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ON THE STABILITY OF FIRST ORDER IMPULSIVE EVOLUTION EQUATIONS

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Abstract. In this paper, concepts of Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for impulsive evolution equations are raised. Ulam-Hyers-Rassias stability results on a compact interval and an unbounded interval are presented by using an impulsive integral inequality of the Gronwall type. Two examples are also provided to illustrate our results. Finally, some extensions of the Ulam-Hyers-Rassias stability for the case with infinite impulses are given.

Keywords: first order, impulsive evolution equations, Ulam-Hyers-Rassias stability.

Mathematics Subject Classification: 34G20, 34D10, 45N05.

1. INTRODUCTION

As stated in Brzdęk *et al*'s paper [6], the stability of functional equations was first presented by Ulam in 1940 at Wisconsin University and concerned approximate homomorphisms. The pursuit of solutions to this problem, to its generalizations and modifications for various classes of difference, functional, differential, integral and impulsive equations, is an expanding area of research and has led to the development of what is now quite often called Ulam's type stability theory or the Ulam-Hyers stability theory. This theory has been the subject of many papers as well as talks presented at various conferences, especially at the series of International Conference on Functional Equations and Inequalities conferences organized by the Department of Mathematics of the Pedagogical University in Cracow since 1984.

For the case of Banach spaces, Hyers [10] answered this question completely in 1941. Thereafter, a new stability concept, Ulam-Hyers stability, was named by researchers. In 1978, Rassias [22] provided an extension of the Ulam-Hyers stability by introducing new function variables. As a result, another new stability concept, Ulam-Hyers-Rassias stability, was named by mathematicians.

Many researchers paid attention to the stability properties of all kinds of equations since 1940. We emphasize that Ulam's type stability problems have been taken up by a large number of mathematicians and the study of this area has grown to be one of the most important subjects in mathematical analysis. For the advanced contributions on such problems, we refer the readers to András and Kolumbán [1], András and Mészáros [2], Burger *et al.* [7], Cădariu [8], Cimpean and Popa [9], Hyers [11], Hegyi and Jung [12], Jung [13,14], Lungu and Popa [16], Miura *et al.* [17,18], Obłoza [19,20], Rassias [23, 24], Rezaei *et al.* [25], Rus [26, 27], Takahasi *et al.* [30] and Wang *et al.* [31–33].

As a matter of fact, Ulam-Hyers-Rassias stability of impulsive evolution equations has not yet been studied as far as we are aware. So motivated by recent works [27,31, 32], we will study the Ulam-Hyers-Rassias stability of the following impulsive evolution equations:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in J' := J \setminus \{t_k\}_{k\in\mathbb{M}}, \\ \Delta x(t_k) = I_k(x(t_k^-)), & k \in \mathbb{M}, \end{cases} \tag{1.1}$$

where the following basic assumptions are imposed:

– either $J = [0, T]$ for some $T > 0$ or $J = \mathbb{R}^+ := [0, \infty)$. If $J = [0, T]$, then $\mathbb{M} = \{1, 2, \ldots, m\}$, if $J = \mathbb{R}^+$ then either $\mathbb{M} = \{1, \ldots, m\}$ or $\mathbb{M} = \mathbb{N}$. We set $\mathbb{M}_0 := \mathbb{M} \cup \{0\}$;
– the linear unbounded operator A: $D(A) \subseteq X \to X$ is the generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on a Banach space X; the corresponding norm on X is denoted by $\|\cdot\|$;
– the function $f : J \times X \to X$ and impulsive operators $I_k : X \to X$ are specified in Section 4;
– we set $t_0 = 0$, and $t_{m+1} = T$ for $J = [0, T]$, and $t_{m+1} = \infty$ for $J = \mathbb{R}^+$; the fixed time sequence $\{t_k\}_{k\in\mathbb{M}_0}$ is increasing, i.e., $t_k < t_{k+1}$ for any $k \in \mathbb{M}_0$;
– $x(t_k^+) = \lim_{\epsilon\to 0^+} x(t_k + \epsilon)$ and $x(t_k^-) = \lim_{\epsilon\to 0^-} x(t_k + \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively;
– $\Delta x(t_k) := x(t_k^+) - x(t_k{}^-)$.

Many processes studied in applied sciences are represented by differential equations. However, the situation is quite different in many modeled phenomena that have a sudden change in their states such as mechanical systems with impact, biological systems such as heart beats, blood flows, population dynamics, theoretical physics, pharmacokinetics, mathematical economy, biotechnology processes, chemistry, engineering, control theory, medicine and so on. Adequate mathematical models of such processes are systems of differential equations with impulses. The impulsive conditions are combinations of traditional initial value problems and short-term perturbations whose duration can be negligible in comparison with the duration of the process. There are many good monographs on impulsive differential equations, like Samoilenko *et al.* [28], Bainov *et al.* [3], and Benchohra *et al.* [5]. Next, our equation (1.1) represents one of the most general form of semilinear evolution equations in Banach spaces with initial value conditions on either finite or infinite time space involving either finite or infinite impulses. In Section 5, we present two simple examples of one-dimensional diffusion processes with sudden changes of either the temperature of the rod or the chemical concentration of the substance.

In order to study our problems, we introduce four new types of Ulam stabilities (see Definitions 3.1–3.4): Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for equation (1.1) in Banach spaces. Here, we remark that we provide an extension of the Ulam-Hyers-Rassias stability by introducing a necessary modifications due to the impulsive conditions $\Delta x(t_k) = I_k(x(t_k^-))$ in equation (1.1). The novelty of our paper is that we consider a new type of equation (1.1) and then present these new Ulam-Hyers-Rassias stability definitions and finally find reasonable conditions on (1.1) for showing that (1.1) is Ulam-Hyers-Rassias stable in the sense of these definitions. We think that the considered conditions on the equation (1.1) are optimal and compare to the above mentioned papers. For the case with infinite impulses, some extensions of Ulam-Hyers-Rassias stability are given in Section 6.

