Journal of Applied Mathematics and Computational Mechanics 2024, 23(1), 83-96
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2024.1.07 e-ISSN 2353-0588

CONTROL OF CHAOS IN THE BURKE-SHAW SYSTEM OF FRACTAL-FRACTIONAL ORDER IN THE SENSE OF CAPUTO-FABRIZIO

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Received: 13 September 2023; Accepted: 3 December 2023

Abstract. For the Burke-Shaw system, we propose a fractal-fractional order in the sense of the Caputo-Fabrizio derivative. The proposed system is solved by utilizing the fractal-fractional derivative operator with an exponential decay kernel. Time-fractional Caputo-Fabrizio fractal fractional derivatives are applied to the Burke-Shaw-type nonlinear chaotic systems. Based on fixed point theory, it has been demonstrated that a fractal-fractional-order model under the Caputo-Fabrizio operator exists and is unique. Using a numerical power series method, we solve the fractional Burke-Shaw model. Using Newton's interpolation polynomial, we solve the equation numerically by implementing a novel numerical scheme based on an efficient polynomial.

MSC 2010: *92D30, 92D25, 92C42, 34C60*
Keywords: *fractional derivatives, nonlinear equations, simulation, numerical results, iterative method*

1. Introduction

In physics and engineering, chaos is one of the most significant findings. A nonlinear dynamic system is said to be chaotic if the future of that system's development in time and space is extremely unpredictable as a consequence of its sensitivity to perturbations in its unexcited state (starting circumstances) or algebraic structure. There is a broad range of applications in the fields of modeling, control, and performance improvement of engineering systems. This is thanks to a deeper comprehension of chaotic dynamics, which is present in both natural and man-made systems. A number of dynamical systems with fractional order have been studied, including the Lorenz systems [1,2], the Chua system [3–6], the Rössler system [7–9], the Newton-Leipnik system [10, 11], the Liu system [12–14], the Lu system [15], the Lin system [16], the Chen system [17–19] are all uncontrollable. The Burke and Shaw, in their article [20] modified the Lorenz's system in a variety of ways. Under the transformation [21], the Burke-Shaw can be described as a topologically equivalent representation of Lorenz. Despite having a similar algebraic structure to the Lorenz system, it is also topologically equivalent to a generalized Lorenz-type system. The Burke-Shaw chaotic attractor can be classified as a system of the 3rd degree based on the equation in (1). In spite of its algebraic similarity to the Lorenz system, the Burke-Shaw system differs from the generalized Lorenz-type system from a topological perspective:

$$\begin{aligned} \frac{d\mathrm{f}(\mathtt{t})}{d\mathtt{t}} &= -\mathrm{V}(\mathrm{f}(\mathtt{t})+\mathrm{g}(\mathtt{t})), \\ \frac{d\mathrm{g}(\mathtt{t})}{d\mathtt{t}} &= -\mathrm{V}\,\mathrm{f}(\mathtt{t})\,\mathrm{h}(\mathtt{t})-\mathrm{g}(\mathtt{t}), \\ \frac{d\mathrm{h}(\mathtt{t})}{d\mathtt{t}} &= \mathrm{V}\,\mathrm{f}(\mathtt{t})\,\mathrm{g}(\mathtt{t})+d, \end{aligned} \tag{1}$$

where f, g and h are the state variables, V and d are constant parameters.

In recent years, there has been an emerging fractal-fractional idea where the operator has two orders, fractional and fractal, see [22–33]. Based on fractal-fractional derivatives and integrals, Atangana and Qureshi studied and predicted the chaotic behavior of some attractors [23]. Due to the high non-linearity of our problem, we used a suitable numerical scheme to solve this system of equations numerically. The Caputo-Fabrizio fractal fractional derivatives have a non-singular kernel which describes various processes accurately. The implicit solutions to the problem are obtained, and the solutions under different fractional orders are compared intuitively through images. By comparing the results obtained in this paper with those produced under Caputo fractional derivatives, it is found that the solutions change relatively gently under Caputo-Fabrizio fractal fractional derivatives.

The novelty of this paper lies in the fact that the Caputo-Fabrizio fractional derivatives have a non-singular kernel which describes various processes in the most accurate way. The implicit solutions to the problem are obtained, and the solutions under different fractional orders are compared intuitively through images. By comparing the results obtained in this paper with those produced under Caputo fractional derivatives, it is found that the solutions change relatively gently under Caputo-Fabrizio fractional derivatives. A significant part of the modeling process is selecting the appropriate fractional derivatives and fractal order. Additionally, the differential equation calculation under the Caputo-Fabrizio derivative is relatively simple and convenient, which is not the case with other fractional derivatives. A novel numerical scheme based on an efficient polynomial is used to solve the Burke-Shaw model numerically using Newton's interpolation polynomial. Moreover, with a numerical power series method, we solve the fractional Burke-Shaw model. Many applications in chaotic systems have been studied in various fields such as encryption, financial models, wind turbines, photo-thermal interactions, thermoelasticity of type III, Thin Slim Strip Non-Gaussian Laser Beam, and the magneto-thermo-viscoelastic medium [34–40].

The remainder of the paper is organized as follows: Section 2, following the introduction, provides some definitions required for the formulation of a fractal-fractional order Burke-Shaw model. Section 3 examines the existence and uniqueness of the proposed model. In Section 4, a numerical scheme is applied to solve the developed model, and numerical simulations are conducted to validate the analytical findings. The power series method used to solve the proposed model is discussed in Section 5. Moreover, this section summarizes all of the major findings of the current study and discusses the behavior of the obtained solutions.

2. Model formulation

Definition 1. [41, 42] In the Liouville-Caputo sense, the Caputo-Fabrizio fractal--fractional derivative of $f(\mathtt{t})$ with order $\rho - \sigma$ is:

$$^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho}\{f(\mathtt{t})\} = \frac{\mathrm{Z}(\sigma)}{1-\sigma}\int_0^{\mathtt{t}} \exp\left(-\frac{\sigma}{1-\sigma}(\mathtt{t}-\mu)\right)\left(\frac{d}{d\mu^{\rho}}f(\mu)\right)d\mu,$$

where $\sigma > 0, \rho \leq m$, $m \in \mathbb{N}$ and $\mathrm{Z}(0) = \mathrm{Z}(1) = 1$. □

Definition 2. [41,42] According to Caputo-Fabrizio, $f(\mathtt{t})$ in order σ has the following fractal-fractional integral:

$$^{\mathtt{FFE}}I_{0,t}^{\sigma,\rho}\{f(\mathtt{t})\} = \frac{\sigma\rho}{\mathrm{Z}(\sigma)}\int_0^{\mathtt{t}} \mu^{\sigma-1}f(\mu)d\mu + \frac{\rho(1-\sigma)\mathtt{t}^{\rho-1}}{\mathrm{Z}(\sigma)}f(\mathtt{t}).$$

In fractal-fractional terms, we can obtain the Burke-Shaw model as follows:

$$\begin{aligned} ^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho}\,\mathrm{f}(\mathtt{t}) &= -\mathrm{V}(\mathrm{f}(\mathtt{t})+\mathrm{g}(\mathtt{t})),\\ ^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho}\,\mathrm{g}(\mathtt{t}) &= -\mathrm{V}\,\mathrm{f}(\mathtt{t})\,\mathrm{h}(\mathtt{t})-\mathrm{g}(\mathtt{t}),\\ ^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho}\,\mathrm{h}(\mathtt{t}) &= \mathrm{V}\,\mathrm{f}(\mathtt{t})\,\mathrm{g}(\mathtt{t})+d. \end{aligned} \tag{2}$$

