsto\int_t^{\eta}\frac{L(x)}{x}dx\in\mathcal{K}.$$

If further $\int_0^{\eta}\frac{L(t)}{t}dt$ converges, then

$$t\mapsto\int_0^{t}\frac{L(x)}{x}dx\in\mathcal{K}.$$

Lemma 4.3 (Karamata's Theorem). *Let $L\in\mathcal{K}$ be defined on $(0,\eta]$ and $\sigma\in\mathbb{R}$. Then*

(i) *if $\sigma>-1$, then $\int_0^{\eta}t^{\sigma}L(t)dt$ converges and*

$$\int_0^t s^{\sigma}L(s)ds\sim\frac{t^{1+\sigma}L(t)}{\sigma+1}\quad as\quad t\longrightarrow 0^+,$$

(ii) *if $\sigma<-1$, then $\int_0^{\eta}t^{\sigma}L(t)dt$ diverges and*

$$\int_t^{\eta} s^{\sigma}L(s)ds\sim-\frac{t^{1+\sigma}L(t)}{\sigma+1}\quad as\quad t\longrightarrow 0^+.$$

Remark 4.4. According to Lemma 4.3, we need to verify condition $\int_0^{\eta}t^{1-\lambda}L(t)dt<\infty$ in hypothesis (H_0), only for $\lambda=2$.

We give here a typical example of functions in $\mathcal{K}$.

Example 4.5. Let $p\in\mathbb{N}^*$, $(\lambda_1,\lambda_2,\dots,\lambda_p)\in\mathbb{R}^p$ and ω be a positive real number sufficiently large such that the function

$$L(t):=\prod_{k=1}^{p}\left(\log_k\left(\frac{\omega}{t}\right)\right)^{\lambda_k}$$

is well defined and positive on $(0,\eta]$, where $\log_k(x)=\log o\log o\dots o\log(x)$ (k times). Using Lemma 4.1 (i), we have clearly that $L\in\mathcal{K}$.

4.2. PROOFS OF MAIN RESULTS

The main idea in order to prove Theorems 2.1, 2.2 and 2.3 is to find functions θ and φ satisfying (3.2). This enable us to construct a subsolution and a supersolution to system (1.1) of the form $(cV(a_1\theta^\alpha\varphi^r), cV(a_2\theta^s\varphi^\beta))$ and to deduce by Proposition 3.3 the existence of a solution (u,v) to (1.1) satisfying

$$u(x) \approx \theta(x) \quad \text{and} \quad v(x) \approx \varphi(x).$$

To this end, we consider the following decoupled system

$$\begin{cases} -\Delta w_1 = p(x) w_1^\alpha, \;\; w_1 > 0 & \text{in } \Omega, \\ -\Delta w_2 = q(x) w_2^\beta, \;\; w_2 > 0 & \text{in } \Omega, \\ w_1 = w_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $p(x) = a_1(x)\varphi^r(x)$ and $q(x) = a_2(x)\theta^s(x)$. The choice of θ and φ depends closely on cases according to the boundary behavior of a solution to the semilinear elliptic problem (1.5) as described in Theorem 1.2. Hence a solution (w_1, w_2) of (4.1) is given by Theorem 1.2 such that $w_1 \approx \theta$ and $w_2 \approx \varphi$. Consequently, the functions θ and φ will satisfy

$$\theta \approx w_1 = V(pw_1^\alpha) \approx V(a_1\theta^\alpha\varphi^r),$$
$$\varphi \approx w_2 = V(qw_2^\beta) \approx V(a_2\theta^s\varphi^\beta),$$

which is our principal aim.

We prove in the following our main results.

4.2.1. Proof of Theorem 2.1

Proof. Assume that

$$\lambda_1 - r\sigma_1 = 2, \;\; \lambda_2 < 2 \quad \text{and} \quad \lambda_2 \neq 1 + \beta.$$

We divide the proof into two cases.

Case 1. $\lambda_1 - r\frac{2-\lambda_2}{1-\beta} = 2, 1+\beta < \lambda_2 < 2$ and $\int_0^\eta \frac{L_1(t)(L_2(t))^{\frac{r}{1-\beta}}}{t} dt < \infty$.

Let

$$\theta(x) := \left(\int_0^{\delta(x)} \frac{L_1(t)(L_2(t))^{\frac{r}{1-\beta}}}{t} dt \right)^{\frac{1-\beta}{\gamma}},$$

$$\varphi(x) := \delta(x)^{\frac{2-\lambda_2}{1-\beta}} (L_2(\delta(x)))^{\frac{1}{1-\beta}} \left(\int_0^{\delta(x)} \frac{L_1(t)(L_2(t))^{\frac{r}{1-\beta}}}{t} dt \right)^{\frac{s}{\gamma}}$$

and consider system (4.1). Using (H), we have

$$p(x) \approx (\delta(x))^{-2}\tilde{L}_1(\delta(x)) \quad \text{and} \quad q(x) \approx (\delta(x))^{-\lambda_2}\tilde{L}_2(\delta(x)),$$

where

$$\tilde{L}_1(t) = L_1(t)(L_2(t))^{\frac{r}{1-\beta}} \left(\int_0^t \frac{L_1(x)(L_2(x))^{\frac{r}{1-\beta}}}{x} dx \right)^{\frac{rs}{\gamma}}$$

and

$$\tilde{L}_2(t) = L_2(t) \left(\int_0^t \frac{L_1(x)(L_2(x))^{\frac{r}{1-\beta}}}{x} dx \right)^{\frac{s(1-\beta)}{\gamma}}.$$