Finally we note that the current results have also practical meaning in the following sense. Consider an evolution process with sudden changes of states at some fixed times which can be modeled by (1.1). Assume that we can measure the state of the process at any time to get a function $x(\cdot)$. Putting this $x(\cdot)$ into (1.1), in general, we do not expect to get a precise solution of (1.1). All that is required is to get a function which satisfies these suitable approximation inequalities (3.1), (3.2) and (3.3) of Section 3, and (6.1) and (6.2) of Section 6. In other words, our results of Sections 4 and 6 guarantee that there is a solution $y(\cdot)$ of (1.1) close to the measured output $x(\cdot)$ and close is defined in the sense of Ulam-Hyers-Rassias stability. This is our main original contribution of this paper. This is quite useful in many applications such as numerical analysis, optimization, biology and economics, where finding the exact solution is quite difficult. If the stochastic effects are small, it also helps to use a deterministic model to approximate a stochastic one.

2. PRELIMINARIES

Throughout this paper, let $B(X)$ be the Banach space of all linear and bounded operators on the Banach space X. The corresponding norm on $B(X)$ is denoted by $\|\cdot\|_{B(X)}$. Since the linear unbounded operator A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on X, there exist $N \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\|_{B(X)} \leq Ne^{\omega t}$ for $t \geq 0$. Denote $M = \sup_{t\in J} \|T(t)\|_{B(X)}$ for $J = [0, T]$. Let us introduce a vector space:

$$\begin{aligned} PC(J, X) := \{&x : J \to X | x \in C((t_k, t_{k+1}], X), \\ &\text{there exist } x(t_k^-) \text{ and } x(t_k^+) \text{ with } x(t_k^-) = x(t_k) \text{ for any } k \in \mathbb{M}_0\}. \end{aligned}$$

We also consider $C(J', D(A))$ with the graph norm on $D(A)$ and the usual $C^1(J', X)$.

Definition 2.1. By a PC-mild solution of the following impulsive Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in J', \\ \Delta x(t_k) = I_k(x(t_k^-)), & k \in \mathbb{M}_0, \\ x(0) = x_0, & x_0 \in X, \end{cases} \tag{2.1}$$

we mean a function $x \in PC(J, X)$ which satisfies

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)), \quad t \in J.$$

Remark 2.2. The existence and uniqueness of a mild solution of the problem (2.1) have been discussed by Liu (see [15, Theorem 2.1]) when f and I_k satisfy the standard Lipschtiz and linear growth conditions respectively.

The following impulsive integral inequality of Gronwall type

$$u(t) \ \leq \ a + \int_0^t b(s)u(s)ds + \sum_{0<t_k<t} \beta_k u(t_k), \quad t \geq 0, \tag{2.2}$$

has been discussed by Samoilenko and Perestyuk [29] to derive a priori bound of the solution of impulsive problems of the form

$$\begin{cases} u'(t) = f(t, u(t)), & t \neq t_k, \\ \Delta u(t_k) = I_k(u(t_k^-)), & t = t_k. \end{cases}$$

In order to deal with Ulam's type stability, we need to extend (2.2) the following result by Bainov and Simeonov (see [4, Theorem 16.4]).

Lemma 2.3. *Let the following inequality hold:*

$$u(t) \leq a(t) + \int_0^t b(s)u(s)ds + \sum_{0<t_k<t} \beta_k u(t_k^-), \quad t \geq 0, \tag{2.3}$$

where u, $a, b \in PC(\mathbb{R}^+, \mathbb{R}^+)$, a is nondecreasing and $b(t) > 0$, $\beta_k > 0$, $k \in \mathbb{M}$. Then, for $t \in \mathbb{R}^+$, the following inequality is valid:

$$u(t) \ \leq \ a(t)\,(1+\beta)^k \exp\left(\int_0^t b(s)ds\right), \quad t \in (t_k, t_{k+1}],\ k \in \mathbb{M}_0, \tag{2.4}$$

where $\beta = \sup_{k \in \mathbb{M}}\{\beta_k\}$.

Remark 2.4. (i) If we replace β_k in (2.3) by nondecreasing functions $\beta_k(t) > 0$ for $t \geq 0$, then (2.4) turns to the following inequality

$$u(t) \;\leq\; a(t) \prod_{0<t_k<t} (1 + \beta_k(t)) \exp\left(\int_0^t b(s)ds \right), \quad t \in (t_k, t_{k+1}],\; k \in \mathbb{M}_0, \tag{2.5}$$

(see [4, Theorem 16.4]).

(ii) As an extension of (2.3), a generalized impulsive singular integral inequality of Gronwall type

$$u(t) \;\leq\; a(t) + b \int_0^t (t-s)^{\alpha-1} u(s)ds + \sum_{0<t_k<t} \beta_k u(t_k^-), \quad \alpha, b > 0,\; t \geq 0,$$

has been reported by Wang *et al.* (see [31, Lemma 2.8]).

3. BASIC CONCEPTS AND REMARKS

In this section, we introduce the concepts of Ulam's type stability for equation (1.1).

