Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XVIII (2013)
41–54

SOLUTIONS TO FRACTIONAL DIFFUSION-WAVE EQUATION IN A CIRCULAR SECTOR

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Abstract

The time-fractional diffusion-wave equation with the Caputo derivative of the order $0 < \alpha \leq 2$ is considered in a domain $0 \leq r < R$, $0 < \varphi < \varphi_0$ under different boundary conditions. The Laplace integral transform with respect to time, the finite Fourier transforms with respect to the angular coordinate, and the finite Hankel transforms with respect to the radial coordinate are used. The fundamental solutions are expressed in terms of the Mittag-Leffler function. The particular cases of the obtained solutions corresponding to the diffusion equation ($\alpha = 1$) and the wave equation ($\alpha = 2$) coincide with those known in the literature.

1. Introduction

The time-fractional diffusion-wave equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\Delta T, \quad 0 < \alpha \leq 2, \tag{1}$$

is a mathematical model of different physical phenomena in amorphous, colloid, glassy and porous materials, random and disordered media, polymers, dielectrics and semiconductors, in medicine and biological systems, etc. This equation covers the whole spectrum from the so-called localized diffusion (the Helmholtz equation when the order of the time-derivative $\alpha \to 0$) through the standard diffusion equation (represented by the particular case $\alpha = 1$) to the ballistic diffusion (the wave equation when $\alpha = 2$).

Starting from the pioneering papers [4], [7], [8], [26], [28], considerable interest has been shown in solutions to time-fractional diffusion-wave equation. Several problems in polar or cylindrical coordinates were studied in [5], [9]–[12], [14]–[20], [22]–[24], among others. In this paper, the time-fractional diffusion-wave equation with the Caputo derivative of the order $0 < \alpha \leq 2$

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is considered in a domain $0 \le r < R$, $0 < \varphi < \varphi_0$ under different boundary conditions.

2. Mathematical preliminaries

Integral transforms technique allows us to remove the partial derivatives from the considered differential equations and to obtain the algebraic equation in a transform domain. For details, see, e.g., [1], [2], [27].

The Laplace transform is defined as

$$\mathcal{L}\left\{f(t)\right\} = f^*(s) = \int_0^\infty f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t, \tag{2}$$

where s is the transform variable.

The inverse Laplace transfrom is carried out according to the Fourier–Mellin formula

$$\mathcal{L}^{-1}\left\{f^*(s)\right\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s)\,\mathrm{e}^{st}\,\mathrm{d}s, \quad t > 0, \tag{3}$$

where c is a positive fixed number.

The finite sin-Fourier transform is the convenient reformulation of the sin-Fourier series in the domain $0 \le x \le L$:

$$\mathcal{F}\{f(x)\} = \tilde{f}(\xi_n) = \int_0^L f(x)\sin(x\xi_n)\,\mathrm{d}x, \tag{4}$$

$$\mathcal{F}^{-1}\{\tilde{f}(\xi_n)\} = f(x) = \frac{2}{L}\sum_{n=1}^{\infty}\tilde{f}(\xi_n)\sin(x\xi_n), \tag{5}$$

where

$$\xi_n = \frac{n\pi}{L}. \tag{6}$$

The finite sin-Fourier transform is used in the case of the Dirichlet boundary condition as for the second derivative of a function we have

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi_n^2\tilde{f}(\xi_n) + \xi_n\big[f(0) - (-1)^n f(L)\big].$$

The finite cos-Fourier transform is the convenient reformulation of the cos-Fourier series in the domain $0 \le x \le L$:

$$\mathcal{F}\{f(x)\} = \tilde{f}(\xi_n) = \int_0^L f(x)\cos(x\xi_n)\,\mathrm{d}x, \tag{7}$$

$$\mathcal{F}^{-1}\{\tilde{f}(\xi_n)\} = f(x) = \frac{2}{L}{\sum_{n=0}^{\infty}}{}'\,\tilde{f}(\xi_n)\cos(x\xi_n), \tag{8}$$

where ξ_n is defined by (6). The prime near the summation symbol in (8) denotes that the term with $n = 0$ should be multiplied by 1/2.