3. Existence and uniqueness

Existence of the solution of the system (2) and its uniqueness will be provided here. The functions $\mathrm{f}(\mathtt{t})$, $\mathrm{g}(\mathtt{t})$, and $\mathrm{h}(\mathtt{t})$ are assumed to be bounded for all $\mathtt{t} \in [0,T]$, such that $\|\mathrm{f}\|_\infty \leq \mathrm{M_f}$, $\|\mathrm{g}\|_\infty \leq \mathrm{M_g}$, $\|\mathrm{h}\|_\infty \leq \mathrm{M_h}$. Model (1) rewritten as follows:

$$\begin{aligned} \mathrm{f}_{\mathtt{x}}(\mathtt{t},\mathrm{f},\mathrm{g},\mathrm{h}) &= -\mathrm{V}(\mathrm{f}(\mathtt{t})+\mathrm{g}(\mathtt{t})),\\ \mathrm{f}_{\mathtt{y}}(\mathtt{t},\mathrm{f},\mathrm{g},\mathrm{h}) &= -\mathrm{V}\,\mathrm{f}(\mathtt{t})\,\mathrm{h}(\mathtt{t})-\mathrm{g}(\mathtt{t}),\\ \mathrm{f}_{\mathtt{z}}(\mathtt{t},\mathrm{f},\mathrm{g},\mathrm{h}) &= \mathrm{V}\,\mathrm{f}(\mathtt{t})\,\mathrm{g}(\mathtt{t})+d. \end{aligned}$$

Bounded variables $\mathtt{x},\mathrm{g},\mathtt{z}$ also imply bound variables $\mathrm{f}_{\mathtt{x}},\mathrm{f}_{\mathtt{y}}$, and $\mathrm{f}_{\mathtt{z}}$. In the case of bounded $\mathtt{x}$, $\mathtt{y}$, and $\mathtt{z}$, $\mathrm{M_f},\mathrm{M_g}$, and $\mathrm{M_h}$ exist such that

$$\sup_{\mathtt{t}\in \mathrm{D}_x}|\mathrm{f}(\mathtt{t})| = \|\mathrm{f}\|_\infty \leq \mathrm{M_f},\ \sup_{\mathtt{t}\in D_y}|\mathrm{g}(\mathtt{t})| \quad = \|\mathrm{g}\|_\infty \leq \mathrm{M_g},\ \sup_{\mathtt{t}\in D_z}|\mathrm{h}(\mathtt{t})| = \|\mathrm{h}\|_\infty \leq \mathrm{M_h}.$$

The linear growth property is first demonstrated for the functions.

$$|\mathrm{f_f}(\mathtt{t},\mathrm{f},\mathrm{g},\mathrm{h})| \leq \mathrm{V}(\sup_{\mathtt{t}\in D_\mathrm{f}}|\mathrm{f}| + \sup_{\mathtt{t}\in D_y}|\mathrm{g}|) \leq \mathrm{V}(\|\mathrm{f}\|_\infty + \|\mathrm{g}\|_\infty) \leq \mathrm{V}(\mathrm{M_f}+\mathrm{M_g}) = \mathrm{M_{f_f}} < \infty,$$
$$\left|\mathrm{f_y}(\mathtt{t},\mathrm{f},\mathrm{g},\mathrm{h})\right| \leq |\mathrm{V}| \sup_{\mathtt{t}\in D_\mathrm{f}}|\mathrm{f}| \sup_{\mathtt{t}\in D_z}|\mathrm{h}| + \sup_{\mathtt{t}\in D_y}|\mathrm{g}| \leq \mathrm{V}\|\mathrm{f}\|_\infty\|\mathrm{h}\|_\infty + \|\mathrm{g}\|_\infty \leq \mathrm{M_{f_y}} < \infty,$$
$$|\mathrm{f_z}(\mathtt{t},\mathrm{f},\mathrm{g},\mathrm{h})| \leq |\mathrm{V}| \sup_{\mathtt{t}\in D_\mathrm{f}}|\mathrm{f}| \sup_{\mathtt{t}\in D_y}|\mathrm{g}| + d \leq \mathrm{V}\|\mathrm{f}\|_\infty\|\mathrm{g}\|_\infty + d \leq V\,\mathrm{M_f}\,\mathrm{M_g} + d = \mathrm{M_{f_z}} < \infty.$$

But on the other hand, we have that

$$|\mathrm{f_f}(\mathtt{t},\mathrm{f_1},\mathrm{g},\mathrm{h}) - \mathrm{f_f}(\mathtt{t},\mathrm{f_2},\mathrm{g},\mathrm{h})| \leq |\mathrm{V}|\,|\mathrm{f_1}-\mathrm{f_2}|,$$
$$\left|\mathrm{f_y}(\mathtt{t},\mathrm{f},\mathrm{g_1},\mathrm{h}) - \mathrm{f_y}(\mathtt{t},\mathrm{f},\mathrm{g_2},\mathrm{h})\right| \leq |\mathrm{g_1}-\mathrm{g_2}|,$$
$$|\mathrm{f_z}(\mathtt{t},\mathrm{f},\mathrm{g},\mathrm{h_1}) - \mathrm{f_z}(\mathtt{t},\mathrm{f},\mathrm{g},\mathrm{h_2})| = 0 \leq |\mathrm{h_1}-\mathrm{h_2}|.$$

$\mathrm{f_f}$, $\mathrm{f_y}$, $\mathrm{f_z}$ verify the Lipschitz condition; however, there is a contraction if $V < 1$. The linear growth and Lipschitz conditions of $\mathrm{f_f}$, $\mathrm{f_y}$, and $\mathrm{f_z}$ will be verified:

$$|\mathrm{f_x}(\mathtt{t},\mathrm{f},\mathrm{g},\mathrm{h})|^2 \leq 3\,\mathrm{V}^2|\mathrm{f}|^2 + 3|\mathrm{g}|^2 \leq \mathrm{K_f}\left(1+|\mathrm{f}|^2\right), \mathrm{K_f} = \frac{\mathrm{V}^2}{3\,\mathrm{M_g^2}} < 1,$$
$$\left|\mathrm{f_y}(\mathtt{t},\mathrm{f},\mathrm{g},\mathrm{h})\right|^2 \leq 3\,\mathrm{V}^2\,\mathrm{M_f^2}\,\mathrm{M_h^2} + 3|\mathrm{g}|^2 \leq \mathrm{K}_y\left(1+|\mathrm{g}|^2\right), \mathrm{K}_y = \frac{1}{\mathrm{V}^2\,\mathrm{M_f^2}\,\mathrm{M_h^2}},$$
$$|\mathrm{f_z}(\mathtt{t},\mathrm{f},\mathrm{g},\mathrm{h})|^2 \leq 3\,\mathrm{V}^2|\mathrm{f}|^2|\mathrm{g}|^2 + 3d^2 \leq 3\,\mathrm{V}^2\,\mathrm{M_f^2}\,\mathrm{M_g^2} + 3d^2 \leq 3\,\mathrm{K}_z,\ \mathrm{K}_z = 3\,\mathrm{V}^2\,\mathrm{M_f^2}\,\mathrm{M_g^2},$$

$$|\mathrm{f_f}(\mathtt{t},\mathrm{f_1},\mathrm{g},\mathrm{h}) - \mathrm{f_f}(\mathtt{t},\mathrm{f_2},\mathrm{g},\mathrm{h})|^2 \leq \bar{K}_\mathrm{f}|\mathrm{f_1}-\mathrm{f_2}|^2, \text{ with } \bar{K}_\mathrm{f} = \frac{3}{2}\mathrm{V}^2,$$
$$\left|\mathrm{f_y}(\mathtt{t},\mathrm{f},\mathrm{g_1},\mathrm{h}) - \mathrm{f_y}(\mathtt{t},\mathrm{f},\mathrm{g_2},\mathrm{h})\right|^2 \leq \bar{K}_y|\mathrm{g_1}-\mathrm{g_2}|^2, \text{ with } \bar{K}_y = \frac{3}{2},$$
$$|\mathrm{f_z}(\mathtt{t},\mathrm{f},\mathrm{g},\mathrm{h_1}) - \mathrm{f_z}(\mathtt{t},\mathrm{f},\mathrm{g},\mathrm{h_2})|^2 \leq \bar{K}_z|\mathrm{h_1}-\mathrm{h_2}|, \text{ with } \bar{K}_z = \frac{3}{2}.$$

Thus, the presented model possesses an existence and uniqueness solution.