Set $PC(J, \mathbb{R}^+) := \{x \in PC(J, \mathbb{R}) : x(t) \geq 0\}$. Let $\varepsilon > 0$, $\psi \geq 0$ and $\varphi \in PC(J, \mathbb{R}^+)$. We consider the following inequalities:

$$\begin{cases} \|y'(t) - Ay(t) - f(t, y(t))\| \leq \varepsilon, & t \in J', \\ \|\Delta y(t_k) - I_k(y(t_k^-))\| \leq \varepsilon, & k \in \mathbb{M}, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \|y'(t) - Ay(t) - f(t, y(t))\| \leq \varphi(t), & t \in J', \\ \|\Delta y(t_k) - I_k(y(t_k^-))\| \leq \psi, & k \in \mathbb{M}, \end{cases} \tag{3.2}$$

and

$$\begin{cases} \|y'(t) - Ay(t) - f(t, y(t))\| \leq \varepsilon\varphi(t), & t \in J', \\ \|\Delta y(t_k) - I_k(y(t_k^-))\| \leq \varepsilon\psi, & k \in \mathbb{M}. \end{cases} \tag{3.3}$$

Definition 3.1. Equation (1.1) is Ulam-Hyers stable if there exists a real number $c_{f,\mathbb{M}} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in PC(J, X) \cap C(J', D(A)) \cap C^1(J', X)$ of the inequality (3.1) there exists a mild solution $x \in PC(J, X)$ of the equation (1.1) with

$$\|y(t) - x(t)\| \leq c_{f,\mathbb{M}}\varepsilon, \quad t \in J.$$

Definition 3.2. The equation (1.1) is generalized Ulam-Hyers stable if there exists $\theta_{f,\mathbb{M}} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\theta_{f,\mathbb{M}}(0) = 0$ such that for each solution $y \in PC(J, X) \cap C(J', D(A)) \cap C^1(J', X)$ of the inequality (3.1) there exists a mild solution $x \in PC(J, X)$ of the equation (1.1) with

$$\|y(t) - x(t)\| \leq \theta_{f,\mathbb{M}}(\varepsilon), \quad t \in J.$$

The following definitions will extend the original (generalized) Ulam-Hyers-Rassias stability concepts for the equations without impulses to equations with impulses.

Definition 3.3. Equation (1.1) is Ulam-Hyers-Rassias stable with respect to (φ, ψ) if there exists $c_{f,\mathbb{M},\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in PC(J, X) \cap C(J', D(A)) \cap C^1(J', X)$ of inequality (3.3) there exists a mild solution $x \in PC(J, X)$ of equation (1.1) with

$$\|y(t) - x(t)\| \leq c_{f,\mathbb{M},\varphi}\varepsilon(\varphi(t) + \psi), \quad t \in J.$$

Definition 3.4. Equation (1.1) is generalized Ulam-Hyers-Rassias stable with respect to (φ, ψ) if there exists $c_{f,\mathbb{M},\varphi} > 0$ such that for each solution $y \in PC(J, X) \cap C(J', D(A)) \cap C^1(J', X)$ of inequality (3.2) there exists a mild solution $x \in PC(J, X)$ of equation (1.1) with

$$\|y(t) - x(t)\| \leq c_{f,\mathbb{M},\varphi}(\varphi(t) + \psi), \quad t \in J.$$

We note that:

(i) Definition 3.3 for $\varphi(t) = \psi = 1 \Longrightarrow$ Definition 3.1 $\Longrightarrow$ Definition 3.2,
(ii) Definition 3.3 $\Longrightarrow$ Definition 3.4.

Remark 3.5. It follows directly from inequality (3.3) that a function $y \in PC(J, X) \cap C(J', D(A)) \cap C^1(J', X)$ is a solution of inequality (3.3) if and only if there is $g \in C(J', X)$, $\psi \geq 0$ and a sequence g_k, $k \in \mathbb{M}$ (which depend on y) such that:

(i) $\|g(t)\| \leq \varepsilon\varphi(t)$ and $\|g_k\| \leq \varepsilon\psi$, $t \in J'$, $k \in \mathbb{M}$,
(ii) $y'(t) - Ay(t) - f(t, y(t)) = g(t)$, $t \in J'$,
(iii) $\Delta y(t_k) - I_k(y(t_k^-)) = g_k$, $k \in \mathbb{M}$.

Remark 3.6. If $y \in PC(J, X) \cap C(J', D(A)) \cap C^1(J', X)$ is a solution of inequality (3.3) then y is a solution of the following integral inequality

$$\begin{aligned} &\left\| y(t) - T(t)y(0) - \int_0^t T(t-s)f(s, y(s))ds - \sum_{i=1}^{k} T(t-t_i)I_i(y(t_i^-)) \right\| \leq \\ &\leq M\varepsilon\left(m\psi + \int_0^t \varphi(s)ds\right), \quad t \in (t_k, t_{k+1}],\ k \in \mathbb{M}_0, \end{aligned} \tag{3.4}$$

where we set $\sum_{i=1}^{0} = 0$.

Proof. It follows from Remark 3.5 that we have

$$\begin{cases} y'(t) = Ay(t) + f(t, y(t)) + g(t), & t \in J', \\ \Delta y(t_k) = I_k(y(t_k^-)) + g_k, & k \in \mathbb{M}. \end{cases}$$

Thus the formula of solution is (see [21, p. 105])

$$y(t) = T(t)y(0) + \sum_{i=1}^{k} T(t-t_i)I_i(y(t_i^-)) + \sum_{i=1}^{k} T(t-t_i)g_i +$$
$$+ \int_0^t T(t-s)f(s,y(s))ds + \int_0^t T(t-s)g(s)ds, \quad t \in (t_k, t_{k+1}],\ k \in \mathbb{M}_0.$$

As a result, we find that

$$\left\| y(t) - T(t)y(0) - \int_0^t T(t-s)f(s,y(s))ds - \sum_{i=1}^{k} T(t-t_i)I_i(y(t_i^-)) \right\| \le$$
$$\le \sum_{i=1}^{k} \|T(t-t_i)\|_{B(X)} \|g_i\| + \int_0^t \|T(t-s)\|_{B(X)} \|g(s)\| ds \le$$
$$\le M\varepsilon \left(m\psi + \int_0^t \varphi(s)ds \right).$$

The proof is complete. □

Similarly to Remark 3.6, we have the following results.