The finite cos-Fourier transform is used in the case of Neumann boundary conditions as

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi_n^2 \tilde{f}(\xi_n) - \left.\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\right|_{x=0} + (-1)^n \left.\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\right|_{x=L}.$$

The Fourier–Bessel and Dini series can be interpreted in terms of finite Hankel transform used in polar (cylindrical) coordinate system in the domain $0 \leq r \leq R$. The form of the finite Hankel transform depends on the type of boundary conditions at $r = R$. For Dirichlet boundary condition with the given boundary value of a function at $r = R$ we have

$$\mathcal{H}\{f(r)\} = \widehat{f}(\xi_{\nu m}) = \int_0^R f(r)\, J_\nu(r\xi_{\nu m})\, r\, \mathrm{d}r, \tag{9}$$

$$\mathcal{H}^{-1}\{\widehat{f}(\xi_{\nu m})\} = f(r) = \frac{2}{R^2} \sum_{m=1}^{\infty} \widehat{f}(\xi_{\nu m}) \frac{J_\nu(r\xi_{\nu m})}{[J_\nu'(R\xi_{\nu m})]^2}, \tag{10}$$

where $J_\nu(r)$ is the bessel function of the first kind of the order ν. Here the prime denotes the derivative of the Bessel function, and $\xi_{\nu m}$ are the positive roots of the transcendental equation

$$J_\nu(R\xi_{\nu m}) = 0. \tag{11}$$

The basic equation for this integral transform reads:

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{\nu^2}{r^2} f(r)\right\} = -\xi_{\nu m}^2 \widehat{f}(\xi_{\nu m}) - R\xi_{\nu m} J_\nu'(R\xi_{\nu m}) f(R).$$

In the case of Neumann boundary conditions with the given boundary value of a normal derivative of a function

$$\mathcal{H}\{f(r)\} = \widehat{f}(\xi_{\nu m}) = \int_0^R f(r)\, J_\nu(r\xi_{\nu m})\, r\, \mathrm{d}r, \tag{12}$$

$$\mathcal{H}^{-1}\{\widehat{f}(\xi_{\nu m})\} = f(r) = 2 \sum_{m=1}^{\infty} \widehat{f}(\xi_{\nu m}) \frac{\xi_{\nu m}^2}{R^2 \xi_{\nu m}^2 - \nu^2} \cdot \frac{J_\nu(r\xi_{\nu m})}{[J_\nu(R\xi_{\nu m})]^2}, \tag{13}$$

where $\xi_{\nu m}$ are the positive roots of the transcendental equation

$$J_\nu'(R\xi_{\nu m}) = 0. \tag{14}$$

It should be noted that for $\nu = 0$ there appears the additional zero root $\xi_{00} = 0$.

The basic equation for this integral transform has the following form:

$$\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2}+\frac{1}{r}\cdot\frac{\mathrm{d}f(r)}{\mathrm{d}r}-\frac{\nu^2}{r^2}\cdot f(r)\right\}=-\xi_{\nu m}^2\widehat{f}(\xi_{\nu m})+RJ_\nu(R\xi_{\nu m})\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)\bigg|_{r=R}.$$

For Robin boundary condition with the given linear combination of values of function and its normal derivative at the boundary we have

$$\mathcal{H}\{f(r)\}=\widehat{f}(\xi_{\nu m})=\int_0^R f(r)\,J_\nu(r\xi_{\nu m})\,r\,\mathrm{d}r, \tag{15}$$

$$\begin{aligned}\mathcal{H}^{-1}\{\widehat{f}(\xi_{\nu m})\}&=f(r)\\ &=2\sum_{m=1}^{\infty}\widehat{f}(\xi_{\nu m})\cdot\frac{\xi_{\nu m}^2}{R^2H^2+R^2\xi_{\nu m}^2-\nu^2}\cdot\frac{J_\nu(r\xi_{\nu m})}{[J_\nu(R\xi_{\nu m})]^2},\end{aligned} \tag{16}$$

where $\xi_{\nu m}$ are the positive roots of the transcendental equation

$$\xi_{\nu m}J_\nu'(R\xi_{\nu m})+HJ_\nu(R\xi_{\nu m})=0 \tag{17}$$

and

$$\begin{aligned}&\mathcal{H}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2}+\frac{1}{r}\cdot\frac{\mathrm{d}f(r)}{\mathrm{d}r}-\frac{\nu^2}{r^2}\,f(r)\right\}\\ &\qquad=-\xi_{\nu m}^2\widehat{f}(\xi_{\nu m})+RJ_\nu(R\xi_{\nu m})\left[\frac{\mathrm{d}f(r)}{\mathrm{d}r}+Hf(r)\right]\bigg|_{r=R}.\end{aligned}$$

Now we recall the basic notions of the fractional calculus [3], [6], [13], [25]. The Riemann–Liouville fractional integral is introduced as a natural generalization of the repeated integral $I^n f(t)$ written in a convolution type form:

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)\,\mathrm{d}\tau, \qquad \alpha>0, \tag{18}$$

where $\Gamma(\alpha)$ is the gamma function.