By Lemma 4.2 and Lemma 4.1 (ii), the functions $\tilde{L}_1$ and $\tilde{L}_2$ are in $\mathcal{K}$ and we have

$$\int_0^\eta \frac{\tilde{L}_1(t)}{t} dt \approx \left(\int_0^\eta \frac{L_1(t)(L_2(t))^{\frac{r}{1-\beta}}}{t} dt \right)^{\frac{(1-\alpha)(1-\beta)}{\gamma}} < \infty.$$

It follows by Theorem 1.2 that system (4.1) has a solution (w_1, w_2) satisfying for each $x \in \Omega$,

$$w_1(x) \approx \left(\int_0^{\delta(x)} \frac{\tilde{L}_1(t)}{t} dt \right)^{\frac{1}{1-\alpha}} \approx \theta(x) \quad \text{and} \quad w_2(x) \approx \delta(x)^{\frac{2-\lambda_2}{1-\beta}} (\tilde{L}_2(\delta(x)))^{\frac{1}{1-\beta}} = \varphi(x).$$

Hence (3.2) is satisfied and so the result holds by using Proposition 3.3.

Case 2. $\lambda_1 - r = 2$, $\lambda_2 < 1 + \beta$ and $\int_0^\eta \frac{L_1(t)}{t} dt < \infty$.

Let

$$\theta(x) := \left(\int_0^{\delta(x)} \frac{L_1(t)}{t} dt \right)^{\frac{1}{1-\alpha}}, \quad \varphi(x) := \delta(x)$$

and consider system (4.1). Using (H), we have

$$p(x) \approx (\delta(x))^{-2} L_1(\delta(x)) \quad \text{and} \quad q(x) \approx (\delta(x))^{-\lambda_2} \tilde{L}_2(\delta(x)),$$

where

$$\tilde{L}_2(t) = L_2(t) \left(\int_0^t \frac{L_1(x)}{x} dx \right)^{\frac{s}{1-\alpha}}.$$

By Lemma 4.2 and Lemma 4.1 (ii), the function $\tilde{L}_2$ is in $\mathcal{K}$. Using that $\int_0^\eta \frac{L_1(t)}{t} dt < \infty$, it follows by Theorem 1.2 that system (4.1) has a solution (w_1, w_2) satisfying for each $x \in \Omega$,

$$w_1(x) \approx \left(\int_0^{\delta(x)} \frac{L_1(t)}{t} dt \right)^{\frac{1}{1-\alpha}} = \theta(x) \quad \text{and} \quad w_2(x) \approx \delta(x) = \varphi(x).$$

Hence (3.2) is satisfied and so the result holds by using Proposition 3.3. This completes the proof.

□

4.2.2. Proof of Theorem 2.2

Proof. Assume that

$$\lambda_1 - r\sigma_2 \leq 1+\alpha, \quad \lambda_2 < 2+s \quad \text{and} \quad (\lambda_1,\lambda_2) \neq (1+\alpha+r, 1+\beta+s).$$

We divide the proof into five cases.

Case 1. $\lambda_1 - r\frac{2-\lambda_2+s}{1-\beta} = 1+\alpha$, $1+\beta < \lambda_2 - s < 2$. Let

$$\theta(x) := \delta(x)\left(\int_{\delta(x)}^{\eta} \frac{L_1(t)(L_2(t))^{\frac{r}{1-\beta}}}{t}dt\right)^{\frac{1-\beta}{\gamma}},$$

$$\varphi(x) := \delta(x)^{\frac{2+s-\lambda_2}{1-\beta}}(L_2(\delta(x)))^{\frac{1}{1-\beta}}\left(\int_{\delta(x)}^{\eta} \frac{L_1(t)(L_2(t))^{\frac{r}{1-\beta}}}{t}dt\right)^{\frac{s}{\gamma}}$$

and consider system (4.1). Using (H), we have

$$p(x) \approx \delta(x)^{-1-\alpha}\tilde{L}_1(\delta(x)) \quad \text{and} \quad q(x) \approx \delta(x)^{-\lambda_2+s}\tilde{L}_2(\delta(x)),$$

where

$$\tilde{L}_1(t) = L_1(t)(L_2(t))^{\frac{r}{1-\beta}}\left(\int_t^{\eta} \frac{L_1(x)(L_2(x))^{\frac{r}{1-\beta}}}{x}dx\right)^{\frac{rs}{\gamma}}$$

and

$$\tilde{L}_2(t) = L_2(t)\left(\int_t^{\eta} \frac{L_1(x)(L_2(x))^{\frac{r}{1-\beta}}}{x}dx\right)^{\frac{s(1-\beta)}{\gamma}}.$$