4. Power series method

During this section, we apply the power series method to the system (2). Using power series solutions: $\mathrm{f} = \sum_{n=0}^{\infty} a_n \mathtt{t}^{n\sigma}$, $\mathrm{g} = \sum_{n=0}^{\infty} b_n \mathtt{t}^{n\sigma}$, $\mathrm{h} = \sum_{n=0}^{\infty} c_n \mathtt{t}^{n\sigma}$. The recurrence relations:

$$
\begin{aligned}
a_{n+1}\frac{\rho((n+1)\sigma)+1}{\rho(n\sigma+1)} &= a(\mathrm{g}_n-\mathrm{f}_n)+\sum_{k=0}^{n} b_\ell c_{n-k}, \quad a_0=3,\\
b_{n+1}\frac{\rho((n+1)\sigma)+1}{\rho(n\sigma+1)} &= (c-a)\,\mathrm{f}_n+c\,\mathrm{g}_n-\sum_{k=0}^{n} a_\ell c_{n-k}, \quad b_0=2,\\
c_{n+1}\frac{\rho((n+1)\sigma)+1}{\rho(n\sigma+1)} &= -b\,\mathrm{h}_n+e\sum_{\ell}^{n-k} b_\ell b_{n-k}, \quad c_0=1.
\end{aligned}
$$

Then, we introduce the solution in the equation:

$$
\begin{aligned}
\mathrm{f} &= 3-38.3177\mathtt{t}^{0.98}+731.795\mathtt{t}^{1.96}-5700.48\mathtt{t}^{2.94}+28380.5\mathtt{t}^{3.92}-41294.6\mathtt{t}^{4.9}\\
&\quad -340243\mathtt{t}^{5.88}+2.85037\times 10^6\mathtt{t}^{6.86}-3.41899*10^6\times 10^7\mathtt{t}^{7.84}-1.44472\times 10^8\mathtt{t}^{8.82},\\
\mathrm{g} &= 2-3.02508\mathtt{t}^{0.98}+314.812\mathtt{t}^{1.96}-3038.56\mathtt{t}^{2.94}+24431.5\mathtt{t}^{3.92}-113236\mathtt{t}^{4.9}\\
&\quad +452133\mathtt{t}^{5.88}-1.70875\times 10^6\mathtt{t}^{6.86}+7.17208\times 10^6\mathtt{t}^{7.84}+1.79\times 10^7\mathtt{t}^{8.82},\\
\mathrm{h} &= 1+7.05853\mathtt{t}^{0.98}-36.8231\mathtt{t}^{1.96}+1381.63\mathtt{t}^{2.94}-12835.1\mathtt{t}^{3.92}+149232\mathtt{t}^{4.9}\\
&\quad -1.45784\times 10^6\mathtt{t}^{5.88}+1.34055\times 10^7\mathtt{t}^{6.86}-1.00228\times 10^8\mathtt{t}^{7.84}+6.30178\times 10^8\mathtt{t}^{8.82}.
\end{aligned}
$$

5. Schemas for the fractal-fractional Burke-Shaw model

5.1. Constant order with exponential decay

The purpose of this section is to search numerical solutions to Burke-Shaw models that include Caputo-Fabrizio fractal-fractional operators. We can write system (2) as:

$$
\begin{cases}
{}^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho}\,\mathrm{f}(\mathtt{t}) = \varphi(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t}),\\
{}^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho}\,\mathrm{g}(\mathtt{t}) = \psi(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t}),\\
{}^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho}\,\mathrm{h}(\mathtt{t}) = \mu(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t}),
\end{cases}
$$

and we can reformulate the above equation as follows

$$
\begin{cases}
{}^{\mathtt{CF}}\mathscr{D}_{0,\mathtt{t}}^{\sigma}\,\mathrm{f}(\mathtt{t}) = \rho t^{\rho-1}\varphi(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t}),\\
{}^{\mathtt{CF}}\mathscr{D}_{0,\mathtt{t}}^{\sigma}\,\mathrm{g}(\mathtt{t}) = \rho t^{\rho-1}\psi(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t}),\\
{}^{\mathtt{CF}}\mathscr{D}_{0,\mathtt{t}}^{\sigma}\,\mathrm{h}(\mathtt{t}) = \rho t^{\rho-1}\mu(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t}).
\end{cases} \tag{3}
$$

If we define as $U(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t}) = \rho t^{\rho-1}\varphi(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t})$, $V(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t}) = \rho t^{\rho-1}\psi(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t})$, $W(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t}) = \rho t^{\rho-1}\mu(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t})$ and integrate the Eq. (4), we can get the following equalities

$$
\begin{aligned}
\mathrm{f}(\mathtt{t}) &= \mathrm{f}(0)+\frac{1-\sigma}{\mathrm{Z}(\sigma)}U(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t})+\frac{\sigma}{\mathrm{Z}(\sigma)}\int_0^t U(\mathrm{f},\mathrm{g},\mathrm{h},\tau)d\tau,\\
\mathrm{g}(\mathtt{t}) &= \mathrm{g}(0)+\frac{1-\sigma}{\mathrm{Z}(\sigma)}V(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t})+\frac{\sigma}{\mathrm{Z}(\sigma)}\int_0^t V(\mathrm{f},\mathrm{g},\mathrm{h},\tau)d\tau,
\end{aligned}
$$

$$\mathrm{h}(\mathtt{t}) = \mathrm{h}(0) + \frac{1-\sigma}{\mathrm{Z}(\sigma)} W(\mathrm{f},\mathrm{g},\mathrm{h},\mathtt{t}) + \frac{\sigma}{\mathrm{Z}(\sigma)} \int_0^t W(\mathrm{f},\mathrm{g},\mathrm{h},\tau) d\tau.$$

We can have the following equalities at point $t = t_{n+1}$,

$$\mathrm{f}(t_{n+1}) = \mathrm{f}(t_n) + \frac{1-\sigma}{\mathrm{Z}(\sigma)} U(\mathrm{f}^n,\mathrm{g}^n,\mathrm{h}^n,t_n) + \frac{\sigma}{\mathrm{Z}(\sigma)} \int_0^{t_{n+1}} U(\mathrm{f},\mathrm{g},\mathrm{h},\tau) d\tau,$$
$$\mathrm{g}(t_{n+1}) = \mathrm{g}(t_n) + \frac{1-\sigma}{\mathrm{Z}(\sigma)} V(\mathrm{f}^n,\mathrm{g}^n,\mathrm{h}^n,t_n) + \frac{\sigma}{\mathrm{Z}(\sigma)} \int_0^{t_{n+1}} V(\mathrm{f},\mathrm{g},\mathrm{h},\tau) d\tau,$$
$$\mathrm{h}(t_{n+1}) = \mathrm{h}(t_n) + \frac{1-\sigma}{\mathrm{Z}(\sigma)} W(\mathrm{f}^n,\mathrm{g}^n,\mathrm{h}^n,t_n) + \frac{\sigma}{\mathrm{Z}(\sigma)} \int_0^{t_{n+1}} W(\mathrm{f},\mathrm{g},\mathrm{h},\tau) d\tau,$$

and at point $t = t_n$

$$\mathrm{f}(t_n) = \mathrm{f}(t_{n-1}) + \frac{1-\sigma}{\mathrm{Z}(\sigma)} U\left(\mathrm{f}^{n-1},\mathrm{g}^{n-1},\mathrm{h}^{n-1},t_{n-1}\right) + \frac{\sigma}{\mathrm{Z}(\sigma)} \int_0^{t_n} U(\mathrm{f},\mathrm{g},\mathrm{h},\tau) d\tau,$$
$$\mathrm{g}(t_n) = \mathrm{g}(t_{n-1}) + \frac{1-\sigma}{\mathrm{Z}(\sigma)} V\left(\mathrm{f}^{n-1},\mathrm{g}^{n-1},\mathrm{h}^{n-1},t_{n-1}\right) + \frac{\sigma}{\mathrm{Z}(\sigma)} \int_0^{t_n} V(\mathrm{f},\mathrm{g},\mathrm{h},\tau) d\tau,$$
$$\mathrm{h}(t_n) = \mathrm{h}(t_{n-1}) + \frac{1-\sigma}{\mathrm{Z}(\sigma)} W\left(\mathrm{f}^{n-1},\mathrm{g}^{n-1},\mathrm{h}^{n-1},t_{n-1}\right) + \frac{\sigma}{\mathrm{Z}(\sigma)} \int_0^{t_n} W(\mathrm{f},\mathrm{g},\mathrm{h},\tau) d\tau.$$