The Laplace transform rule for the fractional integral has the following form:

$$\mathcal{L}\left\{I^\alpha f(t)\right\}=\frac{1}{s^\alpha}\cdot f^*(s). \tag{19}$$

The Riemann–Liouville derivative of the fractional order α is defined as left-inverse to the fractional integral I^α, i.e.

$$D_{RL}^\alpha f(t)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f(\tau)\,\mathrm{d}\tau\right], \quad n-1<\alpha<n, \tag{20}$$

and for its Laplace transform requires knowledge of the initial values of the fractional integral $I^{n-\alpha}$ and its derivatives of the order $k = 1, 2, \ldots, n-1$

$$\mathcal{L}\left\{D_{RL}^{\alpha} f(t)\right\} = s^{\alpha} f^{*}(s) - \sum_{k=0}^{n-1} D^{k} I^{n-\alpha} f(0^{+}) s^{n-1-k}, \quad n-1 < \alpha < n. \tag{21}$$

An alternative definition of the fractional derivative was proposed by Caputo:

$$\frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^{n} f(\tau)}{\mathrm{d}\tau^{n}}\, \mathrm{d}\tau, \quad n-1 < \alpha < n. \tag{22}$$

For its Laplace transform rule, the Caputo fractional derivative requires knowledge of the initial values of the function $f(t)$ and its integer derivatives of the order $k = 1, 2, \ldots, n-1$

$$\mathcal{L}\left\{\frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}}\right\} = s^{\alpha} f^{*}(s) - \sum_{k=0}^{n-1} f^{(k)}(0^{+}) s^{\alpha-1-k}, \quad n-1 < \alpha < n. \tag{23}$$

The Caputo fractional derivative is a regularization in the time origin for the Riemann–Liouville fractional derivative by incorporating the relevant initial conditions. The major utility of the Caputo fractional derivative is caused by the treatment of differential equations of fractional order for physical applications, where the initial conditions are usually expressed in terms of a given function and its derivatives of integer (not fractional) order, even if the governing equation is of fractional order [13]. If care is taken, the results obtained using the Caputo formulation can be recast to the Riemann–Liouville version and vice versa.

The Mittag-Leffler function in one parameter α [3], [6], [13]

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(\alpha n + 1)}, \qquad \alpha > 0, \ z \in C,$$

provides a generalization of the exponential function

$$\mathrm{e}^{z} = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(n+1)}, \qquad z \in C.$$

The generalized Mittag-Leffler function in two parameters α and β is described by the following series representation

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(\alpha n + \beta)}, \qquad \alpha > 0, \ \beta > 0, \ z \in C.$$

The important particular cases of Mittag-Leffler functions are the following:

$$E_{1}\left(-x^{2}\right) = \mathrm{e}^{-x^{2}}, \qquad E_{2}\left(-x^{2}\right) = \cos x, \qquad E_{2,2}\left(-x^{2}\right) = \frac{\sin x}{x}.$$

The essential role of the Mittag-Leffler functions in fractional calculus results from the formula for the inverse Laplace transform [6], [13]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^{\alpha}+b}\right\} = t^{\beta-1} E_{\alpha,\beta}(-bt^{\alpha}). \tag{24}$$

3. The Dirichlet boundary condition. Statement of the problem

Consider the time-fractional diffusion-wave equation in polar coordinates in the domain $0 \le r < R$, $0 < \varphi < \varphi_0$

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}\right) + \Phi(r,\varphi,t) \tag{25}$$

under initial conditions

$$t = 0: \qquad T = f(r,\varphi), \qquad 0 < \alpha \le 2, \tag{26}$$

$$t = 0: \qquad \frac{\partial T}{\partial t} = F(r,\varphi), \qquad 1 < \alpha \le 2, \tag{27}$$

and Dirichlet boundary conditions

$$r = R: \quad T = g_1(\varphi,t), \tag{28}$$

$$\varphi = 0: \quad T = g_2(r,t), \tag{29}$$

$$\varphi = \varphi_0: \quad T = g_3(r,t). \tag{30}$$

The solution reads:

