TRANSNAV
http://www.transnav.eu
the International Journal on Marine Navigation and Safety of Sea Transportation
Volume 13
Number 3
September 2019
DOI: 10.12716/1001.13.03.08

Computation of Positive Realizations for Descriptor Linear Discrete-time Systems

T. Kaczorek
Białystok University of Technology, Białystok, Poland

ABSTRACT: A new method for computation of positive realizations of given transfer matrices of descriptor linear discrete-time linear systems is proposed. Necessary and sufficient conditions for the existence of positive realizations of transfer matrices are given. A procedure for computation of the positive realizations for descriptor discrete-time linear systems is proposed and illustrated by examples.

1 INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in the monographs [2, 14]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc. [2, 14].

The determination of the matrices *A*, *B*, *C*, *D* of the state equations of linear systems for given their transfer matrices is called the realization problem. The realization problem is a classical problem of analysis of linear systems and has been considered in many books and papers [4-6, 12, 13, 23, 25]. A tutorial on the positive realization problem has been given in the paper [1] and in the books [2, 14]. The positive minimal realization problem for linear systems without and with delays has been analyzed in [3, 7, 9, 10, 14-18, 21, 22, 24]. The existence and determination of the set of Metzler matrices for given stable polynomials have been considered in [11]. The realization problem for positive 2D hybrid systems has been addressed in [20]. For fractional linear systems the realization problem has been considered in [4, 19, 23, 25]. A method for computation of positive realizations of descriptor continuous-time linear systems has been proposed in [8].

In this paper a new method for determination of positive realizations of descriptor linear discrete-time systems is proposed.

The paper is organized as follows. In section 2 some definitions and theorems concerning the positive discrete-time linear systems are recalled. A new method for computation of positive realizations for single-input single-output linear systems is proposed in section 3 and for multi-input multi-output systems in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, Z_+ - the set of nonnegative integers, I_n - the $n\times n$ identity matrix.

2 PRELIMINARIES

Consider the discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i, \tag{2.1a}$$

$$y_i = Cx_i + Du_i, \tag{2.1b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Definition 2.1. [2, 14] The system (2.1) is called (internally) positive if $x_i \in \Re_+^n$ and $y_i \in \Re_+^p$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 2.1. [2, 14] The system (2.1) is positive if and only if

$$A \in \Re_+^{n\times n}, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}. \tag{2.2}$$

The transfer matrix of the system (2.1) is given by

$$T(z) = C[I_n z - A]^{-1} B + D. \tag{2.3}$$

The transfer matrix is called proper if

$$\lim_{z\to\infty} T(z) = D \in \Re_+^{p\times m} \tag{2.4}$$

and it is called strictly proper if $D = 0$.

Definition 2.2. [1, 25] The matrices (2.2) are called a positive realization of $T(z)$ if they satisfy the equality (2.3).

Definition 2.3. [1, 25] The matrices (2.2) are called asymptotically stable realization of (2.3) if the matrix $A \in \Re_+^{n\times n}$ is an asymptotically stable matrix (Schur matrix).

Theorem 2.2. [1, 25] The positive realization (2.2) is asymptotically stable if and only if all coefficients of the polynomial

$$p_A(z) = \det[I_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0 \tag{2.5}$$

are positive, i.e. $a_i > 0$ for $i = 0,1,\dots,n-1$.

The positive realization problem for the standard system can be stated as follows. Given a proper transfer matrix $T(z)$ find its positive realization (2.2).

Theorem 2.3. [25] If (2.2) is a positive realization of (2.3) then the matrices

$$\overline{A} = PAP^{-1}, \quad \overline{B} = PB, \quad \overline{C} = CP^{-1}, \quad \overline{D} = D \tag{2.6}$$

are also a positive realization of (2.3) if and only if the matrix $P \in \Re_+^{n\times n}$ is a monomial matrix (in each row and in each column only one entry is positive and the remaining entries are zero).