If we take the difference of above equations, we can write in the form of

$$\mathrm{f}(t_n) = \mathrm{f}(t_{n-1}) + \frac{1-\sigma}{\mathrm{Z}(\sigma)} U\left(\mathrm{f}^{n-1},\mathrm{g}^{n-1},\mathrm{h}^{n-1},t_{n-1}\right) + \frac{\sigma}{\mathrm{Z}(\sigma)} \sum_{r=2}^{n} \int_{t_r}^{t_{r+1}} U(\mathrm{f},\mathrm{g},\mathrm{h},\tau) d\tau,$$
$$\mathrm{g}(t_n) = \mathrm{g}(t_{n-1}) + \frac{1-\sigma}{\mathrm{Z}(\sigma)} V\left(\mathrm{f}^{n-1},\mathrm{g}^{n-1},\mathrm{h}^{n-1},t_{n-1}\right) + \frac{\sigma}{\mathrm{Z}(\sigma)} \sum_{r=2}^{n} \int_{t_r}^{t_{r+1}} V(\mathrm{f},\mathrm{g},\mathrm{h},\tau) d\tau,$$
$$\mathrm{h}(t_n) = \mathrm{h}(t_{n-1}) + \frac{1-\sigma}{\mathrm{Z}(\sigma)} W\left(\mathrm{f}^{n-1},\mathrm{g}^{n-1},\mathrm{h}^{n-1},t_{n-1}\right) + \frac{\sigma}{\mathrm{Z}(\sigma)} \sum_{r=2}^{n} \int_{t_r}^{t_{r+1}} W(\mathrm{f},\mathrm{g},\mathrm{h},\tau) d\tau.$$

We shall apply the Newton polynomial and we obtain the following equalities

$$\mathrm{f}(t_{n+1}) = \mathrm{f}(t_n) + \frac{1-\sigma}{M(x)} \left[U(\mathrm{f}^n,\mathrm{g}^n,\mathrm{h}^n,t_n) - U\left(\mathrm{f}^{n-1},\mathrm{g}^{n-1},\mathrm{h}^{n-1},t_{n-1}\right)\right] U\left(\mathrm{f}^{r-2},\mathrm{g}^{r-2},\mathrm{h}^{r-2},t_{r-2}\right)$$
$$+\frac{\sigma}{\mathrm{Z}(\sigma)} \sum_{r=2}^{n} \int_{t_r}^{t_{r+1}} \left\{ \begin{array}{c} \dfrac{U\left(\mathrm{f}^{r-1},\mathrm{g}^{r-1},\mathrm{h}^{r-1},t_{r-1}\right) - U\left(\mathrm{f}^{r-2},\mathrm{g}^{r-2},\mathrm{h}^{r-2},t_{r-2}\right)}{\Delta t} (\tau - t_{r-2}) + \\ \dfrac{U(\mathrm{f}^r,\mathrm{g}^r,\mathrm{h}^r,t_r) - 2U\left(\mathrm{f}^{r-1},\mathrm{g}^{r-1},\mathrm{h}^{r-1},t_{r-1}\right) + U\left(\mathrm{f}^{r-2},\mathrm{g}^{r-2},\mathrm{h}^{r-2},t_{r-2}\right)}{2\,(\Delta t^2} \\ \times (\tau - t_{r-2})(\tau - t_{r-1}) \end{array} \right\} d\tau,$$

$$\mathrm{g}(t_{n+1}) = \mathrm{g}(t_n) + \frac{1-\sigma}{\mathrm{Z}(\sigma)} \left[V(\mathrm{f}^n,\mathrm{g}^n,\mathrm{h}^n,t_n) - V\left(\mathrm{f}^{n-1},\mathrm{g}^{n-1},\mathrm{h}^{n-1},t_{n-1}\right)\right] V\left(\mathrm{f}^{r-2},\mathrm{g}^{r-2},\mathrm{h}^{r-2},t_{r-2}\right)$$
$$+\frac{\sigma}{\mathrm{Z}(\sigma)} \sum_{r=2}^{n} \int_{t_r}^{t_{r+1}} \left\{ \begin{array}{c} \dfrac{V\left(\mathrm{f}^{r-1},\mathrm{g}^{r-1},\mathrm{h}^{r-1},t_{r-1}\right) - V\left(\mathrm{f}^{r-2},\mathrm{g}^{r-2},\mathrm{h}^{r-2},t_{r-2}\right)}{\Delta t} (\tau - t_{r-2}) + \\ \dfrac{V(\mathrm{f}^r,\mathrm{g}^r,\mathrm{h}^r,t_r) - 2V\left(\mathrm{f}^{r-1},\mathrm{g}^{r-1},\mathrm{h}^{r-1},t_{r-1}\right) + V\left(\mathrm{f}^{r-2},\mathrm{g}^{r-2},\mathrm{h}^{r-2},t_{r-2}\right)}{2(\Delta t)^2} \\ \times (\tau - t_{r-2})(\tau - t_{r-1}) \end{array} \right\} d\tau,$$

$$\mathrm{h}(t_{n+1}) = \mathrm{h}(t_n) + \frac{1-\sigma}{M(x)}\left[W\left(\mathrm{f}^n,\mathrm{g}^n,\mathrm{h}^n,t_n\right) - W\left(\mathrm{f}^{n-1},\mathrm{g}^{n-1},\mathrm{h}^{n-1},t_{n-1}\right)\right] W\left(\mathrm{f}^{r-2},\mathrm{g}^{r-2},\mathrm{h}^{r-2},t_{r-2}\right)$$

$$+\frac{\sigma}{\mathrm{Z}(\sigma)}\sum_{r=2}^{n}\int_{t_r}^{t_{r+1}}\left\{\begin{array}{c}\dfrac{W\left(\mathrm{f}^{r-1},\mathrm{g}^{r-1},\mathrm{h}^{r-1},t_{r-1}\right) - W\left(\mathrm{f}^{-2},\mathrm{g}^{r-2},\mathrm{h}^{-2},t_{r-2}\right)}{\Delta t}(\tau - t_{r-2}) + \\ \dfrac{W\left(\mathrm{f}^{r},\mathrm{g}^{r},\mathrm{h}^{r},t_r\right) - 2W\left(\mathrm{f}^{-1},\mathrm{g}^{r-1},\mathrm{h}^{\prime-1},t_{r-1}\right) + W\left(\mathrm{f}^{r-2},\mathrm{g}^{-2},\mathrm{h}^{r-2},t_{r-2}\right)}{2(\Delta)^2} \\ \times(\tau - t_{r-2})(\tau - t_{r-1})\end{array}\right] d\tau.$$