$$\begin{aligned} T(r,t,\varphi) \;=\; & \int_0^{\varphi_0}\int_0^R f(\rho,\phi)\,\mathcal{G}_f(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ & + \int_0^{\varphi_0}\int_0^R F(\rho,\phi)\,\mathcal{G}_F(r,\varphi,\rho,\phi,t)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ & + \int_0^t\int_0^{\varphi_0}\int_0^R \Phi(\rho,\phi,\tau)\,\mathcal{G}_{\Phi}(r,\varphi,\rho,\phi,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\tau \\ & + \int_0^t\int_0^{\varphi_0} g_1(\phi,\tau)\,\mathcal{G}_{g_1}(r,\varphi,\phi,t-\tau)\,\mathrm{d}\phi\,\mathrm{d}\tau \\ & + \int_0^t\int_0^R g_2(\rho,\tau)\,\mathcal{G}_{g_2}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau \\ & + \int_0^t\int_0^R g_3(\rho,\tau)\,\mathcal{G}_{g_3}(r,\varphi,\rho,t-\tau)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\tau, \end{aligned} \tag{31}$$

where $\mathcal{G}_f(r,\varphi,\rho,\phi,t)$ is the fundamental solution to the first Cauchy problem, $\mathcal{G}_F(r,\varphi,\rho,\phi,t)$ is the fundamental solution to the second Cauchy problem, $\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t)$ is the fundamental solution to the source problem, $\mathcal{G}_{g_1}(r,\varphi,\phi,t)$ is the fundamental solution to the first Dirichlet problem, $\mathcal{G}_{g_2}(r,\varphi,\rho,t)$ is the fundamental solution to the second Dirichlet problem, $\mathcal{G}_{g_3}(r,\varphi,\rho,t)$ is the fundamental solution to the third Dirichlet problem.

3.1. The fundamental solution to the first Cauchy problem under zero Dirichlet boundary condition

In this case we have

$$f(r,\varphi)=\frac{\delta(r-\rho)}{r}\cdot\delta(\varphi-\phi),\quad F(r,\varphi)=0,\quad \Phi(r,\varphi,t)=0,$$

$$g_1(\varphi,t)=0,\quad g_2(r,t)=0,\quad g_3(r,t)=0,$$

where $\delta(x)$ is the Dirac delta function. It should be noted that the two-dimensional Dirac delta function in Cartesian coordinates after passing to polar coordinates takes the form $\frac{1}{2\pi r}\delta(r)$, but for the sake of simplicity we have omitted the factor $\frac{1}{2\pi}$ in the delta term as well as the factor 2π in the solution (31).

The Laplace transform with respect to time t gives

$$s^\alpha\mathcal{G}_f^*-s^{\alpha-1}\cdot\frac{\delta(r-\rho)}{r}\cdot\delta(\varphi-\phi)=a\left(\frac{\partial^2\mathcal{G}_f^*}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\mathcal{G}_f^*}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2\mathcal{G}_f^*}{\partial\varphi^2}\right),$$

$$r=R:\quad \mathcal{G}_f^*=0,$$

$$\varphi=0:\quad \mathcal{G}_f^*=0,$$

$$\varphi=\varphi_0:\quad \mathcal{G}_f^*=0.$$

Next we use the finite sin-Fourier transform (4) with respect to the angular coordinate φ, thus obtaining

$$s^\alpha\tilde{\mathcal{G}}_f^*-s^{\alpha-1}\cdot\frac{\delta(r-\rho)}{r}\cdot\sin\left(\frac{n\pi\phi}{\varphi_0}\right)=a\left[\frac{\partial^2\tilde{\mathcal{G}}_f^*}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\tilde{\mathcal{G}}_f^*}{\partial r}-\frac{(n\pi/\varphi_0)^2}{r^2}\cdot\tilde{\mathcal{G}}_f^*\right],$$

$$r=R:\quad \tilde{\mathcal{G}}_f^*=0.$$

The finite Hankel transform (9) with respect to the radial variable r with $\nu=n\pi/\varphi_0$ leads to the solution in the transform domain

$$\widehat{\tilde{\mathcal{G}}}_f^*=J_{n\pi/\varphi_0}(\rho\xi_{nm})\cdot\sin\left(\frac{n\pi\phi}{\varphi_0}\right)\cdot\frac{s^{\alpha-1}}{s^\alpha+a\xi_{nm}^2},$$

where ξ_{nm} are the positive roots of the transcendental equation

$$J_{n\pi/\varphi_0}(R\xi_{nm})=0.$$

For the sake of simplicity, we have used the notation ξ_{nm} for the roots (not $\xi_{n\pi/\varphi_0,m}$).