Proof. Proof follows immediately from the fact that $P^{-1} \in \Re_+^{n\times n}$ if and only if P is a monomial matrix. □

Theorem 2.4. The polynomial $p_n(z)$ with zeros z_k, $\operatorname{Re} z_k > 0$, $k = 1,\dots,n$ has the form

$$p_n(z) = z^n - a_{n-1}z^{n-1} + a_{n-2}z^{n-2} - a_{n-3}z^{n-3} + \dots + (-1)^n a_0 \tag{2.7}$$

and its real coefficients a_k satisfy the condition

$$a_k > 0 \text{ for } k = 0,1,\dots,n-1. \tag{2.8}$$

Proof. Proof will be accomplished by induction. The hypothesis is true for $n = 1$ and $n = 2$ since

$$p_1(z) = z - a_0, \quad a_0 > 0$$

and

$$p_2(z) = (z - z_1)(z - z_2) = (z - \alpha + j\beta)(z - \alpha - j\beta) = z^2 - 2\alpha z + \alpha^2 + \beta^2.$$

Assuming that the hypothesis is true for k we shall show that it is also valid for $k+1$:

$$\begin{aligned} p_{k+1}(z) &= p_k(z)(z-\alpha) = \\ &(z^k - a_{k-1}z^{k-1} + a_{k-2}z^{k-2} - \dots + (-1)^k a_0)(z-\alpha) \\ &= z^{k+1} - (a_{k-1}+\alpha)z^k + (a_{k-2}+\alpha)z^{k-1} - \dots + (-1)^{k+1}a_0\alpha. \end{aligned}$$

Therefore, the hypothesis is true for any k. The proof for a pair of complex conjugate zeros is similar. □

3 COMPUTATION OF POSITIVE REALIZATIONS OF DESCRIPTOR SINGLE-INPUT SINGLE-OUTPUT SYSTEMS

Consider the descriptor descriptor-time linear system

$$Ex_{i+1} = Ax_i + Bu_i, \tag{3.1a}$$

$$y_i = Cx_i, \tag{3.1b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

It is assumed that $\det E = 0$ and the pencil of (E, A) is regular, i.e.

$\det[Ez - A] \neq 0$ for some $z \in \mathbf{C}$ (the field of complex numbers). (3.2)

Definition 3.1. The descriptor system (3.1) is called (internally) positive if $x_i \in \Re_+^n$, $y_i \in \Re_+^p$, $i \in Z_+$ for any consistent initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i = 0,1,\dots,q$.

The transfer matrix of the system (3.1)

$$T(z) = C[Ez - A]^{-1} B \in \Re^{p\times m}(z) \tag{3.3}$$

can be decomposed in the polynomial part $P(z)$ and strictly proper part $T_{sp}(z)$, i.e.

$$T(z) = P(z) + T_{sp}(z), \tag{3.4a}$$

where

$$P(z) = P_0 + P_1 z + ... + P_q z^q \in \Re^{p\times m}[z] \tag{3.4b}$$

and

$$T_{sp}(z) = \overline{C}[I_n z - \overline{A}]^{-1}\overline{B}. \tag{3.4c}$$

First the new method for computation of a positive realization of given transfer function will be presented.

Theorem 3.1. There exists the positive realization

$$\overline{A} = \begin{bmatrix} z_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & z_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 1 & z_n \end{bmatrix},\quad \overline{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

$$\overline{C} = [0 \ \cdots \ 0 \ 1] \tag{3.5}$$

of the transfer function

$$T_{sp}(s) = \frac{\overline{m}_{n-1} z^{n-1} + ... + \overline{m}_1 z + \overline{m}_0}{z^n + d_{n-1} z^{n-1} + ... + d_1 z + d_0} \tag{3.6}$$

if and only if

$$\overline{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & z_1 & z_1 z_2 & \cdots & z_1 z_2 ... z_{n-1} \\ 0 & 1 & z_1 + z_2 & \cdots & z_1 + z_2 + ... + z_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}^{-1} \begin{bmatrix} \overline{m}_0 \\ \overline{m}_1 \\ \vdots \\ \overline{m}_{n-1} \end{bmatrix} \in \Re_+^n, \tag{3.7}$$

where z_k, $k = 1,...,n$ are the zeros of the denominator

$$d(z) = z^n + d_{n-1} z^{n-1} + ... + d_1 z + d_0 = (z + z_1)(z + z_2)...(z + z_n) \tag{3.8}$$

which are nonnegative, i.e. $z_k \ge 0$, $k = 1,...,n$.