Then, the numerical solution of the Burke-Shaw attractor with Caputo-Fabrizio derivative is given by

$$\begin{aligned}
\mathrm{f}_{n+1} &= u_n + \frac{\rho(1-\sigma)}{\mathrm{Z}(\sigma)}\left[t_n^{\rho-1}\varphi_n - t_{n-1}^{\rho-1}\varphi_{n-1}\right] \\
&\quad + \frac{\sigma\rho}{\mathrm{Z}(\sigma)}\sum_{r=2}^{n}\left[t_{r-2}^{\rho-1}\frac{5}{12}\varphi_{r-2}\Delta t - t_{r-1}^{\rho-1}\frac{4}{3}\varphi_{r-1}\Delta t + t_r^{\rho-1}\frac{23}{12}\varphi_r\Delta t\right], \\
\mathrm{g}_{n+1} &= v_n + \frac{\rho(1-\sigma)}{\mathrm{Z}(\sigma)}\left[t_n^{\rho-1}\psi_n - t_{n-1}^{\rho-1}\psi_{n-1}\right] \\
&\quad + \frac{\sigma\rho}{\mathrm{Z}(\sigma)}\sum_{r=2}^{n}\left[t_{r-2}^{\rho-1}\frac{5}{12}\psi_{r-2}\Delta t - t_{r-1}^{\rho-1}\frac{4}{3}\psi_{r-1}\Delta t + t_r^{\rho-1}\frac{23}{12}\psi_r\Delta t\right], \\
\mathrm{h}_{n+1} &= w_n + \frac{\rho(1-\sigma)}{\mathrm{Z}(\sigma)}\left[t_n^{\rho-1}\mu_n - t_{n-1}^{\rho-1}\mu_{n-1}\right] \\
&\quad + \frac{\sigma\rho}{\mathrm{Z}(\sigma)}\sum_{r=2}^{n}\left[t_{r-2}^{\rho-1}\frac{5}{12}\mu_{r-2}\Delta t - t_{r-1}^{\rho-1}\frac{4}{3}\mu_{r-1}\Delta t + t_r^{\rho-1}\frac{23}{12}\mu_r\Delta t\right].
\end{aligned}$$

5.2. Variable order with exponential decay

Suppose $\mathrm{f}(\mathtt{t})$ is a difractal-fractionalerential function with order σ and fractal variable dimension $\rho(\mathtt{t})$, then a fractional derivative of $\mathrm{f}(\mathtt{t})$ is given by σ and $\rho(\mathtt{t})$ with order σ.

$${}^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho(\mathtt{t})}\mathrm{f}(\mathtt{t}) = \frac{\mathrm{Z}(\sigma)}{1-\sigma}\frac{d}{d\mathtt{t}^{\rho(\mathtt{t})}}\int_0^{\mathtt{t}} u(\nu)\exp\left(-\frac{\sigma}{1-\sigma}(\mathtt{t}-\nu)\right)d\nu,$$

where $\dfrac{d\rho(\nu)}{d\mathtt{t}^{\rho(\mathtt{t})}} = \lim\limits_{\mathtt{t}\to\nu}\dfrac{\rho(\mathtt{t})-\rho(\nu)}{\mathtt{t}^{\rho(\mathtt{t})}-\nu^{\rho(\nu)}}$. In order to define a new fractional integral based on the exponential decay kernel, the following definition is provided:

$$\begin{aligned}
{}^{\mathtt{FFE}}I^{\sigma,\rho(\mathtt{t})}\mathrm{f}(\mathtt{t}) &= \frac{\sigma}{\mathrm{Z}(\sigma)}\int_0^{\mathtt{t}} u(\nu)\left[\rho\prime(\nu)\ln(\nu) + \frac{\rho(\nu)}{\nu}\right]\nu^{\rho(\nu)}d\nu \\
&\quad + \frac{1-\sigma}{\mathrm{Z}(\sigma)}\mathtt{t}^{\rho(\mathtt{t})}\left[\rho\prime(\mathtt{t})\ln(\mathtt{t}) + \frac{\rho(\mathtt{t})}{\mathtt{t}}\right]\mathrm{f}(\mathtt{t}).
\end{aligned}$$

For the purpose of determining the numerical scheme, the following new class of Cauchy problems is considered:

$${}^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho(\mathtt{t})}\mathrm{f}(\mathtt{t}) = \hbar(\mathtt{t},\mathrm{f}(\mathtt{t})), \quad \mathrm{f}(0) = \mathrm{f}_0\,.$$

By rewriting the equation above with an exponential decay kernel, we get the new fractional integral

$$\begin{aligned}\mathrm{f}(\mathtt{t}) &= \frac{\sigma}{\mathrm{Z}(\sigma)}\int_0^{\mathtt{t}} \hbar(\nu,u(\nu))\left[\rho\prime(\nu)\ln(\nu)+\frac{\rho(\nu)}{\nu}\right]\nu^{\rho(\nu)}d\nu \\ &+\frac{1-\sigma}{\mathrm{Z}(\sigma)}\mathtt{t}^{\rho(\mathtt{t})}\left[\rho\prime(\mathtt{t})\ln(\mathtt{t})+\frac{\rho(\mathtt{t})}{\mathtt{t}}\right]\hbar(\mathtt{t},\mathrm{f}(\mathtt{t})).\end{aligned}$$

The following is the result when $\mathtt{t}_{\ell+1}=(k+1)\Delta\mathtt{t}$:

$$\begin{aligned}u(\mathtt{t}_{\ell+1}) &= \frac{\sigma}{\mathrm{Z}(\sigma)}\int_0^{\mathtt{t}_{\ell+1}} \hbar(\nu,u(\nu))\left[\rho\prime(\nu)\ln(\nu)+\frac{\rho(\nu)}{\nu}\right]\nu^{\rho(\nu)}d\nu \\ &+\frac{1-\sigma}{\mathrm{Z}(\sigma)}\mathtt{t}_{\ell}^{\rho(\mathtt{t}_\ell)}\left[\frac{\rho(\mathtt{t}_{\ell+1})-\rho(\mathtt{t}_\ell)}{\Delta\mathtt{t}}\ln\mathtt{t}_\ell+\frac{\rho(\mathtt{t}_\ell)}{\mathtt{t}_\ell}\right]\hbar\left(\mathtt{t}_\ell,\mathrm{f}^\ell\right).\end{aligned}$$

In order to keep things simple, we can take the following approach:

$$g(\nu,u(\nu))=\hbar(\nu,u(\nu))\left[\rho\prime(\nu)\ln(\nu)+\frac{\rho(\nu)}{\nu}\right]\nu^{\rho(\nu)}.$$

As a result, at $\mathtt{t}_{\ell+1}=(k+1)\Delta\mathtt{t}$ and $\mathtt{t}_\ell=k\Delta\mathtt{t}$, and taking fractal-fractional of equations,