The inverse integral transforms result in

$$\mathcal{G}_f(r,\varphi,\rho,\phi,t) = \frac{4}{\varphi_0 R^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{\alpha}\left(-a\xi_{nm}^2 t^{\alpha}\right)\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\phi}{\varphi_0}\right)$$
$$\times\frac{J_{n\pi/\varphi_0}(r\xi_{nm})\,J_{n\pi/\varphi_0}(\rho\xi_{nm})}{\left[J'_{n\pi/\varphi_0}(R\xi_{nm})\right]^2}.$$

3.2. The fundamental solution to the second Cauchy problem under zero Dirichlet boundary condition

This solution is obtained for

$$f(r,\varphi)=0,\quad F(r,\varphi)=\frac{\delta(r-\rho)}{r}\cdot\delta(\varphi-\phi),\quad \Phi(r,\varphi,t)=0,$$
$$g_1(\varphi,t)=0,\quad g_2(r,t)=0,\quad g_3(r,t)=0,$$

and has the form

$$\mathcal{G}_F(r,\varphi,\rho,\phi,t) = \frac{4t}{\varphi_0 R^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{\alpha,2}\left(-a\xi_{nm}^2 t^{\alpha}\right)\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\phi}{\varphi_0}\right)$$
$$\times\frac{J_{n\pi/\varphi_0}(r\xi_{nm})\,J_{n\pi/\varphi_0}(\rho\xi_{nm})}{\left[J'_{n\pi/\varphi_0}(R\xi_{nm})\right]^2}.$$

3.3. The fundamental solution to the source problem under zero Dirichlet boundary condition

In this case

$$f(r,\varphi)=0,\quad F(r,\varphi)=0,\quad \Phi(r,\varphi,t)=\frac{\delta(r-\rho)}{r}\cdot\delta(\varphi-\phi)\,\delta(t),$$
$$g_1(\varphi,t)=0,\quad g_2(r,t)=0,\quad g_3(r,t)=0,$$

and

$$\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) = \frac{4t^{\alpha-1}}{\varphi_0 R^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{\alpha,\alpha}\left(-a\xi_{nm}^2 t^{\alpha}\right)\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\phi}{\varphi_0}\right)$$
$$\times\frac{J_{n\pi/\varphi_0}(r\xi_{nm})\,J_{n\pi/\varphi_0}(\rho\xi_{nm})}{\left[J'_{n\pi/\varphi_0}(R\xi_{nm})\right]^2}.$$

3.4. The fundamental solution to the first Dirichlet problem under zero initial conditions

This solution corresponds to the choice

$$f(r,\varphi)=0,\quad F(r,\varphi)=0,\quad \Phi(r,\varphi,t)=0,$$
$$g_1(\varphi,t)=\delta(\varphi-\phi)\,\delta(t),\quad g_2(r,t)=0,\quad g_3(r,t)=0,$$

and is expressed as

$$\mathcal{G}_{g_1}(r,\varphi,\phi,t) = -\frac{4at^{\alpha-1}}{\varphi_0 R}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E_{\alpha,\alpha}\left(-a\xi_{nm}^2 t^{\alpha}\right)\sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\sin\left(\frac{n\pi\phi}{\varphi_0}\right) \times \frac{\xi_{nm}\, J_{n\pi/\varphi_0}(r\xi_{nm})}{J'_{n\pi/\varphi_0}(R\xi_{nm})}.$$

3.5. The fundamental solution to the second Dirichlet problem under zero initial conditions

In this case

$$f(r,\varphi)=0,\quad F(r,\varphi)=0,\quad \Phi(r,\varphi,t)=0,$$
$$g_1(\varphi,t)=0,\quad g_2(r,t)=\frac{\delta(r-\rho)}{r}\cdot\delta(t),\quad g_3(r,t)=0,$$

and

$$\mathcal{G}_{g_2}(r,\varphi,\rho,t) = \frac{4at^{\alpha-1}}{\varphi_0 R^2\rho^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n\pi}{\varphi_0}\, E_{\alpha,\alpha}\left(-a\xi_{nm}^2 t^{\alpha}\right)\sin\left(\frac{n\pi\varphi}{\varphi_0}\right) \times \frac{J_{n\pi/\varphi_0}(r\xi_{nm})\, J_{n\pi/\varphi_0}(\rho\xi_{nm})}{\left[J'_{n\pi/\varphi_0}(R\xi_{nm})\right]^2}.$$