Remark 3.7. If $y \in PC(J,X) \cap C(J', D(A)) \cap C^1(J', X)$ is a solution of inequality (3.1) then y is a solution of the following integral inequality

$$\left\| y(t) - T(t)y(0) - \int_0^t T(t-s)f(s,y(s))ds - \sum_{i=1}^{k} T(t-t_i)I_i(y(t_i^-)) \right\| \le$$
$$\le M\,(m+t)\,\varepsilon, \quad t \in (t_k, t_{k+1}],\ k \in \mathbb{M}_0.$$

Remark 3.8. If $y \in PC(J,X) \cap C(J', D(A)) \cap C^1(J', X)$ is a solution of inequality (3.2) then y is a solution of the following integral inequality

$$\left\| y(t) - T(t)y(0) - \int_0^t T(t-s)f(s,y(s))ds - \sum_{i=1}^{k} T(t-t_i)I_i(y(t_i^-)) \right\| \le$$
$$\le M \left(m\psi + \int_0^t \varphi(s)ds \right), \quad t \in (t_k, t_{k+1}],\ k \in \mathbb{M}_0.$$

4. MAIN RESULTS

In this section, we will present Ulam-Hyers-Rassias stability results for equation (1.1) on a compact interval J and unbounded interval $\mathbb{R}^+$.

4.1. ULAM-HYERS-RASSIAS STABILITY RESULTS ON $J = [0, T]$

We introduce the following assumptions:

(H1) $f : J \times X \to X$ satisfies the Carathéodory conditions and there exists a function $L_f \in C(J, \mathbb{R}^+)$ such that

$$\|f(t,u) - f(t,v)\| \le L_f(t)\|u - v\|$$

for almost each (a.e.) $t \in J$ and all $u, v \in X$.

(H2) I_k: $X \to X$ and there exist constants $\rho_k > 0$ such that

$$\|I_k(u) - I_k(v)\| \le \rho_k \|u - v\|$$

for all $u, v \in X$ and $k = 1, 2, \ldots, m$.

Remark 4.1. One can use the standard methods via Banach contraction principle in Liu [15] to derive the existence and uniqueness of mild solutions of equation (1.1) with initial value condition $x(0) = x_0$ under the assumptions (H1) and (H2).

(H3) There exists a constant $\lambda_\varphi > 0$ and a nondecreasing function $\varphi \in PC(J, \mathbb{R}^+)$ such that

$$\int_0^t \varphi(s)ds \le \lambda_\varphi \varphi(t) \text{ for each } t \in J.$$

Under the above assumptions, we consider equation (1.1) and inequality (3.3).

Theorem 4.2. *Assume* (H1)–(H3) *are satisfied. Then equation* (1.1) *is Ulam-Hyers--Rassias stable with respect to* (φ, ψ).

Proof. Let $y \in PC(J, X) \cap C(J', D(A)) \cap C^1(J', X)$ be a solution of inequality (3.3). Let x be the unique mild solution of the impulsive Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in J', \\ \Delta x(t_k) = I_k(x(t_k^-)), & k = 1, 2, \ldots, m, \\ x(0) = y(0). \end{cases}$$

Then we have

$$x(t) = \begin{cases} T(t)y(0) + \int\limits_0^t T(t-s)f(s, x(s))ds, & t \in [0, t_1], \\ T(t)y(0) + T(t-t_1)I_1(x(t_1^-)) + \int\limits_0^t T(t-s)f(s, x(s))ds, & t \in (t_1, t_2], \\ \vdots \\ T(t)y(0) + \sum_{k=1}^m T(t-t_k)I_k(x(t_k^-)) + \int\limits_0^t T(t-s)f(s, x(s))ds, & t \in (t_m, T]. \end{cases}$$

Like in (3.4), by inequality (3.3), for each $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned}&\left\| y(t) - T(t)y(0) - \int_0^t T(t-s)f(s,y(s))ds - \sum_{i=1}^{k} T(t-t_i)I_i(y(t_i^-)) \right\| \leq \\ &\leq M\left(\sum_{i=1}^{m}\|g_i\| + \int_0^t \varphi(s)ds\right) \leq \\ &\leq M\varepsilon(m+\lambda_\varphi)(\varphi(t)+\psi).\end{aligned}$$

Thus, for each $t \in (t_k, t_{k+1}]$, we obtain

$$\begin{aligned}\|y(t) - x(t)\| \leq &\left\| y(t) - T(t)y(0) - \sum_{i=1}^{k} T(t-t_i)I_i(y(t_i^-)) - \int_0^t T(t-s)f(s,y(s))ds \right\| + \\ &+ \sum_{i=1}^{k} \|T(t-t_i)\|_{B(X)}\|I_i(x(t_i^-)) - I_i(y(t_i^-))\| + \\ &+ \int_0^t \|T(t-s)\|_{B(X)}\|f(s,y(s)) - f(s,x(s))\|ds \leq \\ \leq\, &M\varepsilon(m+\lambda_\varphi)(\varphi(t)+\psi) + \int_0^t ML_f(s)\,\|y(s)-x(s)\|\,ds + \\ &+ \sum_{i=1}^{k} M\rho_i\|y(t_i^-) - x(t_i^-)\|.\end{aligned}$$

Denote $\rho = \max\{\rho_1, \rho_2, \ldots, \rho_m\}$. By Lemma 2.3, we obtain

$$\begin{aligned}\|y(t) - x(t)\| &\leq M\varepsilon(m+\lambda_\varphi)(\varphi(t)+\psi)(1+M\rho)^k \exp\left(M\int_0^t L_f(s)ds\right) \leq \\ &\leq c_{f,\mathbb{M},\varphi}\varepsilon(\varphi(t)+\psi), \quad t \in (t_k, t_{k+1}],\end{aligned}$$

where

$$c_{f,\mathbb{M},\varphi} := M(m+\lambda_\varphi)\,(1+M\rho)^m \exp\left(M\int_0^T L_f(s)ds\right) > 0. \tag{4.1}$$

Thus, equation (1.1) is Ulam-Hyers-Rassias stable with respect to (φ, ψ). The proof is complete. □

One can proceed as in the proof of Theorem 4.2 to show the following results.