Proof. The proof is given in [6].

The realization is asymptotically stable if and only if $0 \le z_k < 1$ for $k = 1,...,n$.

Remark 3.1. The positive realization (3.5) is asymptotically stable if and only if all coefficients of the polynomial

$$d(z+1) = (z+1)^n + d_{n-1}(z+1)^{n-1} + ... + d_1(z+1) + d_0 = z^n + \overline{d}_{n-1} z^{n-1} + ... + \overline{d}_1 z + \overline{d}_0$$

are positive, i.e. $\overline{d}_k > 0$, $k = 0,1,...,n-1$ [6].

Theorem 3.1 and Remark 3.1 can be easily extended to the multi-input multi-output linear systems [6].

Theorem 3.2. There always exists the positive asymptotically stable realization (3.5) of the transfer function

$$T_{sp}(s) = \frac{\overline{m}_0}{z^n + d_{n-1} z^{n-1} + ... + d_1 z + d_0}. \tag{3.9}$$

if and only if $\overline{m}_0 > 0$ and the zeros of (3.8) satisfy the condition $0 \le z_k < 1$, $k = 1,...,n$.

Proof. From (3.7) it follows that if $\overline{m}_k = 0$ for $k = 1,...,n$ then $b_1 = \overline{m}_0$, $b_k = 0$, $k = 2,...,n$ and $\overline{B} = [\overline{m}_0 \ 0 \ \cdots \ 0]^T \in \Re_+^n$. The positive realization is asymptotically stable if and only if $0 \le z_k < 1$ for $k = 1,...,n$. □

Remark 3.2. The Theorems 3.1 and 3.2 are also valid if the matrix $\overline{A}$ has multiple eigenvalues.

Example 3.1. Compute the positive realization (3.5) of the transfer function

$$T_{sp}(s) = \frac{\overline{m}_2 z^2 + \overline{m}_1 z + \overline{m}_0}{z^3 + d_2 z^2 + d_1 z + d_0} = \frac{z^2 + 2z + 1}{z^3 - 0.6z^2 + 0.11z - 0.006}. \tag{3.10}$$

The denominator

$$d(z) = z^3 - 0.6z^2 + 0.11z - 0.006 = (z - 0.1)(z - 0.2)(z - 0.3) \tag{3.11}$$

has the real positive zeros $z_1 = 0.1$, $z_2 = 0.2$, $z_3 = 0.3$ and the matrix $\overline{A}$ is the Schur matrix of the form

$$\overline{A} = \begin{bmatrix} z_1 & 0 & 0 \\ 1 & z_2 & 0 \\ 0 & 1 & z_3 \end{bmatrix} = \begin{bmatrix} 0.1 & 0 & 0 \\ 1 & 0.2 & 0 \\ 0 & 1 & 0.3 \end{bmatrix}. \tag{3.12}$$

Note that the polynomial (3.11) satisfies the conditions of Theorem 2.4.

Using (3.7) and (3.11) we obtain

$$\overline{B} = \begin{bmatrix} 1 & z_1 & z_1 z_2 \\ 0 & 1 & z_1 + z_2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \overline{m}_0 \\ \overline{m}_1 \\ \overline{m}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0.1 & 0.02 \\ 0 & 1 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.81 \\ 1.7 \\ 1 \end{bmatrix}. \tag{3.13}$$

In this case the matrix $\overline{C}$ has the form

$$\overline{C} = [0 \ \ 0 \ \ 1]. \tag{3.14}$$

The positive asymptotically stable realization of (3.10) is given by (3.12) – (3.14).

It is easy to check that the matrices

$$\hat{A} = \begin{bmatrix} 0.1 & 1 & 0 \\ 0 & 0.2 & 1 \\ 0 & 0 & 0.3 \end{bmatrix},\quad \hat{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\quad \hat{C} = [0.81 \ \ 1.7 \ \ 1] \tag{3.15}$$

are also the positive asymptotically stable realization of the transfer function (3.10).