3.6. The fundamental solution to the third Dirichlet problem under zero initial conditions

This type of fundamental solution is obtained for

$$f(r,\varphi)=0,\quad F(r,\varphi)=0,\quad \Phi(r,\varphi,t)=0,$$
$$g_1(\varphi,t)=0,\quad g_2(r,t)=0,\quad g_3(r,t)=\frac{\delta(r-\rho)}{r}\cdot\delta(t),$$

and reads

$$\mathcal{G}_{g_3}(r,\varphi,\rho,t) = \frac{4at^{\alpha-1}}{\varphi_0 R^2\rho^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+1}\frac{n\pi}{\varphi_0}\, E_{\alpha,\alpha}\left(-a\xi_{nm}^2 t^{\alpha}\right) \times \sin\left(\frac{n\pi\varphi}{\varphi_0}\right)\frac{J_{n\pi/\varphi_0}(r\xi_{nm})\, J_{n\pi/\varphi_0}(\rho\xi_{nm})}{\left[J'_{n\pi/\varphi_0}(R\xi_{nm})\right]^2}.$$

4. The Neumann boundary condition

4.1. The fundamental solution to the first Cauchy problem under zero Neumann boundary condition

To solve the problems under Neumann boundary condition the Laplace transform (2) with respect to time t, the finite cos-Fourier transform (7) we respect to the angular coordinate φ, and the finite Hankel transform (12) of the order $\nu = n\pi/\varphi_0$ with respect to the radial coordinate r are used. The solution has the following form

$$\mathcal{G}_f(r,\varphi,\rho,\phi,t) = \frac{2}{R^2\varphi_0} + \frac{4}{\varphi_0}{\sum_{n=0}^{\infty}}' \sum_{m=1}^{\infty} E_\alpha\left(-a\xi_{nm}^2 t^\alpha\right) \cdot \frac{\xi_{nm}^2}{R^2\xi_{nm}^2 - (n\pi/\varphi_0)^2}$$
$$\times \frac{J_{n\pi/\varphi_0}(r\xi_{nm})\, J_{n\pi/\varphi_0}(\rho\xi_{nm})}{\left[J_{n\pi/\varphi_0}(R\xi_{nm})\right]^2} \cdot \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \cos\left(\frac{n\pi\phi}{\varphi_0}\right),$$

where ξ_{nm} are the positive roots of the transcendental equation

$$J'_{n\pi/\varphi_0}(R\xi_{nm}) = 0.$$

4.2. The fundamental solution to the second Cauchy problem under zero Neumann boundary condition

$$\mathcal{G}_F(r,\varphi,\rho,\phi,t) = \frac{2t}{R^2\varphi_0} + \frac{4t}{\varphi_0}{\sum_{n=0}^{\infty}}' \sum_{m=1}^{\infty} E_{\alpha,2}\left(-a\xi_{nm}^2 t^\alpha\right) \cdot \frac{\xi_{nm}^2}{R^2\xi_{nm}^2 - (n\pi/\varphi_0)^2}$$
$$\times \frac{J_{n\pi/\varphi_0}(r\xi_{nm})\, J_{n\pi/\varphi_0}(\rho\xi_{nm})}{\left[J_{n\pi/\varphi_0}(R\xi_{nm})\right]^2} \cdot \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \cos\left(\frac{n\pi\phi}{\varphi_0}\right).$$

4.3. The fundamental solution to the source problem under zero Neumann boundary condition

$$\mathcal{G}_\Phi(r,\varphi,\rho,\phi,t) = \frac{2t^{\alpha-1}}{R^2\varphi_0\Gamma(\alpha)} + \frac{4t^{\alpha-1}}{\varphi_0}{\sum_{n=0}^{\infty}}' \sum_{m=1}^{\infty} E_{\alpha,\alpha}\left(-a\xi_{nm}^2 t^\alpha\right)$$
$$\times \frac{\xi_{nm}^2}{R^2\xi_{nm}^2 - (n\pi/\varphi_0)^2} \cdot \frac{J_{n\pi/\varphi_0}(r\xi_{nm})\, J_{n\pi/\varphi_0}(\rho\xi_{nm})}{\left[J_{n\pi/\varphi_0}(R\xi_{nm})\right]^2} \cdot \cos\left(\frac{n\pi\varphi}{\varphi_0}\right) \cos\left(\frac{n\pi\phi}{\varphi_0}\right).$$