In view of Lemma 4.2 and Lemma 4.1 (ii), the functions $\tilde{L}_1$ and $\tilde{L}_2$ are in $\mathcal{K}$. We deduce from Theorem 1.2 that system (4.1) has a solution (w_1, w_2) satisfying for each $x \in \Omega$,

$$w_1(x) \approx \delta(x)\left(\int_{\delta(x)}^{\eta} \frac{\tilde{L}_1(t)}{t}dt\right)^{\frac{1}{1-\alpha}} \approx \delta(x)\left(\int_{\delta(x)}^{\eta} \frac{L_1(t)(L_2(t))^{\frac{r}{1-\beta}}}{t}dt\right)^{\frac{1-\beta}{\gamma}} = \theta(x)$$

and

$$w_2(x) \approx \delta(x)^{\frac{2+s-\lambda_2}{1-\beta}}(\tilde{L}_2(\delta(x)))^{\frac{1}{1-\beta}} = \varphi(x).$$

Case 2. $\lambda_1 - r = 1+\alpha$ and $\lambda_2 - s < 1+\beta$. Let

$$\theta(x) := \delta(x)\left(\int_{\delta(x)}^{\eta} \frac{L_1(t)}{t}dt\right)^{\frac{1}{1-\alpha}}, \quad \varphi(x) := \delta(x)$$

and consider system (4.1). Using (H), we have

$$p(x) \approx \delta(x)^{-\lambda_1+r}\tilde{L}_1(\delta(x)) \quad \text{and} \quad q(x) \approx \delta(x)^{-\lambda_2+s}L_2(\delta(x)),$$

where

$$\tilde{L}_1(t) = L_1(t)\left(\int_{t}^{\eta} \frac{L_2(x)}{x}dx\right)^{\frac{r}{1-\beta}}.$$

In view of Lemma 4.2 and Lemma 4.1 (ii), $\tilde{L}_1 \in \mathcal{K}$. So by Theorem 1.2, system (4.1) has a solution (w_1, w_2) satisfying for each $x \in \Omega$,

$$w_1(x) \approx \delta(x) = \theta(x)$$

and

$$w_2(x) \approx \delta(x)\left(\int_{\delta(x)}^{\eta} \frac{L_2(t)}{t}dt\right)^{\frac{1}{1-\beta}} = \varphi(x).$$

Case 3. $\lambda_1 - r\frac{2-\lambda_2+s}{1-\beta} < 1+\alpha$, $1+\beta < \lambda_2 - s < 2$. Let

$$\theta(x) := \delta(x), \quad \varphi(x) := \delta(x)^{\frac{2-\lambda_2+s}{1-\beta}}(L_2(\delta(x)))^{\frac{1}{1-\beta}}$$

and consider system (4.1). Using (H), we have

$$p(x) \approx \delta(x)^{-\lambda_1+r\frac{2-\lambda_2+s}{1-\beta}}L_1(\delta(x))(L_2(\delta(x)))^{\frac{r}{1-\beta}} \quad \text{and} \quad q(x) \approx \delta(x)^{-\lambda_2+s}L_2(\delta(x)).$$

In view of Lemma 4.1 (ii) and Theorem 1.2, system (4.1) has a solution (w_1, w_2) satisfying for each $x \in \Omega$,

$$w_1(x) \approx \delta(x) = \theta(x) \quad \text{and} \quad v(x) \approx \delta(x)^{\frac{2-\lambda_2+s}{1-\beta}}(L_2(\delta(x)))^{\frac{1}{1-\beta}} = \varphi(x).$$

Case 4. $\lambda_1 - r < 1+\alpha$ and $\lambda_2 - s = 1+\beta$. Interchanging the role of u and v, the proof is the same as in Case 2 above.

Case 5. $\lambda_1 - r < 1+\alpha$ and $\lambda_2 - s < 1+\beta$. Let

$$\theta(x) = \varphi(x) = \delta(x)$$

and consider system (4.1). Using (H), we have

$$p(x) \approx \delta(x)^{-\lambda_1+r}L_1(\delta(x)) \quad \text{and} \quad q(x) \approx \delta(x)^{-\lambda_2+s}L_2(\delta(x)).$$

It follows by Theorem 1.2 that system (4.1) has a solution (w_1, w_2) satisfying for each $x \in \Omega$,

$$w_1(x) \approx \delta(x) = \theta(x)$$

and

$$w_2(x) \approx \delta(x) = \varphi(x).$$

Then the result follows by Proposition 3.3. This completes the proof. □

4.2.3. Proof of Theorem 2.3

Proof. Suppose that

$$1 + \beta < \lambda_2 - s\sigma_3 < 2 \quad \text{and} \quad 1 + \alpha < \lambda_1 - r\sigma_4 < 2.$$