Corollary 4.3. *Under assumptions* (H1)–(H3) *equation* (1.1) *is generalized Ulam-Hyers-Rassias stable with respect to* (φ, ψ).

Corollary 4.4. *Under assumptions* (H1) *and* (H2) *equation* (1.1) *is Ulam-Hyers stable.*

4.2. ULAM-HYERS-RASSIAS STABILITY RESULTS ON $J = \mathbb{R}^+$

Now, we discuss the following impulsive equation (1.1) on the unbounded interval $J = \mathbb{R}^+$. To achieve our aim, we suppose that the linear equation $x' = Ax$ is stable, i.e., we need a restriction $\omega < 0$ in this section. Then we state the following assumptions:

(H4) $f \in C(\mathbb{R}^+ \times X, X)$ and there exists a function $L_f \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\|f(t,u) - f(t,v)\| \le L_f(t)\|u - v\|$$

for each $t \in \mathbb{R}^+$ and all $u, v \in X$. Moreover, we suppose

$$\int_0^t L_f(s)ds \le \omega_f t + \Omega_f$$

for any $t \ge 0$ and some $\omega_f \ge 0$, $\Omega_f \ge 0$ satisfying $N\omega_f + \omega < 0$.

(H5) I_k: $X \to X$ and there exist constants $\rho_k > 0$ such that

$$\|I_k(u) - I_k(v)\| \le \rho_k\|u - v\|, \quad k \in \mathbb{M}$$

for each $t \in \mathbb{R}^+$ and all $u, v \in X$. Moreover, we assume

$$\rho := \sup_{k\in\mathbb{M}} \prod_{i=1}^{k}(1 + N\rho_i) < \infty,$$

which means some stability for impulsive conditions when $\mathbb{M} = \mathbb{N}$.

(H6) There exists a constant $\lambda_\varphi > 0$ and a function $\varphi \in PC(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\int_0^t e^{\omega(t-s)+N\omega_f t}\varphi(s)ds \le \lambda_\varphi\varphi(t) \quad \text{for each} \quad t \in \mathbb{R}^+.$$

(H7) Set

$$M_1 := \sup_{k\in\mathbb{M}} \sum_{i=1}^{k} e^{\omega(t_k - t_i)+N\omega_f t_k},$$

while for $\mathbb{M} = \mathbb{N}$ we suppose that $M_1 < \infty$.

Under the above assumptions, we arrive at the following result.

Theorem 4.5. *Assume* (H4)–(H7) *are satisfied. Then equation* (1.1) *is Ulam-Hyers--Rassias stable with respect to* (φ, ψ).

Proof. Let $y \in PC(\mathbb{R}^+, X) \cap C(\mathbb{R}^{+'}, D(A)) \cap C^1(\mathbb{R}^{+'}, X)$ be a solution of the inequality (3.3). Let x be the unique mild solution of the impulsive Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in J', \\ \Delta x(t_k) = I_k(x(t_k^-)), & k \in \mathbb{M}, \\ x(0) = y(0). \end{cases}$$

Like in (3.4), by differential inequality (3.3), for each $t \in (t_k, t_{k+1}]$, $k \in \mathbb{M}_0$ we have

$$\left\| y(t) - T(t)y(0) - \int_0^t T(t-s)f(s, y(s))ds - \sum_{i=1}^k T(t-t_i)I_i(y(t_i^-)) \right\| \leq$$

$$\leq \sum_{i=1}^k \|T(t-t_i)\|_{B(X)} \|g_i\| + \int_0^t \|T(t-s)\|_{B(X)} \|g(s)\| ds \leq$$

$$\leq N \left(\sum_{i=1}^k e^{\omega(t-t_i)} \varepsilon\psi + \int_0^t e^{\omega(t-s)} \varepsilon\varphi(s) ds \right).$$

Hence for each $t \in (t_k, t_{k+1}]$, it follows that

$$\begin{aligned} \|y(t) - x(t)\| \leq{}& N \left(\sum_{i=1}^k e^{\omega(t-t_i)} \varepsilon\psi + \int_0^t e^{\omega(t-s)} \varepsilon\varphi(s) ds \right) + \\ &+ \int_0^t N e^{\omega(t-s)} L_f(s) \|y(s) - x(s)\| ds + \\ &+ \sum_{i=1}^k N\rho_i e^{\omega(t-t_i)} \|y(t_i^-) - x(t_i^-)\|. \end{aligned}$$

Set $\bar{y}(t) := e^{-\omega t} y(t)$ and $\bar{x}(t) := e^{-\omega t} x(t)$, then we derive

$$\begin{aligned} \|\bar{y}(t) - \bar{x}(t)\| \leq{}& N\varepsilon \left(\sum_{i=1}^k e^{-\omega t_i} \psi + \int_0^t e^{-\omega s} \varphi(s) ds \right) + \\ &+ \int_0^t N L_f(s) \|\bar{y}(s) - \bar{x}(s)\| ds + \sum_{i=1}^k N\rho_i \|\bar{y}(t_i^-) - \bar{x}(t_i^-)\|. \end{aligned}$$