4.4. The fundamental solution to the first mathematical Neumann problem under zero initial conditions

In this case the boundary condition at $r = R$ is formulated for the normal derivative of the function

$$r = R: \quad \frac{\partial G_{g_1}}{\partial r} = \delta(\varphi - \phi)\,\delta(t).$$

The solution reads:

$$\mathcal{G}_{g_1}(r,\varphi,\phi,t) = \frac{2at^{\alpha-1}}{R^2\varphi_0\Gamma(\alpha)} + \frac{4aRt^{\alpha-1}}{\varphi_0}{\sum_{n=0}^{\infty}}' \sum_{m=1}^{\infty} E_{\alpha,\alpha}\left(-a\xi_{nm}^2 t^\alpha\right)$$
$$\times \frac{\xi_{nm}^2}{R^2\xi_{nm}^2-(n\pi/\varphi_0)^2}\cdot\frac{J_{n\pi/\varphi_0}(r\xi_{nm})}{J_{n\pi/\varphi_0}(R\xi_{nm})}\cdot\cos\left(\frac{n\pi\varphi}{\varphi_0}\right)\cos\left(\frac{n\pi\phi}{\varphi_0}\right).$$

4.5. The fundamental solution to the first physical Neumann problem under zero initial conditions

The physical Neumann condition at $r = R$ is formulated in terms of the heat flux at the boundary:

$$r = R: \quad D_{RL}^{1-\alpha}\frac{\partial G_{g_1}}{\partial r} = \delta(\varphi-\phi)\,\delta(t), \quad 0<\alpha\le 1,$$
$$r = R: \quad I^{\alpha-1}\frac{\partial G_{g_1}}{\partial r} = \delta(\varphi-\phi)\,\delta(t), \quad 1<\alpha\le 2.$$

The difference between the mathematical and physical boundary conditions (as well as the difference between the solutions) disappear in the case of standard diffusion (heat conduction) equation corresponding to $\alpha = 1$. For details see [21], [22], [23].

The solution has the following form:

$$\mathcal{G}_{g_1}(r,\varphi,\phi,t) = \frac{2a}{R^2\varphi_0} + \frac{4aR}{\varphi_0}{\sum_{n=0}^{\infty}}' \sum_{m=1}^{\infty} E_{\alpha}\left(-a\xi_{nm}^2 t^\alpha\right)$$
$$\times \frac{\xi_{nm}^2}{R^2\xi_{nm}^2-(n\pi/\varphi_0)^2}\cdot\frac{J_{n\pi/\varphi_0}(r\xi_{nm})}{J_{n\pi/\varphi_0}(R\xi_{nm})}\cdot\cos\left(\frac{n\pi\varphi}{\varphi_0}\right)\cos\left(\frac{n\pi\phi}{\varphi_0}\right).$$

4.6. The fundamental solution to the second mathematical Neumann problem under zero initial conditions

In this case the bounadry condition at $\varphi = 0$ is formulated for the normal derivative of the function

$$\varphi = 0: \quad -\frac{1}{r}\cdot\frac{\partial G_{g_2}}{\partial\varphi} = \frac{\delta(r-\rho)}{r}\cdot\delta(t).$$

The solution has the form:

$$\mathcal{G}_{g_2}(r,\varphi,\rho,t) = \frac{2at^{\alpha-1}}{R^2\rho\varphi_0\Gamma(\alpha)} + \frac{4at^{\alpha-1}}{\rho\varphi_0}{\sum_{n=0}^{\infty}}' \sum_{m=1}^{\infty} E_{\alpha,\alpha}\left(-a\xi_{nm}^2 t^\alpha\right)$$
$$\times \frac{\xi_{nm}^2}{R^2\xi_{nm}^2-(n\pi/\varphi_0)^2}\cdot\frac{J_{n\pi/\varphi_0}(r\xi_{nm})\,J_{n\pi/\varphi_0}(\rho\xi_{nm})}{\left[J_{n\pi/\varphi_0}(R\xi_{nm})\right]^2}\cdot\cos\left(\frac{n\pi\varphi}{\varphi_0}\right).$$