Let

$$\theta(x) := \delta(x)^{\sigma_3} (L_1(\delta(x)))^{\frac{(1-\beta)}{\gamma}} (L_2(\delta(x)))^{\frac{r}{\gamma}},$$

$$\varphi(x) := \delta(x)^{\sigma_4} (L_2(\delta(x)))^{\frac{(1-\alpha)}{\gamma}} (L_1(\delta(x)))^{\frac{s}{\gamma}}$$

and consider system (4.1). Using (H), we have

$$p(x) \approx (\delta(x))^{r\sigma_4 - \lambda_1} \tilde{L}_1(\delta(x)) \quad \text{and} \quad q(x) \approx (\delta(x))^{s\sigma_3 - \lambda_2} \tilde{L}_2(\delta(x)),$$

where

$$\tilde{L}_1(t) = (L_2(t))^{\frac{r(1-\alpha)}{\gamma}} (L_1(t))^{\frac{(1-\alpha)(1-\beta)}{\gamma}} \quad \text{and} \quad \tilde{L}_2(t) = (L_1(t))^{\frac{s(1-\beta)}{\gamma}} (L_2(t))^{\frac{(1-\alpha)(1-\beta)}{\gamma}}.$$

By Lemma 4.1 (ii), the functions $\tilde{L}_1$ and $\tilde{L}_2$ are in $\mathcal{K}$. Put $\mu_1 = \lambda_1 - r\sigma_4$ and $\mu_2 = \lambda_2 - r\sigma_3$. Since $\mu_1 \in (1 + \alpha, 2)$ and $\mu_2 \in (1 + \beta, 2)$, then by Theorem 1.2, we deduce that system (4.1) has a solution (w_1, w_2) satisfying for each $x \in \Omega$,

$$w_1(x) \approx \delta(x)^{\frac{2-\mu_1}{1-\alpha}} (\tilde{L}_1(\delta(x)))^{\frac{1}{1-\alpha}}$$

and

$$w_2(x) \approx \delta(x)^{\frac{2-\mu_2}{1-\beta}} (\tilde{L}_2(\delta(x)))^{\frac{1}{1-\beta}}.$$

By calculus, we have $\frac{2-\mu_1}{1-\alpha} = \sigma_3$ and $\frac{2-\mu_2}{1-\beta} = \sigma_4$. This implies that $w_1 \approx \theta$ and $w_2 \approx \varphi$. The result is given by Proposition 3.3. This completes the proof. □

5. EXTREMAL CASES

In some examples, we can extend results of Theorems 2.1 and 2.2 by reaching some extremal cases. To illustrate this, we discuss in this section the example of functions a_i $(i = 1, 2)$ in system (1.1) satisfying

$$a_i(x) \approx \delta(x)^{-\lambda_i} \Big(\log \frac{\omega}{\delta(x)} \Big)^{-\mu_i},$$

where $\omega > 2\,\mathrm{diam}(\Omega)$ and $\mu_i \in \mathbb{R}$.

Namely, we apply Theorems 2.1, 2.2 and 2.3 to give explicitly the asymptotic behavior of solutions of system (1.1) in this example. Moreover, we prove the existence and we describe the asymptotic behavior of solutions to system (1.1) in some extremal cases. This is stated in Theorem 6 below.

First, we give the following elementary lemma.

Lemma 5.1. *Let $\mu \in \mathbb{R}$, $\omega > 2\operatorname{diam}(\Omega)$ and $\operatorname{diam}(\Omega) < \eta < \omega$, then for $x \in \Omega$ we have*

$$\int_{\delta(x)}^{\eta} \frac{1}{t}\left(\log\frac{\omega}{t}\right)^{-\mu} dt \approx \begin{cases} 1, & \text{if } \mu > 1, \\ \log\left(\log\frac{\omega}{\delta(x)}\right), & \text{if } \mu = 1, \\ \left(\log\frac{\omega}{\delta(x)}\right)^{1-\mu}, & \text{if } \mu < 1. \end{cases} \tag{5.1}$$

If we suppose further $\mu > 1$, then for $x \in \Omega$ we have

$$\int_{0}^{\delta(x)} \frac{1}{t}\left(\log\frac{\omega}{t}\right)^{-\mu} dt \approx \left(\log\frac{\omega}{\delta(x)}\right)^{1-\mu}. \tag{5.2}$$

Theorem 5.2. *Let $L_i(t) = (\log\frac{\omega}{t})^{-\mu_i}, (i = 1, 2)$. Assume (H) and that the exponents $\alpha, \beta < 1$ and $r, s \in \mathbb{R}$ satisfy (1.2).*

(i) *Assume that $\lambda_1 = \lambda_2 = 2$ and suppose that*

$$(1-\beta)(1-\mu_1) + r(1-\mu_2) < 0 \quad \text{and} \quad (1-\alpha)(1-\mu_2) + s(1-\mu_1) < 0.$$

Then, system (1.1) has a classical solution (u, v) satisfying for each $x \in \Omega$,

$$u(x) \approx \left(\log\frac{\omega}{\delta(x)}\right)^{\frac{(1-\beta)(1-\mu_1)+r(1-\mu_2)}{\gamma}}$$

and

$$v(x) \approx \left(\log\frac{\omega}{\delta(x)}\right)^{\frac{(1-\alpha)(1-\mu_2)+s(1-\mu_1)}{\gamma}}.$$