$$\begin{aligned}u(\mathtt{t}_{\ell+1}) &= u(\mathtt{t}_\ell)-\frac{1-\sigma}{\mathrm{Z}(\sigma)}\mathtt{t}_{\ell-1}^{\rho(\mathtt{t}_{\ell-1})}\left[\frac{\rho(\mathtt{t}_\ell)-\rho(\mathtt{t}_{\ell-1})}{\Delta\mathtt{t}}\ln\mathtt{t}_{\ell-1}+\frac{\rho(\mathtt{t}_{\ell-1})}{\mathtt{t}_{\ell-1}}\right] \\ &\times\hbar\left(\mathtt{t}_{\ell-1},\mathrm{f}^{\ell-1}\right)+\frac{\sigma}{\mathrm{Z}(\sigma)}\int_0^{\mathtt{t}_{\ell+1}}g(\nu,u(\nu))d\nu \\ &+\frac{1-\sigma}{\mathrm{Z}(\sigma)}\mathtt{t}_{\ell}^{\rho(\mathtt{t}_\ell)}\left[\frac{\rho(\mathtt{t}_{\ell+1})-\rho(\mathtt{t}_\ell)}{\Delta\mathtt{t}}\ln\mathtt{t}_\ell+\frac{\rho(\mathtt{t}_\ell)}{\mathtt{t}_\ell}\right]\hbar\left(\mathtt{t}_\ell,\mathrm{f}^\ell\right) \\ &= u(\mathtt{t}_\ell)+\frac{1-\sigma}{\mathrm{Z}(\sigma)}\mathtt{t}_{\ell}^{\rho(\mathtt{t}_\ell)}\left[\frac{\rho(\mathtt{t}_{\ell+1})-\rho(\mathtt{t}_\ell)}{\Delta\mathtt{t}}\ln\mathtt{t}_\ell+\frac{\rho(\mathtt{t}_\ell)}{\mathtt{t}_\ell}\right]\hbar\left(\mathtt{t}_\ell,\mathrm{f}^\ell\right) \\ &-\frac{1-\sigma}{\mathrm{Z}(\sigma)}\mathtt{t}_{\ell-1}^{\rho(\mathtt{t}_{\ell-1})}\left[\frac{\rho(\mathtt{t}_\ell)-\rho(\mathtt{t}_{\ell-1})}{\Delta\mathtt{t}}\ln\mathtt{t}_{\ell-1}+\frac{\rho(\mathtt{t}_{\ell-1})}{\mathtt{t}_{\ell-1}}\right]\times\hbar\left(\mathtt{t}_{\ell-1},\mathrm{f}^{\ell-1}\right) \\ &+\frac{\sigma}{\mathrm{Z}(\sigma)}\int_{\mathtt{t}_\ell}^{\mathtt{t}_{\ell+1}}g(\nu,u(\nu))d\nu.\end{aligned} \tag{4}$$

Using Eq. (4), one can rephrase the above equation as follows:

$$\begin{aligned}u_{\ell+1} &= u_\ell-\frac{1-\sigma}{\mathrm{Z}(\sigma)}\mathtt{t}_{\ell-1}^{\rho(\mathtt{t}_{\ell-1})}\left[\frac{\rho(\mathtt{t}_\ell)-\rho(\mathtt{t}_{\ell-1})}{\Delta\mathtt{t}}\ln\mathtt{t}_{\ell-1}+\frac{\rho(\mathtt{t}_{\ell-1})}{\mathtt{t}_{\ell-1}}\right]\times\hbar\left(\mathtt{t}_{\ell-1},\mathrm{f}^{\ell-1}\right) \\ &+\frac{\sigma}{\mathrm{Z}(\sigma)}\int_{\mathtt{t}_\ell}^{\mathtt{t}_{\ell+1}}\left\{\frac{g\left(\mathtt{t}_\ell,\mathrm{f}^\ell\right)}{\Delta\mathtt{t}}(\nu-\mathtt{t}_{\ell-1})-\frac{g\left(\mathtt{t}_{\ell-1},\mathrm{f}^{\ell-1}\right)}{\Delta\mathtt{t}}(\nu-\mathtt{t}_\ell)\right\}d\nu \\ &+\frac{1-\sigma}{\mathrm{Z}(\sigma)}\mathtt{t}_{\ell}^{\rho(\mathtt{t}_\ell)}\left[\frac{\rho(\mathtt{t}_{\ell+1})-\rho(\mathtt{t}_\ell)}{\Delta\mathtt{t}}\ln\mathtt{t}_\ell+\frac{\rho(\mathtt{t}_\ell)}{\mathtt{t}_\ell}\right]\hbar\left(\mathtt{t}_\ell,\mathrm{f}^\ell\right).\end{aligned}$$

The above equation can be organized as follows:

$$\begin{aligned}u_{\ell+1}=u_{\ell}-&\frac{1-\sigma}{Z(\sigma)}\mathtt{t}_{\ell-1}^{\rho(\mathtt{t}_{\ell-1})}\left[\frac{\rho\left(\mathtt{t}_{\ell}\right)-\rho\left(\mathtt{t}_{\ell-1}\right)}{\Delta\mathtt{t}}\ln\mathtt{t}_{\ell-1}+\frac{\rho\left(\mathtt{t}_{\ell-1}\right)}{\mathtt{t}_{\ell-1}}\right]\times\hbar\left(\mathtt{t}_{\ell-1},\mathrm{f}^{\ell-1}\right)\\&+\frac{\sigma}{Z(\sigma)}\frac{g\left(\mathtt{t}_{\ell},\mathrm{f}^{\ell}\right)}{\Delta\mathtt{t}}\int_{\mathtt{t}_{\ell}}^{\mathtt{t}_{\ell+1}}(\nu-\mathtt{t}_{\ell-1})\,d\nu-\frac{\sigma}{Z(\sigma)}\frac{g\left(\mathtt{t}_{\ell-1},\mathrm{f}^{\ell-1}\right)}{\Delta\mathtt{t}}\int_{\mathtt{t}_{\ell}}^{\mathtt{t}_{\ell+1}}(\nu-\mathtt{t}_{\ell})\,d\nu\\&+\frac{1-\sigma}{Z(\sigma)}\mathtt{t}_{\ell}^{\rho(\mathtt{t}_{\ell})}\left[\frac{\rho\left(\mathtt{t}_{\ell+1}\right)-\rho\left(\mathtt{t}_{\ell}\right)}{\Delta\mathtt{t}}\ln\mathtt{t}_{\ell}+\frac{\rho\left(\mathtt{t}_{\ell}\right)}{\mathtt{t}_{\ell}}\right]\hbar\left(\mathtt{t}_{\ell},\mathrm{f}^{\ell}\right).\end{aligned}\tag{5}$$

As a result of (5), we have

$$\int_{\mathtt{t}_{\ell}}^{\mathtt{t}_{\ell+1}}(\nu-\mathtt{t}_{\ell-1})\,d\nu=\frac{3(\Delta\mathtt{t})^{2}}{2},\quad\int_{\mathtt{t}_{\ell}}^{\mathtt{t}_{\ell+1}}(\nu-\mathtt{t}_{\ell})\,d\nu=\frac{(\Delta\mathtt{t})^{2}}{2}.\tag{6}$$

Our approximation for the Eq. (5) can be obtained as follows:

$$\begin{aligned}u_{\ell+1}=u_{\ell}-&\frac{1-\sigma}{Z(\sigma)}\mathtt{t}_{\ell-1}^{\rho(\mathtt{t}_{\ell-1})}\left[\frac{\rho\left(\mathtt{t}_{\ell}\right)-\rho\left(\mathtt{t}_{\ell-1}\right)}{\Delta\mathtt{t}}\ln\mathtt{t}_{\ell-1}+\frac{\rho\left(\mathtt{t}_{\ell-1}\right)}{\mathtt{t}_{\ell-1}}\right]\hbar\left(\mathtt{t}_{\ell-1},\mathrm{f}^{\ell-1}\right)\\&+\frac{\sigma}{Z(\sigma)}g\left(\mathtt{t}_{\ell},\mathrm{f}^{\ell}\right)\frac{3\Delta\mathtt{t}}{2}-\frac{\sigma}{Z(\sigma)}g\left(\mathtt{t}_{\ell-1},\mathrm{f}^{\ell-1}\right)\frac{\Delta\mathtt{t}}{2}\\&+\frac{1-\sigma}{Z(\sigma)}\mathtt{t}_{\ell}^{\rho(\mathtt{t}_{\ell})}\left[\frac{\rho\left(\mathtt{t}_{\ell+1}\right)-\rho\left(\mathtt{t}_{\ell}\right)}{\Delta\mathtt{t}}\ln\mathtt{t}_{\ell}+\frac{\rho\left(\mathtt{t}_{\ell}\right)}{\mathtt{t}_{\ell}}\right]\hbar\left(\mathtt{t}_{\ell},\mathrm{f}^{\ell}\right).\end{aligned}$$