Since

$$a(t) := N\varepsilon \left(\sum_{i=1}^k e^{-\omega t_i} \psi + \int_0^t e^{-\omega s} \varphi(s) ds \right), \quad t \in (t_k, t_{k+1}],$$

is nondecreasing (note that $\omega < 0$) and $a \in PC(\mathbb{R}^+, \mathbb{R}^+)$, by Remark 2.4 (i), we obtain

$$\|\bar{y}(t) - \bar{x}(t)\| \le N\rho\varepsilon \left(\sum_{i=1}^{k} e^{-\omega t_i}\psi + \int_0^t e^{-\omega s}\varphi(s)ds \right) \exp\left(N \int_0^t L_f(s)ds \right),$$

which gives

$$\begin{aligned}
\|y(t) - x(t)\| &\le N\rho\varepsilon \left(\sum_{i=1}^{k} e^{\omega(t-t_i)}\psi + \int_0^t e^{\omega(t-s)}\varphi(s)ds \right) \exp\left(N \int_0^t L_f(s)ds \right) \le \\
&\le N\rho\varepsilon \left(\sum_{i=1}^{k} e^{\omega(t-t_i)+N(\omega_f t+\Omega_f)}\psi + \int_0^t e^{\omega(t-s)+N(\omega_f t+\Omega_f)}\varphi(s)ds \right) \le \\
&\le N\rho\varepsilon e^{N\Omega_f} (M_1 + \lambda_\varphi) (\varphi(t) + \psi) = \\
&= c_{f,\mathbb{M},\varphi}\varepsilon(\varphi(t) + \psi), \quad t \ge 0,
\end{aligned}$$

where

$$c_{f,\mathbb{M},\varphi} := N\rho e^{N\Omega_f} (M_1 + \lambda_\varphi) > 0. \tag{4.2}$$

Thus, equation (1.1) is Ulam-Hyers-Rassias stable with respect to (φ, ψ). The proof is complete. □

One can proceed as in the proof of Theorem 4.5 to show the following results.

Corollary 4.6. *Under assumptions* (H4)–(H7) *the equation* (1.1) *is generalized Ulam-Hyers-Rassias stable with respect to* (φ, ψ).

Corollary 4.7. *Under assumptions* (H4), (H5) *and* (H7) *equation* (1.1) *is Ulam--Hyers stable.*

5. EXAMPLES

In this section, we give two examples to illustrate our abstract results above. We consider one-dimensional diffusion processes with sudden changes of states. These examples can explain either the evolution of the temperature of the rod or the chemical concentration of the substance. Of course, more general examples could be presented but we think that our examples are suitable for demonstrating our theory.

Example 5.1. Consider a distribution of the temperature $x(t, y)$ on the rod with a sudden change of temperature at time $t = \frac{1}{3}$. The corresponding impulsive partial differential equation is given by

$$\begin{cases} \frac{\partial}{\partial t}x(t, y) = -\frac{\partial^2}{\partial y^2}x(t, y), & y \in (0, 1),\ t \in [0, \frac{1}{3}) \cup (\frac{1}{3}, 1], \\ \frac{\partial}{\partial y}x(t, 0) = \frac{\partial}{\partial y}x(t, 1) = 0, & t \in [0, \frac{1}{3}) \cup (\frac{1}{3}, 1], \\ \Delta x(\frac{1}{3}, y) = \lambda x(\frac{1}{3}^-, y), & \lambda \in \mathbb{R},\ y \in (0, 1). \end{cases} \tag{5.1}$$

Hence $J = [0,1]$, $m = 1$ and $t_1 = \frac{1}{3}$. Let $X = L^2(0,1)$. Define $Ax = -\frac{\partial^2}{\partial y^2}x$ for $x \in D(A)$ with

$$D(A) = \Big\{x \in X : \frac{\partial x}{\partial y}, \frac{\partial^2 x}{\partial y^2} \in X,\ x(0) = x(1) = 0\Big\}.$$

Then, A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ in X. Moreover, $\|T(t)\|_{B(X)} \leq 1 = M$ for all $t \geq 0$.

Denote $x(\cdot)(y) = x(\cdot, y)$, $f(\cdot, x)(y) = 0$ and $I_1(x(\frac{1}{3}^-))(y) = \lambda x(\frac{1}{3}^-, y)$, then the problem (5.1) can be abstracted into

$$\begin{cases} x'(t) = Ax(t), & t \in [0, \frac{1}{3}) \cup (\frac{1}{3}, 1], \\ \Delta x(\frac{1}{3}) = I_1(x(\frac{1}{3}^-)) = \lambda x(\frac{1}{3}^-). \end{cases} \tag{5.2}$$

Clearly, (H1) and (H2) hold. Set $\varphi(t) = 3t^2$ and $\psi = 1$. Then (H3) holds if $\lambda_\varphi = \frac{1}{3}$. Thus, by Theorem 4.2, equation (5.1) is Ulam-Hyers-Rassias stable with respect to $(3t^2, 1)$ on $[0,1]$ and $c_{f,\mathbb{M},\varphi} = \frac{4}{3}(1 + |\lambda|)$ (see (4.1)).