(ii) *Assume that $\lambda_1 - r\sigma_1 = 2$ and $\lambda_2 < 2$.*

(*a*) *If $1+\beta < \lambda_2 < 2$ and $(1-\beta)(1-\mu_1) - r\mu_2 < 0$, then system (1.1) has a classical solution (u, v) satisfying for each $x \in \Omega$,*

$$u(x) \approx \left(\log\frac{\omega}{\delta(x)}\right)^{\frac{(1-\beta)(1-\mu_1)-r\mu_2}{\gamma}}$$

and

$$v(x) \approx \delta(x)^{\frac{2-\lambda_2}{1-\beta}}\left(\log\frac{\omega}{\delta(x)}\right)^{\frac{s(1-\mu_1)-\mu_2(1-\alpha)}{\gamma}}.$$

(*b*) *If* $\lambda_2 = 1+\beta$, $(1-\beta)(1-\mu_1)+r(1-\mu_2) < 0$ *and* $(1-\alpha)(1-\mu_2)+s(1-\mu_1) > 0$, *then system* (1.1) *has a classical solution* (u,v) *satisfying for each* $x \in \Omega$,

$$u(x) \approx \Big(\log \frac{\omega}{\delta(x)}\Big)^{\frac{(1-\beta)(1-\mu_1)+r(1-\mu_2)}{\gamma}}$$

and

$$v(x) \approx \delta(x)\Big(\log \frac{\omega}{\delta(x)}\Big)^{\frac{(1-\alpha)(1-\mu_2)+s(1-\mu_1)}{\gamma}}.$$

(*c*) *If* $\lambda_2 = 1+\beta$, $\mu_1 > 1$ *and* $(1-\alpha)(1-\mu_2)+s(1-\mu_1) < 0$, *then system* (1.1) *has a classical solution* (u,v) *satisfying for each* $x \in \Omega$,

$$u(x) \approx \Big(\log \frac{\omega}{\delta(x)}\Big)^{\frac{1-\mu_1}{1-\alpha}} \quad \text{and} \quad v(x) \approx \delta(x).$$

(*d*) *If* $\lambda_2 < 1+\beta$ *and* $\mu_1 > 1$, *then system* (1.1) *has a classical solution* (u,v) *satisfying for each* $x \in \Omega$,

$$u(x) \approx \Big(\log \frac{\omega}{\delta(x)}\Big)^{\frac{1-\mu_1}{1-\alpha}} \text{ and } v(x) \approx \delta(x).$$

(iii) *Assume that* $\lambda_1 - r\sigma_2 = 1+\alpha$ *and* $\lambda_2 - s < 2$.

(*a*) *If* $1+\beta < \lambda_2 - s < 2$, *then system* (1.1) *has a classical solution* (u,v) *satisfying for each* $x \in \Omega$,

$$u(x) \approx \delta(x) \begin{cases} \Big(\log \frac{\omega}{\delta(x)}\Big)^{\frac{(1-\beta)(1-\mu_1)-r\mu_2}{\gamma}}, & \text{if } (1-\beta)(1-\mu_1)-r\mu_2 > 0,\\ \Big(\log\Big(\log \frac{\omega}{\delta(x)}\Big)\Big)^{\frac{1-\beta}{\gamma}}, & \text{if } (1-\beta)(1-\mu_1)-r\mu_2 = 0,\\ 1, & \text{if } (1-\beta)(1-\mu_1)-r\mu_2 < 0 \end{cases}$$

and

$$v(x) \approx \delta(x)^{\frac{2+s-\lambda_2}{1-\beta}} \begin{cases} \Big(\log \frac{\omega}{\delta(x)}\Big)^{\frac{s(1-\mu_1)-\mu_2(1-\alpha)}{\gamma}}, & \text{if } (1-\beta)(1-\mu_1)-r\mu_2 > 0,\\ \Big(\log \frac{\omega}{\delta(x)}\Big)^{\frac{-\mu_2}{1-\beta}} (\log(\log \frac{\omega}{\delta(x)}))^{\frac{s}{\gamma}}, & \text{if } (1-\beta)(1-\mu_1)-r\mu_2 = 0,\\ \Big(\log \frac{\omega}{\delta(x)}\Big)^{\frac{-\mu_2}{1-\beta}}, & \text{if } (1-\beta)(1-\mu_1)-r\mu_2 < 0. \end{cases}$$

(*b*) *If* $\lambda_2 - s = 1+\beta$, $(1-\beta)(1-\mu_1)+r(1-\mu_2) > 0$ *and* $(1-\alpha)(1-\mu_2)+s(1-\mu_1) > 0$, *then system* (1.1) *has a classical solution* (u,v) *satisfying for each* $x \in \Omega$,

$$u(x) \approx \delta(x)\Big(\log \frac{\omega}{\delta(x)}\Big)^{\frac{(1-\beta)(1-\mu_1)+r(1-\mu_2)}{\gamma}}$$

and

$$v(x) \approx \delta(x)\Big(\log \frac{\omega}{\delta(x)}\Big)^{\frac{(1-\alpha)(1-\mu_2)+s(1-\mu_1)}{\gamma}}.$$