Remark 3.3. If the matrices (3.5) are positive realization of (3.6) then the matrices

$$\hat{A}=\begin{bmatrix} z_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & z_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z_{n-1} & 1 \\ 0 & 0 & 0 & \cdots & 0 & z_n \end{bmatrix},\quad \hat{B}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\quad \hat{C}=[b_1 \quad b_2 \quad \cdots \quad b_n] \tag{3.16}$$

are also the positive realization of (3.6).

Theorem 3.3. Let the matrices (3.5) be a positive realization of the strictly proper transfer function (3.6) then the matrices

$$E=\begin{bmatrix} I_n & 0 \\ 0 & N \end{bmatrix}\in\Re_+^{\bar{n}\times\bar{n}},\quad N=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}\in\Re^{(q+1)\times(q+1)}$$

$$A=\begin{bmatrix} \bar{A} & \bar{B} & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}\in\Re_+^{\bar{n}\times\bar{n}},\quad B=\begin{bmatrix} 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\in\Re^{\bar{n}\times 1}, \tag{3.17}$$

$$C=[\bar{C} \quad P_0 \quad P_1 \quad \cdots \quad P_q]\in\Re_+^{1\times\bar{n}},\quad \bar{n}=n+q+1$$

are a positive realization of the transfer function (3.3) if and only if

$$P_k\in\Re_+ \text{ for } k=0,1,\ldots,q \text{ and } z_k\geq 0 \text{ for } k=1,\ldots,n \tag{3.18}$$

Proof. Using (3.17) it is easy to verify that

$$C[Ez-A]^{-1}B=[\bar{C} \quad P_0 \quad P_1 \quad \cdots \quad P_q]\begin{bmatrix} I_nz-\bar{A} & -\bar{B} & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & z & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z & -1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$
$$=[\bar{C} \quad P_0 \quad P_1 \quad \cdots \quad P_q]\begin{bmatrix} [I_nz-\bar{A}]^{-1}\bar{B} \\ 1 \\ z \\ \vdots \\ z^q \end{bmatrix}=\bar{C}[I_nz-\bar{A}]^{-1}\bar{B}+P_0+P_1z+\ldots+P_qz^q. \tag{3.19}$$

Therefore, the matrices (3.17) are the positive realization of the transfer function (3.3). □

Remark 3.4. Note that the positive realization (3.17) for the descriptor linear systems has the matrix $-B\in\Re_+^{\bar{n}}$.

Remark 3.5. The positive realization (3.17) is asymptotically stable if and only if the matrix $\bar{A}\in\Re_+^{n\times n}$ is Schur matrix with only nonnegative real parts of eigenvalues.

Example 3.2. Compute the positive realization (3.17) of the transfer function

$$T(z)=\frac{z^4+1.4z^3-0.09z^2+2.16z+0.88}{z^3-0.6z^2+0.11z-0.006}. \tag{3.20}$$

The transfer function (3.20) can be decomposed as follows

$$T(z)=P(z)+T_{sp}(z), \tag{3.21}$$

where

$$P(z)=P_0+P_1z=2+z,$$
$$T_{sp}(z)=\frac{z^2+2z+1}{z^3-0.6z^2+0.11z-0.006}. \tag{3.22}$$

The positive realization of $T_{sp}(z)$ given by (3.22) has been found in Example 3.1 and it is given by (3.12) – (3.14).

The conditions of Theorem 3.3 for the existence of the positive realization of (3.20) are satisfied since in this case $P_0=2$ and $P_1=1$.

Therefore, by Theorem 3.3 the desired positive realization of the transfer function (3.20) has the form

$$E=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},\quad A=\begin{bmatrix} 0.1 & 0 & 0 & 0.81 & 0 \\ 1 & 0.2 & 0 & 1.7 & 0 \\ 0 & 1 & 0.3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix},\quad C=[0 \quad 0 \quad 1 \quad 1 \quad 2]. \tag{3.20}$$

4 COMPUTATION OF POSITIVE REALIZATIONS OF DESCRIPTOR MIMO SYSTEMS

In this section the method presented in section 3 will be extended to multi-input multi-output (MIMO) linear discrete-time systems.