Example 5.2. Consider

$$\begin{cases} \frac{\partial}{\partial t}x(t,y) = (\Delta_y - 2I)x(t,y) + \sin t, & y \in \Omega,\ t > 0,\ t \neq \mathbb{N}, \\ \frac{\partial}{\partial y}x(t,y) = 0, & y \in \partial\Omega,\ t > 0,\ t \neq \mathbb{N}, \\ \Delta x(i,y) = \frac{1}{i^2}x(i^-, y), & y \in \Omega,\ i \in \mathbb{N}, \end{cases} \tag{5.3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, Δ_y is the Laplace operator in $\mathbb{R}^2$, and $\partial\Omega \in C^2$. Note now $J = \mathbb{R}^+$, $t_i = i$ and $\mathbb{M} = \mathbb{N}$. Here we consider infinitely many impulses on the infinite time interval $\mathbb{R}^+$.

Let $X = L_2(\Omega)$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Define $Ax = (\Delta_y - 2I)x$, $x \in D(A)$. By Theorem 2.5 of [21], A is just the infinitesimal generator of a contraction C_0-semigroup in $L_2(\Omega)$, that is, $\|T(t)\|_{B(X)} \leq e^{-2t}$ for $t \geq 0$, so $N = 1$ and $\omega = -2 < 0$.

Denote $x(\cdot)(y) = x(\cdot, y)$, $f(t,x)(y) = \sin t$ and $I_i(x(i^-)) = \frac{1}{i^2}x(i^-)$, then problem (5.3) can be abstracted into

$$\begin{cases} x'(t) = Ax(t) + \sin t, & t \in [0, \infty) \setminus \mathbb{N}, \\ \Delta x(i) = I_i(x(i^-)) = \frac{1}{i^2}x(i^-). \end{cases} \tag{5.4}$$

Clearly, (H4) and (H5) hold with $\omega_f = 0$ and $\rho = \prod\limits_{i=1}^{\infty}(1 + \frac{1}{i^2}) \leq e^{\sum_{i=1}^{\infty}\frac{1}{i^2}} = e^{\frac{\pi^2}{6}}$. Next, $\Omega_f = 0$. Set $\varphi(t) = e^t$ and $\psi = 1$. Then (H6) holds if $\lambda_\varphi = \frac{1}{3}$. Moreover, (H7) is satisfied with $M_1 = \frac{e^2}{e^2-1}$.

Thus, applying Theorem 4.5, equation (5.3) is Ulam-Hyers-Rassias stable with respect to $(e^t, 1)$ on $\mathbb{R}^+$ with $c_{f,\mathbb{M},\varphi} = e^{\frac{\pi^2}{6}}\left(\frac{1}{3} + \frac{e^2}{e^2-1}\right)$ (see (4.2)).

6. EXTENSIONS FOR THE CASE WITH INFINITE IMPULSES

For the case $\mathbb{M} = \mathbb{N}$, it is natural to consider changing inequalities (3.2) and (3.3) into the following ones:

$$\begin{cases} \|y'(t) - A(t)y - f(t, y(t))\| \leq \varphi(t), & t \in J', \\ \|\Delta y(t_k) - I_k(y(t_k^-))\| \leq \psi_k, & k \in \mathbb{N}, \end{cases} \tag{6.1}$$

and

$$\begin{cases} \|y'(t) - Ay(t) - f(t, y(t))\| \leq \varepsilon\varphi(t), & t \in J', \\ \|\Delta y(t_k) - I_k(y(t_k^-))\| \leq \varepsilon\psi_k, & k \in \mathbb{N} \end{cases} \tag{6.2}$$

for $\varphi(\cdot)$ like above but $\psi := \{\psi_k\}_{k\in\mathbb{N}}$ is now a nonconstant sequence of nonnegative numbers $\psi_k \geq 0$ for all $k \in \mathbb{N}$. Then the inequalities of Definitions 3.3 and 3.4 are replaced with

$$\|y(t) - x(t)\| \leq c_{f,\mathbb{M},\varphi}\varepsilon(\varphi(t) + \psi_{k+1}), \quad t \in (t_k, t_{k+1}],\ k \geq 0,$$

and

$$\|y(t) - x(t)\| \leq c_{f,\mathbb{M},\varphi}\varepsilon(\varphi(t) + \psi_{k+1}), \quad t \in (t_k, t_{k+1}],\ k \geq 0,$$

respectively. We call these as extended generalized Ulam-Hyers-Rassias stability and extended Ulam-Hyers-Rassias stability, respectively. They provide more flexibility for studying stability.

Then we state the following weaker assumptions:

(H8) $f \in C(\mathbb{R}^+ \times X, X)$ and there exists a function $L_f \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\|f(t, u) - f(t, v)\| \leq L_f(t)\|u - v\|$$

for each $t \in \mathbb{R}^+$ and all $u, v \in X$.

(H9) I_k: $X \to X$ and there exist constants $\rho_k > 0$ such that

$$\|I_k(u) - I_k(v)\| \leq \rho_k\|u - v\|, \quad k \in \mathbb{N}$$

for each $t \in \mathbb{R}^+$ and all $u, v \in X$.

(H10) There exist constants $\lambda_\varphi > 0$, $\lambda_\psi > 0$ and a function $\varphi \in PC(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\prod_{i=1}^{k}(1 + N\rho_i) \max_{t\in[t_k, t_{k+1}]} \sum_{i=1}^{k} e^{\omega(t-t_i)+N\int_0^t L_f(s)ds}\psi_i \leq \lambda_\psi\psi_{k+1} \text{ for each } k \geq 0, \tag{6.3}$$

and

$$\prod_{i=1}^{k}(1 + N\rho_i)e^{N\int_0^t L_f(s)ds} \int_0^t e^{\omega(t-s)}\varphi(s)ds \leq \lambda_\varphi\varphi(t) \text{ for each } t \in (t_k, t_{k+1}],\ k \geq 0. \tag{6.4}$$

Under the above assumptions, we can repeat the proof of Theorem 4.5 to get the following result.