(c) *If* $\lambda_2 - s = 1+\beta$, $\mu_1 > 1$ *and* $\mu_2 > 1$, *then system* (1.1) *has a classical solution* (u,v) *satisfying for each* $x \in \Omega$,

$$u(x) \approx \delta(x) \quad \text{and} \quad v(x) \approx \delta(x).$$

(d) *If* $\lambda_2 - s = 1+\beta$, $\mu_1 < 1$ *and* $(1-\alpha)(1-\mu_2) + s(1-\mu_1) < 0$, *then system* (1.1) *has a classical solution* (u,v) *satisfying for each* $x \in \Omega$,

$$u(x) \approx \delta(x)\Big(\log\frac{\omega}{\delta(x)}\Big)^{\frac{1-\mu_1}{1-\alpha}} \quad \text{and} \quad v(x) \approx \delta(x).$$

(e) *If* $\lambda_2 - s = 1+\beta$, $\mu_2 < 1$ *and* $(1-\beta)(1-\mu_1) + r(1-\mu_2) < 0$, *then system* (1.1) *has a classical solution* (u,v) *such that*

$$u(x) \approx \delta(x) \quad \text{and} \quad v(x) \approx \delta(x)\Big(\log\frac{\omega}{\delta(x)}\Big)^{\frac{1-\mu_2}{1-\beta}}.$$

(f) *If* $\lambda_2 - s < 1+\beta$, *then system* (1.1) *has a classical solution* (u,v) *satisfying for each* $x \in \Omega$,

$$u(x) \approx \delta(x)\begin{cases} \Big(\log\frac{\omega}{\delta(x)}\Big)^{\frac{1-\mu_1}{1-\beta}}, & \text{if } \mu_1 < 1,\\ \Big(\log\Big(\log\frac{\omega}{\delta(x)}\Big)\Big)^{\frac{1}{1-\beta}}, & \text{if } \mu_1 = 1,\\ 1, & \text{if } \mu_1 > 1\end{cases}$$

and

$$v(x) \approx \delta(x).$$

(iv) *Assume that* $\lambda_1 - r\sigma_2 < 1+\alpha, \lambda_2 - s < 2$.

(a) *If* $1+\beta < \lambda_2 - s < 2$, *then system* (1.1) *has a classical solution* (u,v) *satisfying for each* $x \in \Omega$,

$$u(x) \approx \delta(x) \quad \text{and} \quad v(x) \approx \delta(x)^{\frac{2+s-\lambda_2}{1-\beta}}\Big(\log\frac{\omega}{\delta(x)}\Big)^{\frac{-\mu_2}{1-\beta}}.$$

(b) *If* $\lambda_2 - s = 1+\beta$, *then system* (1.1) *has a classical solution* (u,v) *satisfying for each* $x \in \Omega$,

$$u(x) \approx \delta(x)$$

and

$$v(x) \approx \delta(x)\begin{cases} \Big(\log\frac{\omega}{\delta(x)}\Big)^{\frac{1-\mu_2}{1-\alpha}}, & \text{if } \mu_2 < 1,\\ \Big(\log\Big(\log\frac{\omega}{\delta(x)}\Big)\Big)^{\frac{1}{1-\alpha}}, & \text{if } \mu_2 = 1,\\ 1, & \text{if } \mu_2 > 1.\end{cases}$$

(c) *If* $\lambda_2 - s < 1+\beta$, *then system* (1.1) *has a classical solution* (u,v) *satisfying for each* $x \in \Omega$,

$$u(x) \approx \delta(x) \quad \text{and} \quad v(x) \approx \delta(x).$$

(v) *Assume that* $1+\beta<\lambda_2-s\sigma_3<2$ *and* $1+\alpha<\lambda_1-r\sigma_4<2$, *then system* (1.1) *has a classical solution* (u,v) *satisfying for each* $x\in\Omega$,

$$u(x)\approx\delta(x)^{\sigma_3}\Big(\log\frac{\omega}{\delta(x)}\Big)^{\frac{-(1-\beta)\mu_1-r\mu_2}{\gamma}}$$

and

$$v(x)\approx\delta(x)^{\sigma_4}\Big(\log\frac{\omega}{\delta(x)}\Big)^{\frac{-(1-\alpha)\mu_2-s\mu_1}{\gamma}}.$$

Proof. The cases

- (ii) (a) and (d) follow by Theorem 2.1,
- (iii) (a) and (f) follow by Theorem 2.2,
- (iv) follows by Theorem 2.2,
- (v) follows by Theorem 2.3.

So, we shall prove existence of solutions to system (1.1) only in the extremal cases.