The strictly proper transfer matrix (3.4c) can be written in the form with common least row denominator

$$T_{sp}(s)=\begin{bmatrix} \frac{\bar{m}_{11}(z)}{\bar{d}_1(z)} & \cdots & \frac{\bar{m}_{1m}(z)}{\bar{d}_1(z)} \\ \vdots & \ddots & \vdots \\ \frac{\bar{m}_{p1}(z)}{\bar{d}_p(z)} & \cdots & \frac{\bar{m}_{pm}(z)}{\bar{d}_p(z)} \end{bmatrix},\quad \bar{m}_{ik}(z)=\bar{m}_{ikn-1}z^{n-1}+\ldots+\bar{m}_{ik1}z+\bar{m}_{ik0},$$
$$\bar{d}_i(z)=z^n+\bar{d}_{in-1}z^{n-1}+\ldots+\bar{d}_{i1}z+\bar{d}_{i0},\ i=1,\ldots,p;\ k=1,\ldots,m \tag{4.1}$$

or with common least column denominator

$$T_{sp}(z)=\begin{bmatrix}\frac{\hat{m}_{11}(z)}{\hat{d}_1(z)} & \cdots & \frac{\hat{m}_{1m}(z)}{\hat{d}_m(z)}\\ \vdots & \ddots & \vdots \\ \frac{\hat{m}_{p1}(z)}{\hat{d}_1(z)} & \cdots & \frac{\hat{m}_{pm}(z)}{\hat{d}_m(z)}\end{bmatrix},\ \hat{m}_{ik}(z)=\hat{m}_{ikn-1}z^{n-1}+...+\hat{m}_{ik1}z+\hat{m}_{ik0},$$
$$\hat{d}_k(z)=z^n+\hat{d}_{kn-1}z^{n-1}+...+\hat{d}_{k1}z+\hat{d}_{k0},\ i=1,...,p;\ k=1,...,m. \quad (4.2)$$

Further we shall consider in details only the first case (4.1) since the considerations for (4.2) are similar (dual).

The matrices $\overline{A}$ and $\overline{B}$ of the desired realization have the forms

$$\overline{A}=\text{blockdiag}[\overline{A}_1 \ \cdots \ \overline{A}_p], \quad (4.3a)$$

where

$$\overline{A}_i=\begin{bmatrix} z_{i1} & 0 & 0 & \cdots & 0 & 0\\ 1 & z_{i2} & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & z_{in-1} & 0\\ 0 & 0 & 0 & \cdots & 1 & z_{in}\end{bmatrix}\in\Re_+^{n\times n},\ z_{ik}\ge 0$$

for $i=1,...,p$, $k=1,...,n$ (4.3b)

and

$$\overline{B}=\begin{bmatrix}\overline{B}_{11} & \cdots & \overline{B}_{1m}\\ \vdots & \ddots & \vdots\\ \overline{B}_{p1} & \cdots & \overline{B}_{pm}\end{bmatrix}\in\Re_+^{np\times m},\ \overline{B}_{ik}=\begin{bmatrix}b_{ik1}\\ b_{ik2}\\ \vdots\\ b_{ikn_i}\end{bmatrix},$$

$i=1,...,p$, $k=1,...,m$. (4.4)

The entries of the matrices $\overline{B}_{ik}$ are computed in the same way as of the matrix $\overline{B}$ in section 3 using the equation

$$B_i=Z_i^{-1}M_i\in\Re_+^{n_i},\ i=1,...,p, \quad (4.5a)$$

where

$$Z_i=\begin{bmatrix}1 & z_{i1} & z_{i1}z_{i2} & \cdots & z_{i1}z_{i2}...z_{in-1}\\ 0 & 1 & z_{i1}+z_{i2} & \cdots & z_{i1}+z_{i2}+...+z_{in-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix},$$

$i=1,...,p$, (4.5b)

$$M_i=\begin{bmatrix}\overline{m}_{ik0}\\ \overline{m}_{ik1}\\ \vdots\\ \overline{m}_{ikn_{i-1}}\end{bmatrix},\ i=1,...,p,\ k=1,...,m. \quad (4.5c)$$

The matrix $\overline{C}$ is given by

$$\overline{C}=\text{blockdiag}[C_1 \ \cdots \ C_p],$$
$$C_i=[0 \ \cdots \ 0 \ 1]\in\Re_+^{1\times n_i}. \quad (4.6)$$