Theorem 6.1. *Assume* (H8)–(H10) *are satisfied. Then equation* (1.1) *with* $J = \mathbb{R}^+$ *and* $\mathbb{M} = \mathbb{N}$ *is extended Ulam-Hyers-Rassias stable with respect to* (φ, ψ).

Proof. Let $y \in PC(\mathbb{R}^+, X) \cap C(\mathbb{R}^{+'}, D(A)) \cap C^1(\mathbb{R}^{+'}, X)$ be a solution of inequality (6.2), and let x be the unique mild solution of the impulsive Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), & t \in \mathbb{R}', \\ \Delta x(t_k) = I_k(x(t_k^-)), & k \in \mathbb{N}, \\ x(0) = y(0). \end{cases}$$

Following the proof of Theorem 4.5, for $t \in (t_k, t_{k+1}]$, we obtain

$$\begin{aligned} &\|y(t) - x(t)\| \leq \\ &\leq N\varepsilon \prod_{i=1}^{k}(1 + N\rho_i) \left(\sum_{i=1}^{k} e^{\omega(t-t_i)}\psi_i + \int_0^t e^{\omega(t-s)}\varphi(s)ds \right) \exp\left(N \int_0^t L_f(s)ds \right) \leq \\ &\leq N\varepsilon \left(\lambda_\psi \psi_{k+1} + \lambda_\varphi \varphi(t)\right) = \\ &= c_{f,\mathbb{M},\varphi}\varepsilon(\varphi(t) + \psi_{k+1}), \end{aligned}$$

where

$$c_{f,\mathbb{M},\varphi} := N\left(\lambda_\psi + \lambda_\varphi\right) > 0. \tag{6.5}$$

Thus, equation (1.1) is extended Ulam-Hyers-Rassias stable with respect to (φ, ψ). The proof is complete. □

One can proceed as in the proof of Theorem 6.1 to show the following results.

Corollary 6.2. *Under assumptions* (H8)–(H10) *equation* (1.1) *is extended generalized Ulam-Hyers-Rassias stable with respect to* (φ, ψ).

Finally, we give some explaination of our assumptions (H4)–(H7) and (H8)–(H10).

Remark 6.3. Certainly (H4)–(H7) imply (H8)–(H10), but (H4)–(H7) are easier to verify than (H8)–(H10). On the other hand, assuming (H8), (H9) and using (6.3), (6.4), constants λ_ψ, λ_φ, sequence ψ and function φ can be constructed step by step on $(t_k, t_{k+1}]$, $k \geq 0$ so that satisfy (H10).

A very rough construct is as follows: first we take $\lambda_\psi = \lambda_\varphi = 1$. Then starting with some ψ_1, the rest terms of sequence ψ are done inductively by (6.3). Now we construct $\varphi(t)$. To begin, we take any $\varphi \in C([0, t_1], \mathbb{R}^+)$. Then assume by induction that we already have $\varphi \in C([0,t_k],\mathbb{R}^+)$ satisfying (6.4). We look for $\varphi(t) = e^{\Omega_k t}$ for $t \in (t_k, t_{k+1}]$ when $\Omega_k \in \mathbb{R}$ must be specified. For $t \in (t_k, t_{k+1}]$, we compute

$$\prod_{i=1}^{k}(1+N\rho_i)e^{N\int_0^t L_f(s)ds}\int_0^t e^{\omega(t-s)}\varphi(s)ds =$$

$$= \prod_{i=1}^{k}(1+N\rho_i)e^{N\int_0^t L_f(s)ds}\int_0^{t_k} e^{\omega(t-s)}\varphi(s)ds +$$

$$+ \prod_{i=1}^{k}(1+N\rho_i)e^{N\int_0^t L_f(s)ds}\int_{t_k}^{t} e^{\omega(t-s)}\varphi(s)ds \leq$$

$$\leq A_k + B_k\int_{t_k}^{t} e^{\omega(t-s)}\varphi(s)ds$$

for

$$A_k = \max_{t\in[t_k,t_{k+1}]}\prod_{i=1}^{k}(1+N\rho_i)e^{N\int_0^t L_f(s)ds}\int_0^{t_k} e^{\omega(t-s)}\varphi(s)ds,$$

$$B_k = \max_{t\in[t_k,t_{k+1}]}\prod_{i=1}^{k}(1+N\rho_i)e^{N\int_0^t L_f(s)ds}.$$

If $\Omega_k > \omega$, then we have

$$A_k + B_k\int_{t_k}^{t} e^{\omega(t-s)}\varphi(s)ds = A_k + B_k\int_{t_k}^{t} e^{\omega(t-s)+\Omega_k s}ds \leq$$

$$\leq A_k + \frac{B_k}{\Omega_k-\omega}e^{\Omega_k t} \leq e^{\Omega_k t}$$

for any $t \in (t_k, t_{k+1}]$, where we choose

$$\Omega_k := \max\left\{\omega + 2B_k, \frac{\ln(2A_k)}{t_k}\right\} + 1.$$

Indeed, then we have $\Omega_k > \omega$, $1 - \frac{B_k}{\Omega_k-\omega} > \frac{1}{2}$ and so for any $t \geq t_k$, we derive

$$e^{\Omega_k t}\left(1 - \frac{B_k}{\Omega_k-\omega}\right) > \frac{1}{2}e^{\Omega_k t_k} > A_k,$$

which implies the desired

$$A_k + \frac{B_k}{\Omega_k-\omega}e^{\Omega_k t}ds \leq e^{\Omega_k t}, \quad t \in (t_k, t_{k+1}].$$

Summarizing, we see that (H10) is reasonable, but obtained formulas are very awkward in general. For this reason we derive more simple and applicable results in Section 4.

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