Thus

$$\begin{aligned}u_{\ell+1}=u_{\ell}-&\frac{1-\sigma}{Z(\sigma)}\mathtt{t}_{\ell-1}^{\rho(\mathtt{t}_{\ell-1})}\left[\frac{\rho\left(\mathtt{t}_{\ell}\right)-\rho\left(\mathtt{t}_{\ell-1}\right)}{\Delta\mathtt{t}}\ln\mathtt{t}_{\ell-1}+\frac{\rho\left(\mathtt{t}_{\ell-1}\right)}{\mathtt{t}_{\ell-1}}\right]\times\hbar\left(\mathtt{t}_{\ell-1},\mathrm{f}^{\ell-1}\right)\\&+\frac{\sigma\rho}{Z(\sigma)}\mathtt{t}_{\ell}^{\rho(\mathtt{t}_{\ell})}\left[\frac{\rho\left(\mathtt{t}_{\ell+1}\right)-\rho\left(\mathtt{t}_{\ell}\right)}{\Delta\mathtt{t}}\ln\mathtt{t}_{\ell}+\frac{\rho\left(\mathtt{t}_{\ell}\right)}{\mathtt{t}_{\ell}}\right]\times\hbar\left(\mathtt{t}_{\ell},\mathrm{f}^{\ell}\right)\frac{3\Delta\mathtt{t}}{2}\\&-\frac{\sigma\rho}{Z(\sigma)}\mathtt{t}_{\ell-1}^{\rho(\mathtt{t}_{\ell-1})}\times\left[\frac{\rho\left(\mathtt{t}_{\ell}\right)-\rho\left(\mathtt{t}_{\ell-1}\right)}{\Delta\mathtt{t}}\ln\mathtt{t}_{\ell-1}+\frac{\rho\left(\mathtt{t}_{\ell-1}\right)}{\mathtt{t}_{\ell-1}}\right]\hbar\left(\mathtt{t}_{\ell-1},\mathrm{f}^{\ell-1}\right)\frac{\Delta\mathtt{t}}{2}\\&+\frac{1-\sigma}{Z(\sigma)}\mathtt{t}_{\ell}^{\rho(\mathtt{t}_{\ell})}\left[\frac{\rho\left(\mathtt{t}_{\ell+1}\right)-\rho\left(\mathtt{t}_{\ell}\right)}{\Delta\mathtt{t}}\ln\mathtt{t}_{\ell}+\frac{\rho\left(\mathtt{t}_{\ell}\right)}{\mathtt{t}_{\ell}}\right]\hbar\left(\mathtt{t}_{\ell},\mathrm{f}^{\ell}\right).\end{aligned}$$

5.3. Numerical simulation

The system behaves chaotically when parameters are $V=10$, $d=13$ or 4.272. The initial conditions are as follows: $\mathrm{f}(0)=0.1$, $\mathrm{g}(0)=0.1$, $\mathrm{h}(0)=0.1$.

Example 1

$$\begin{aligned}{}^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho(\mathtt{t})}\,\mathrm{f}(\mathtt{t})&=-10(\mathrm{f}(\mathtt{t})+\mathrm{g}(\mathtt{t})),\\{}^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho(\mathtt{t})}\,\mathrm{g}(\mathtt{t})&=-10\,\mathrm{f}(\mathtt{t})\,\mathrm{h}(\mathtt{t})-\mathrm{g}(\mathtt{t}),\\{}^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho(\mathtt{t})}\,\mathrm{h}(\mathtt{t})&=10\,\mathrm{f}(\mathtt{t})\,\mathrm{g}(\mathtt{t})+13.\end{aligned}\tag{7}$$

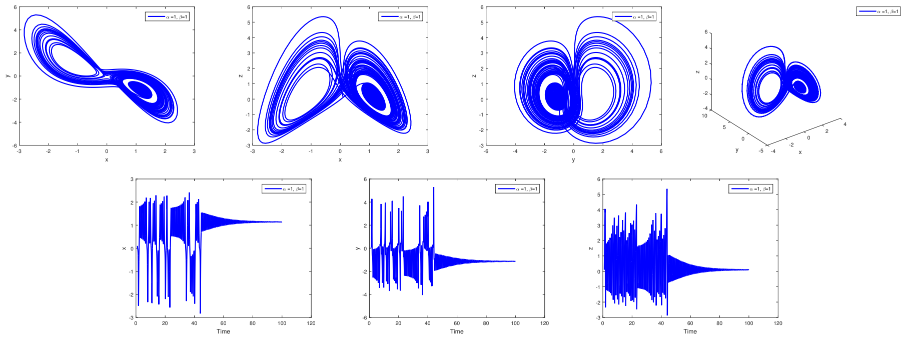

Fig. 1. Simulation of the fractal-fractional Burke-Shaw system (7) with $\sigma = 1$, $\rho = 1$

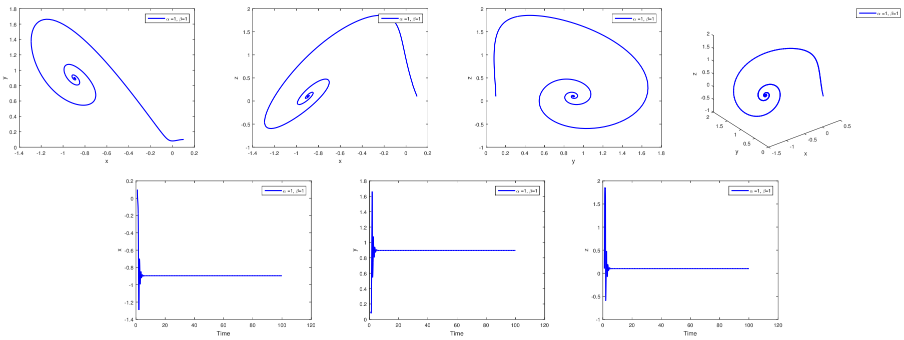

Fig. 2. Simulation of the control fractal-fractional Burke-Shaw system (8) with $\sigma = 1$, $\rho = 1$

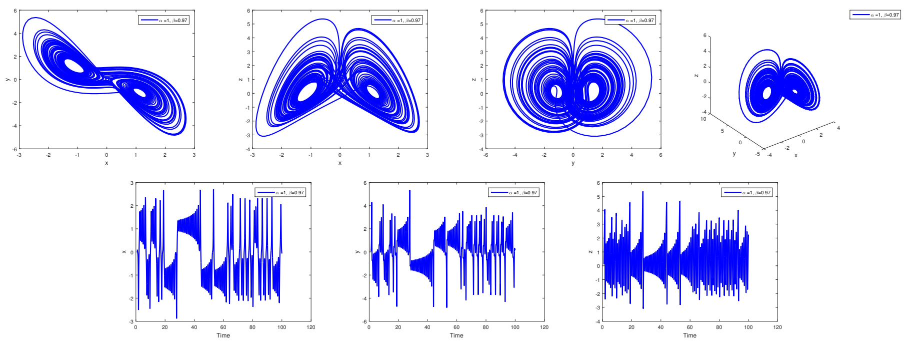

Fig. 3. Simulation of the fractal-fractional Burke-Shaw system (7) with $\sigma = 1$, $\rho = 1$