Set

$$\tilde{\mu_1}=\frac{(1-\beta)(1-\mu_1)+r(1-\mu_2)}{\gamma}\text{ and }\tilde{\mu_2}=\frac{(1-\alpha)(1-\mu_2)+s(1-\mu_1)}{\gamma}.$$

Case (i). In this case we have $\tilde{\mu_1}<0$ and $\tilde{\mu_2}<0$. Let

$$\theta(x):=\Big(\log\frac{\omega}{\delta(x)}\Big)^{\tilde{\mu_1}},\quad\varphi(x):=\Big(\log\frac{\omega}{\delta(x)}\Big)^{\tilde{\mu_2}}$$

and consider system (4.1). Using (H), we have

$$p(x)\approx(\delta(x))^{-2}\Big(\log\frac{\omega}{\delta(x)}\Big)^{-\mu_1+r\tilde{\mu_2}}\quad\text{and}\quad q(x)\approx(\delta(x))^{-2}\Big(\log\frac{\omega}{\delta(x)}\Big)^{-\mu_2+s\tilde{\mu_1}}.$$

It follows by Theorem 1.2 that system (4.1) has a solution (w_1,w_2) satisfying for each $x\in\Omega$,

$$w_1(x)\approx\left(\int_0^{\delta(x)}\frac{1}{t}\Big(\log\frac{\omega}{t}\Big)^{-\mu_1+r\tilde{\mu_2}}dt\right)^{\frac{1}{1-\alpha}}$$

and

$$w_2(x)\approx\left(\int_0^{\delta(x)}\frac{1}{t}\Big(\log\frac{\omega}{t}\Big)^{-\mu_2+s\tilde{\mu_1}}dt\right)^{\frac{1}{1-\beta}}.$$

Since $\tilde{\mu_1}=1-\mu_1+r\tilde{\mu_2}$ and $\tilde{\mu_2}=1-\mu_2+s\tilde{\mu_1}$, it follows by (5.2) that

$$w_1(x)\approx\theta(x)\quad\text{and}\quad w_2(x)\approx\varphi(x).$$

Hence (3.2) is satisfied and so the result holds by using Proposition 3.3.

Case (ii)(b). In this case we have $\tilde{\mu_1} < 0$ and $\tilde{\mu_2} > 0$. Let

$$\theta(x) := \Big(\log \frac{\omega}{\delta(x)}\Big)^{\tilde{\mu_1}}, \quad \varphi(x) := \delta(x)\Big(\log \frac{\omega}{\delta(x)}\Big)^{\tilde{\mu_2}}$$

and consider system (4.1). Using (H), we have

$$p(x) \approx (\delta(x))^{-2}\Big(\log \frac{\omega}{\delta(x)}\Big)^{-\mu_1 + r\tilde{\mu_2}} \quad \text{and} \quad q(x) \approx (\delta(x))^{-1-\beta}\Big(\log \frac{\omega}{\delta(x)}\Big)^{-\mu_2 + s\tilde{\mu_1}}.$$

It follows by Theorem 1.2 that system (4.1) has a solution (w_1, w_2) satisfying for each $x \in \Omega$,

$$w_1(x) \approx \left(\int_0^{\delta(x)} \frac{1}{t}\Big(\log \frac{\omega}{t}\Big)^{-\mu_1 + r\tilde{\mu_2}} dt\right)^{\frac{1}{1-\alpha}}$$

and

$$w_2(x) \approx \delta(x)\left(\int_{\delta(x)}^{\eta} \frac{1}{t}\Big(\log \frac{\omega}{t}\Big)^{-\mu_2 + s\tilde{\mu_1}} dt\right)^{\frac{1}{1-\beta}}.$$

Using (5.2) and (5.1) we deduce that

$$w_1(x) \approx \theta(x) \text{ and } w_2(x) \approx \varphi(x).$$

Hence (3.2) is satisfied and so the result holds by using Proposition 3.3.
Case (ii)(c). In this case we have $\tilde{\mu_2} < 0$. Let

$$\theta(x) := \Big(\log \frac{\omega}{\delta(x)}\Big)^{\frac{1-\mu_1}{1-\alpha}}, \quad \varphi(x) := \delta(x)$$

and consider system (4.1). Using (H), we have

$$p(x) \approx (\delta(x))^{-2}\Big(\log \frac{\omega}{\delta(x)}\Big)^{-\mu_1} \quad \text{and} \quad q(x) \approx (\delta(x))^{-1-\beta}\Big(\log \frac{\omega}{\delta(x)}\Big)^{-\mu_2 + s\frac{1-\mu_1}{1-\alpha}}.$$

It follows by Theorem 1.2 that system (4.1) has a solution (w_1, w_2) satisfying for each $x \in \Omega$,

$$w_1(x) \approx \left(\int_0^{\delta(x)} \frac{1}{t}\Big(\log \frac{\omega}{t}\Big)^{-\mu_1} dt\right)^{\frac{1}{1-\alpha}}$$

and

$$w_2(x) \approx \delta(x)\left(\int_{\delta(x)}^{\eta} \frac{1}{t}\Big(\log \frac{\omega}{t}\Big)^{-\mu_2 + s\frac{1-\mu_1}{1-\alpha}} dt\right)^{\frac{1}{1-\beta}}.$$