Theorem 4.1. If the matrices (4.3), (4.4) and (4.6) are a positive realization of the strictly proper transfer matrix (4.1) then the matrices

$$\tilde{E}=\begin{bmatrix}I_n & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & I_m & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & I_m & 0\end{bmatrix}\in\Re_+^{\bar{n}\times\bar{n}},\ \tilde{A}=\begin{bmatrix}\overline{A} & \overline{B} & 0 & \cdots & 0 & 0\\ 0 & I_m & 0 & \cdots & 0 & 0\\ 0 & 0 & I_m & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & I_m\end{bmatrix}\in\Re_+^{\bar{n}\times\bar{n}}, \quad (4.7)$$
$$\tilde{B}=\begin{bmatrix}0\\ -I_m\\ 0\\ \vdots\\ 0\end{bmatrix}\in\Re^{\bar{n}\times m},\ \tilde{C}=[\overline{C} \ \ P_0 \ \ P_1 \ \cdots \ P_q]\in\Re_+^{p\times\bar{n}},\ \bar{n}=n+(q+1)m$$

are a positive realization of the transfer matrix (3.3) if and only if

$$P_k\in\Re_+^{p\times m} \text{ for } k=0,1,...,q \text{ and } z_{ik}\ge 0,$$
$$i=1,...,p,\ k=1,...,n \quad (4.8)$$

Proof. The proof is similar to the proof of Theorem 3.2.

From the above considerations we have the following procedure for computation of the positive realization (4.7) of the given transfer matrix $T(z)$.

Procedure 4.1.

Step 1. Decompose the given matrix $T(z)$ in the polynomial part (3.4b) and strictly proper part (3.4c).

Step 2. Compute the zeros z_{ij}, $i=1,...,p$, $j=1,...,n_j$ of the denominator $d_i(z)$, $i=1,...,p$ and find the matrices (4.3b), (4.3a).

Step 3. Using (4.5b) and (4.5c) compute the matrices Z_i, M_i and check the conditions (4.5a). If the conditions (4.5a) are satisfied then there exists $\overline{B}\in\Re_+^{np\times m}$ and the positive realization of $T(z)$.

The desired positive realization is given by (4.7).

Example 4.1. Compute the positive realization (4.7) of the transfer matrix

$$T(z)=\begin{bmatrix}\frac{2z^3+0.2z^2+0.66z+0.23}{z^2-0.4z+0.03}\\ \frac{3z^3+0.5z^2+0.18z+1.12}{z^2-0.5z+0.06}\end{bmatrix}. \quad (4.9)$$

Using Procedure 4.1 we obtain the following.

Step 1. The matrix (4.9) can be decomposed in the polynomial part

$$P(s)=\begin{bmatrix}2\\3\end{bmatrix}z+\begin{bmatrix}1\\2\end{bmatrix} \tag{4.10}$$

and strictly proper part

$$T_{sp}(z)=\begin{bmatrix}\dfrac{z+0.2}{z^2-0.4z+0.03}\\ \dfrac{z+1}{z^2-0.5z+0.06}\end{bmatrix}. \tag{4.11}$$

Step 2. The zeros of the first denominator

$$d_1(z)=z^2-0.4z+0.03 \tag{4.12}$$

are: $z_{11}=0.1$, $z_{12}=0.3$ and of the second denominator

$$d_2(z)=z^2-0.5z+0.06 \tag{4.13}$$

are: $z_{21}=0.2$, $z_{22}=0.3$.

Therefore, the matrix $\overline{A}$ has the form

$$\overline{A}=\begin{bmatrix}\overline{A}_1 & 0\\ 0 & \overline{A}_2\end{bmatrix}=\begin{bmatrix}0.1&0&0&0\\1&0.3&0&0\\0&0&0.2&0\\0&0&1&0.3\end{bmatrix}. \tag{4.14}$$