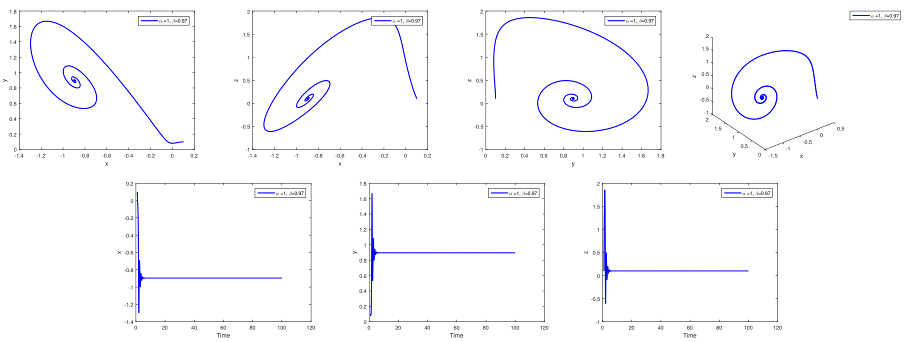

Fig. 4. Simulation of the control fractal-fractional Burke-Shaw system (8) with $\sigma = 1$, $\rho = 1$

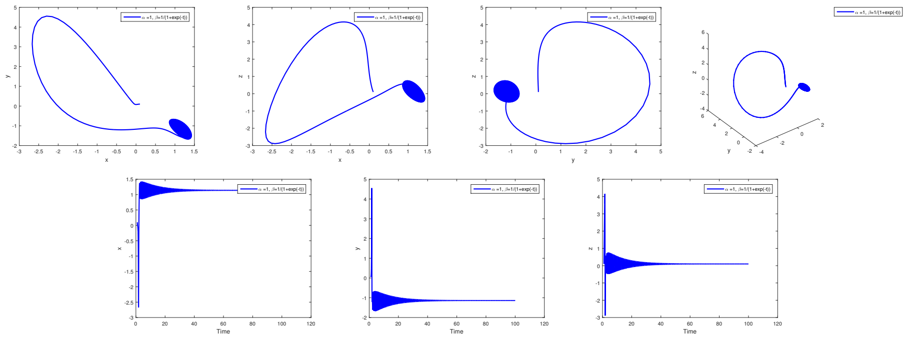

Fig. 5. Simulation of the fractal-fractional Burke-Shaw system (7) with $\sigma = 1$, $\rho = 1/(1+exp(-t))$

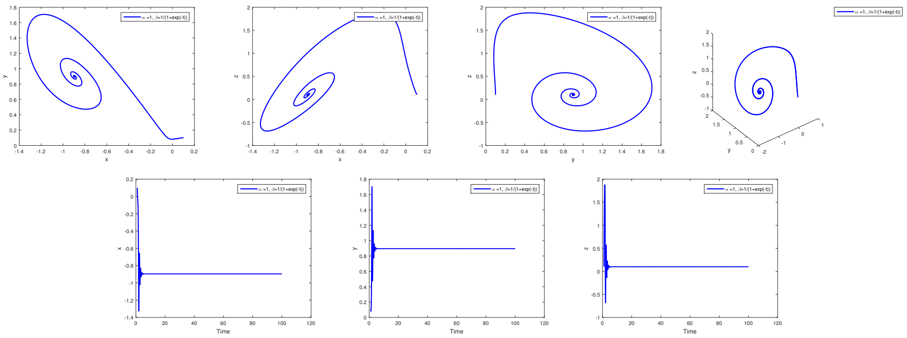

Fig. 6. Simulation of the control fractal-fractional Burke-Shaw system (8) with $\sigma = 1$, $\rho = 1/(1+exp(-t))$

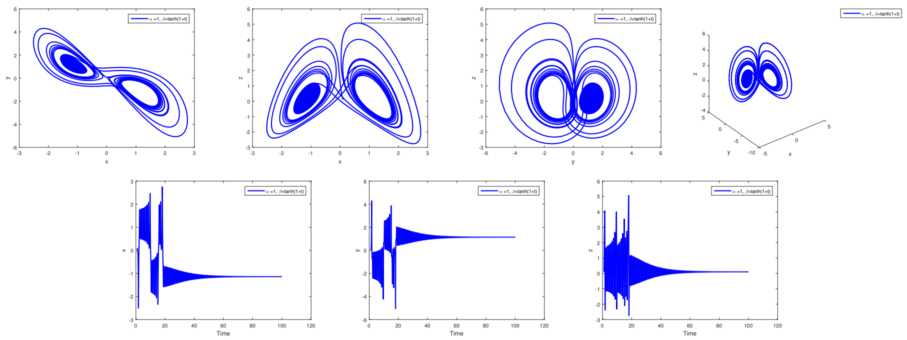

Fig. 7. Simulation of the fractal-fractional Burke-Shaw system (7) with $\sigma = 1$, $\rho = tanh(1+t)$

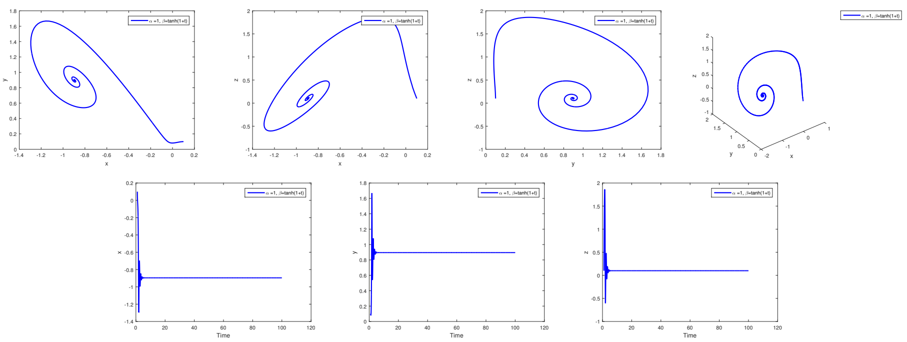

Fig. 8. Simulation of the control fractal-fractional Burke-Shaw system (8) with $\sigma = 1$, $\rho = tanh(1+t)$

Figures 1, 3, 5 and 7 show numerical simulations of a fractal-fractional order Burke--Shaw system (7) in the sense of the Caputo-Fabrizio derivative for $\sigma = 1, \rho = 1$, $\sigma = 1, \rho = 0.97$, $\sigma = 1, \rho = 1/(1+exp(-t))$, $\sigma = 1, \rho = tanh(1+t)$, respectively.

Example 2

$$\begin{aligned}
{}^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho(\mathtt{t})}\,\mathrm{f}(\mathtt{t}) &= -10(\mathrm{f}(\mathtt{t})+\mathrm{g}(\mathtt{t})),\\
{}^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho(\mathtt{t})}\,\mathrm{g}(\mathtt{t}) &= -10\,\mathrm{f}(\mathtt{t})\,\mathrm{h}(\mathtt{t})-\mathrm{g}(\mathtt{t}),\\
{}^{\mathtt{FFE}}\mathscr{D}_{0,\mathtt{t}}^{\sigma,\rho(\mathtt{t})}\,\mathrm{h}(\mathtt{t}) &= 10\,\mathrm{f}(\mathtt{t})\,\mathrm{g}(\mathtt{t})+13-5(\mathrm{g}(\mathtt{t})+\mathrm{h}(\mathtt{t})).
\end{aligned} \tag{8}$$

Figures 2, 4, 6 and 8 show numerical simulations of a fractal-fractional order Burke--Shaw system (8) in the sense of the Caputo-Fabrizio derivative for $\sigma = 1$, $\rho = 1$, $\sigma = 1$, $\rho = 0.97$, $\sigma = 1$, $\rho = 1/(1+exp(-t))$, $\sigma = 1$, $\rho = tanh(1+t)$, respectively.

6. Conclusions

The numerical solution of the fractal-fractional Burke-Shaw model with an exponential decay kernel is a major area of research in applied mathematics. Various numerical methods can be employed to solve this complex model, including the finite fractal-fractional method, the finite element method, the spectral method, and others. The choice of method depends on the specific requirements and characteristics of the problem. It is crucial to select an appropriate method to obtain accurate and reliable results. Solutions are obtained for the Burke-Shaw model using a fractal--fractional operator with an exponential decay kernel. Uniqueness and boundedness for solutions are proved through fixed point theory [43].

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