Using (5.2) and (5.1) we deduce that

$$w_1(x) \approx \theta(x) \quad \text{and} \quad w_2(x) \approx \varphi(x).$$

Hence (3.2) is satisfied and so the result holds by using Proposition 3.3.
Case (iii)(b). In this case we have $\tilde{\mu_1} > 0$ and $\tilde{\mu_2} > 0$. Let

$$\theta(x) := \delta(x)\Big(\log\frac{\omega}{\delta(x)}\Big)^{\tilde{\mu_1}}, \quad \varphi(x) := \delta(x)\Big(\log\frac{\omega}{\delta(x)}\Big)^{\tilde{\mu_2}},$$

and consider system (4.1). Using (H), we have

$$p(x) \approx (\delta(x))^{-1-\alpha}\Big(\log\frac{\omega}{\delta(x)}\Big)^{-\mu_1+r\tilde{\mu_2}} \quad \text{and} \quad q(x) \approx (\delta(x))^{-1-\beta}\Big(\log\frac{\omega}{\delta(x)}\Big)^{-\mu_2+s\tilde{\mu_1}}.$$

It follows by Theorem 1.2 and (5.1) that system (4.1) has a solution (w_1, w_2) satisfying for each $x \in \Omega$,

$$w_1(x) \approx \delta(x)\Bigg(\int\limits_{\delta(x)}^{\eta} \frac{1}{t}\Big(\log\frac{\omega}{t}\Big)^{-\mu_1+r\tilde{\mu_2}} dt\Bigg)^{\frac{1}{1-\alpha}} \approx \theta(x)$$

and

$$w_2(x) \approx \delta(x)\Bigg(\int\limits_{\delta(x)}^{\eta} \frac{1}{t}\Big(\log\frac{\omega}{t}\Big)^{-\mu_2+s\tilde{\mu_1}} dt\Bigg)^{\frac{1}{1-\beta}} \approx \varphi(x).$$

Hence (3.2) is satisfied and so the result holds by using Proposition 3.3.
Case (iii)(c). Let

$$\theta(x) = \varphi(x) := \delta(x)$$

and consider system (4.1). Using (H), we have

$$p(x) \approx (\delta(x))^{-1-\alpha}\Big(\log\frac{\omega}{\delta(x)}\Big)^{-\mu_1} \quad \text{and} \quad q(x) \approx (\delta(x))^{-1-\beta}\Big(\log\frac{\omega}{\delta(x)}\Big)^{-\mu_2}.$$

It follows by Theorem 1.2 and (5.1) that system (4.1) has a solution (w_1, w_2) satisfying for each $x \in \Omega$,

$$w_1(x) \approx \delta(x)\Bigg(\int\limits_{\delta(x)}^{\eta} \frac{1}{t}\Big(\log\frac{\omega}{t}\Big)^{-\mu_1} dt\Bigg)^{\frac{1}{1-\alpha}} \approx \theta(x)$$

and

$$w_2(x) \approx \delta(x)\Bigg(\int\limits_{\delta(x)}^{\eta} \frac{1}{t}\Big(\log\frac{\omega}{t}\Big)^{-\mu_2} dt\Bigg)^{\frac{1}{1-\beta}} \approx \varphi(x).$$

Hence (3.2) is satisfied and so the result holds by using Proposition 3.3.
Case (iii)(d). In this case we have $\tilde{\mu_2} < 0$. Let

$$\theta(x) := \delta(x)\Big(\log\frac{\omega}{\delta(x)}\Big)^{\frac{1-\mu_1}{1-\alpha}}, \quad \varphi(x) := \delta(x)$$

and consider system (4.1). Using (H), we have

$$p(x) \approx (\delta(x))^{-1-\alpha}\Big(\log \frac{\omega}{\delta(x)}\Big)^{-\mu_1} \quad \text{and} \quad q(x) \approx (\delta(x))^{-1-\beta}\Big(\log \frac{\omega}{\delta(x)}\Big)^{-\mu_2+s\frac{1-\mu_1}{1-\alpha}}.$$

It follows by Theorem 1.2 and (5.1) that system (4.1) has a solution (w_1, w_2) satisfying for each $x \in \Omega$,

$$w_1(x) \approx \delta(x)\Bigg(\int_{\delta(x)}^{\eta} \frac{1}{t}\Big(\log \frac{\omega}{t}\Big)^{-\mu_1} dt\Bigg)^{\frac{1}{1-\alpha}} \approx \theta(x)$$

and

$$w_2(x) \approx \delta(x)\Bigg(\int_{\delta(x)}^{\eta} \frac{1}{t}\Big(\log \frac{\omega}{t}\Big)^{-\mu_2+s\frac{1-\mu_1}{1-\alpha}} dt\Bigg)^{\frac{1}{1-\beta}} \approx \varphi(x).$$

Hence (3.2) is satisfied and so the result holds by using Proposition 3.3.
Case (iii)(e). In this case we have $\tilde{\mu_1} < 0$. Interchanging the role of u and v, the proof is the same as in Case (iii)(d) above.

This completes the proof. □

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