Step 3. In this case

$$\overline{B}=\begin{bmatrix}\overline{B}_1\\ \overline{B}_2\end{bmatrix},\ \overline{B}_1=\begin{bmatrix}b_{11}\\ b_{12}\end{bmatrix},\ \overline{B}_2=\begin{bmatrix}b_{21}\\ b_{22}\end{bmatrix} \tag{4.15a}$$

and using (4.5a) we obtain

$$\overline{B}_1=\begin{bmatrix}1 & z_{11}\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}\overline{m}_{10}\\ \overline{m}_{11}\end{bmatrix}=\begin{bmatrix}1 & -0.1\\ 0 & 1\end{bmatrix}\begin{bmatrix}0.2\\1\end{bmatrix}=\begin{bmatrix}0.1\\1\end{bmatrix} \tag{4.15b}$$

and

$$\overline{B}_2=\begin{bmatrix}1 & z_{21}\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}\overline{m}_{20}\\ \overline{m}_{21}\end{bmatrix}=\begin{bmatrix}1 & -0.2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0.8\\1\end{bmatrix}. \tag{4.15c}$$

Therefore, the matrix

$$\overline{B}=\begin{bmatrix}\overline{B}_1\\ \overline{B}_2\end{bmatrix}=\begin{bmatrix}0.1\\1\\0.8\\1\end{bmatrix} \tag{4.16}$$

and the matrix

$$\overline{C}=\begin{bmatrix}\overline{C}_1 & 0\\ 0 & \overline{C}_2\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&0&1\end{bmatrix}. \tag{4.17}$$

The desired positive realization of (4.9) is given by

$$\widetilde{E}=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\end{bmatrix},\ \widetilde{A}=\begin{bmatrix}0.1&0&0&0&0.1&0\\1&0.3&0&0&1&0\\0&0&0.2&0&0.8&0\\0&0&1&0.3&1&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix} \tag{4.18}$$

$$\widetilde{B}=\begin{bmatrix}0\\0\\0\\-1\\0\end{bmatrix},\ \widetilde{C}=\begin{bmatrix}0&1&0&0&2&1\\0&0&0&1&3&2\end{bmatrix}.$$

Now let us consider the strictly proper transfer matrix (4.11) as the matrix with least common column denominator

$$T_{sp}(s)=\frac{1}{z^3-0.6z^2+0.11z-0.006}\begin{bmatrix}z^2+0.3z+0.02\\ z^2+0.9z+0.1\end{bmatrix}, \tag{4.19}$$

where

$$d(z)==z^3-0.6z^2+0.11z-0.006=(z-0.1)(z-0.2)(z-0.3) \tag{4.20}$$

has the zeros: $z_1=0.1$, $z_2=0.2$, $z_3=0.3$.

Therefore, the matrix $\overline{A}$ has the form

$$\overline{A}=\begin{bmatrix}0.1&1&0\\0&0.2&1\\0&0&0.3\end{bmatrix}. \tag{4.21}$$

In this case the matrix $\overline{B}$ is given by

$$\overline{B}=\begin{bmatrix}0\\0\\1\end{bmatrix}. \tag{4.22}$$

Using the dual method to the method for computation of the matrix $\overline{B}$ we obtain

$$\overline{C}=\begin{bmatrix}0&0&1\\0.02&0.6&1\end{bmatrix}. \tag{4.23}$$

Therefore, the desired positive realization of (4.9) has the form

$$\hat{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \ \hat{A} = \begin{bmatrix} 0.1 & 1 & 0 & 0 & 0 \\ 0 & 0.2 & 1 & 0 & 0 \\ 0 & 0 & 0.3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (4.24)$$

$$\hat{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \ \hat{C} = \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0.02 & 0.6 & 1 & 3 & 2 \end{bmatrix}.$$

5 CONCLUDING REMARKS

A new method for determination of positive realizations of transfer matrices of descriptor linear discrete-time systems has been proposed. Necessary and sufficient conditions for the existence of the positive realizations have been established (Theorems 3.1, 3.2 and 4.1). A procedure for computation of the positive realizations has been proposed and illustrated by an example (Example 4.1). The presented method can be extended to fractional descriptor linear continuous-time discrete-time systems.

The presented method can be considered as an extension of the method presented in [8] for continuous-time systems to the discrete-time systems. Between the methods we have the following essential differences:

1 The method presented in this paper can be applied only to discrete-time linear systems with zeros satisfying the condition (3.18).
2 For discrete-time systems the matrix B may have negative entries (see Remark 3.4).

ACKNOWLEDGMENT

The studies have been carried out in the framework of work No. S/WE/1/2016 and financed from the funds for science by the Polish Ministry of Science and Higher Education.

REFERENCES

[1] Benvenuti L., Farina L.: A tutorial on the positive realization problem. IEEE Trans. on Automatic Control, vol. 49, no. 5, 2004, 651-664.

[2] Farina L., Rinaldi S.: Positive Linear Systems; Theory and Applications. J. Wiley, New York, 2000.

[3] Kaczorek T.: A modified state variable diagram method for determination of positive realizations of linear continuous-time systems with delays. Int. J. Appl. Math. Comput. Sci., vol. 22, no. 4, 2012, 897-905.

[4] Kaczorek T.: A new method for computation of positive realizations of fractional linear continuous-time systems, 2017.

[5] Kaczorek T.: A new method for computation of positive realizations of linear discrete-time systems, 2017.

[6] Kaczorek T.: A new method for determination of positive realizations of linear continuous-time systems, 2017.

[7] Kaczorek T.: A realization problem for positive continuous-time linear systems with reduced numbers of delays. Int. J. Appl. Math. Comput. Sci., vol. 16, no. 3, 2006, 325-331.

[8] Kaczorek T.: Computation of positive realizations of descriptor linear continuous-time systems, 2017.

[9] Kaczorek T.: Computation of positive stable realizations for linear continuous-time systems. Bull. Pol. Acad. Techn. Sci., vol. 59, no. 3, 2011, 273-281.

[10] Kaczorek T.: Computation of realizations of discrete-time cone systems. Bull. Pol. Acad. Sci. Techn., vol. 54, no. 3, 2006, 347-350.

[11] Kaczorek T.: Existence and determination of the set of Metzler matrices for given stable polynomials. Int. J. Appl. Math. Comput. Sci., vol. 22, no. 2, 2012, 389-399.

[12] Kaczorek T.: Linear Control Systems: Analysis of Multivariable Systems. J. Wiley & Sons, New York, 1992.

[13] Kaczorek T.: Polynomial and Rational Matrices. Springer-Verlag, London, 2007.

[14] Kaczorek T.: Positive 1D and 2D Systems. Springer-Verlag, London, 2002.

[15] Kaczorek T.: Positive minimal realizations for singular discrete-time systems with delays in state and delays in control. Bull. Pol. Acad. Sci. Techn., vol. 53, no. 3, 2005, 293-298.

[16] Kaczorek T.: Positive stable realizations of continuous-time linear systems. Proc. Conf. Int. Inf. and Eng. Syst., Krynica-Zdrój, Poland, 17-21 September, 2012.

[17] Kaczorek T.: Positive stable realizations for fractional descriptor continuous-time linear systems. Archives of Control Sciences, vol. 22, no. 3, 2012, 255-265.

[18] Kaczorek T.: Positive stable realizations with system Metzler matrices. Archives of Control Sciences, vol. 21, no. 2, 2011, 167-188.

[19] Kaczorek T.: Realization problem for fractional continuous-time systems. Archives of Control Sciences, vol. 18, no. 1, 2008, 43-58.

[20] Kaczorek T.: Realization problem for positive 2D hybrid systems. COMPEL, vol. 27, no. 3, 2008, 613-623.

[21] Kaczorek T.: Realization problem for positive discrete-time systems with delays. System Science, vol. 30, no. 4, 2004, 117-130.

[22] Kaczorek T.: Realization problem for positive multivariable discrete-time linear systems with delays in the state vector and inputs. Int. J. Appl. Math. Comput. Sci., vol. 16,no. 2, 2006, 169-174.

[23] Kaczorek T.: Selected Problems of Fractional Systems Theory. Springer-Verlag, 2011.

[24] Kaczorek T., Busłowicz M.: Minimal realization for positive multivariable linear systems with delay. Int. J. Appl. Math. Comput. Sci., vol. 14, no. 2, 2004, 181-187.

[25] Kaczorek T., Sajewski Ł.: Realization Problem for Positive and Fractional Systems, Springer, 2014.