4.7. The fundamental solution to the second physical Neumann problem under zero initial conditions

For the physical Neumann problem, the bounadry conditions at $\varphi = 0$ is formulated in terms of the normal component of the heat flux:

$$\varphi = 0: \quad -\frac{1}{r} \cdot D_{RL}^{1-\alpha} \frac{\partial G_{g2}}{\partial \varphi} = \frac{\delta(r-\rho)}{r} \cdot \delta(t), \quad 0 < \alpha \leq 1,$$

$$\varphi = 0: \quad -\frac{1}{r} \cdot I^{\alpha-1} \frac{\partial G_{g2}}{\partial \varphi} = \frac{\delta(r-\rho)}{r} \cdot \delta(t), \quad 1 < \alpha \leq 2.$$

The solution is expressed as

$$\mathcal{G}_{g_2}(r, \varphi, \rho, t) = \frac{2a}{R^2 \rho \varphi_0} + \frac{4a}{\rho \varphi_0} \sum_{n=0}^{\infty}{}' \sum_{m=1}^{\infty} E_\alpha\left(-a\xi_{nm}^2 t^\alpha\right)$$
$$\times \frac{\xi_{nm}^2}{R^2 \xi_{nm}^2 - (n\pi/\varphi_0)^2} \cdot \frac{J_{n\pi/\varphi_0}(r\xi_{nm})\, J_{n\pi/\varphi_0}(\rho\xi_{nm})}{\left[J_{n\pi/\varphi_0}(R\xi_{nm})\right]^2} \cdot \cos\left(\frac{n\pi\varphi}{\varphi_0}\right).$$

It should be emphasized that in fundamental solutions to the mathematical and physical Neumann problems there appear different Mittag-Leffler functions: $E_{\alpha,\alpha}\left(-a\xi_{nm}^2 t^\alpha\right)$ and $E_\alpha\left(-a\xi_{nm}^2 t^\alpha\right)$, respectively.

5. Mixed boundary value problems

There are several possibilities to formulate the mixed boundary value problem with the boundary condition of one type at $r = R$ and the boundary conditions of another type at $\varphi = 0$ and $\varphi = \varphi_0$. As an example, we consider the boundary-value problem with the mathematical Robin boundary condition at $r = R$

$$r = R: \quad \frac{\partial \mathcal{G}_{g_1}}{\partial r} + H\mathcal{G}_{g_1} = \delta(\varphi - \phi)\,\delta(t)$$

and zero Dirichlet boundary conditions at $\varphi = 0$ and $\varphi = \varphi_0$

$$\varphi = 0: \quad \mathcal{G}_{g_1} = 0,$$

$$\varphi = \varphi_0: \quad \mathcal{G}_{g_1} = 0.$$

The Laplace transform (2) with respect to time t, the finite sin-Fourier transform (4) with respect to the angular coordinate φ,the finite Hankel transform (15) with respect to the radial coordinate r allow us to get the solution

$$\mathcal{G}_{g_1}(r, \varphi, \phi, t) = \frac{4aRt^{\alpha-1}}{\varphi_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{\alpha,\alpha}\left(-a\xi_{nm}^2 t^\alpha\right) \cdot \sin\left(\frac{n\pi\varphi}{\varphi_0}\right) \cdot \sin\left(\frac{n\pi\phi}{\varphi_0}\right)$$
$$\times \frac{\xi_{nm}}{R^2 H^2 + R^2 \xi_{nm}^2 - (n\pi/\varphi_0)^2} \cdot \frac{J_{n\pi/\varphi_0}(r\xi_{nm})}{J_{n\pi/\varphi_0}(R\xi_{nm})},$$

where ξ_{nm} are the positive roots of the transcendental equation

$$\xi_{nm} J^{'}_{n\pi/\varphi_0}(R\xi_{nm}) + H J_{n\pi/\varphi_0}(R\xi_{nm}) = 0.$$

References

[1] G. Doetsch, *Anleitung zum praktischen Gebrauch der Laplace-Transformation und der Z-Transformation*. Springer, München, 1967.

[2] A. S. Galitsyn, A. N. Zhukovsky, *Integral Transforms and Special Functions in Heat Conduction Problems*. Naukova Dumka, Kiev, 1976 (In Russian).

[3] R. Gorenflo, F. Mainardi, *Fractional calculus: Integral and differential equations of fractional order*, In: A. Carpinteri, F. Mainardi (eds.) *Fractals and Fractional Calculus in Continuum Mechanics*, pp. 223-276. Springer, Wien, 1997.

[4] Y. Fujita, *Integrodifferential equation which interpolates the heat equation and the wave equation*, Osaka J. Math. **27** (1990), 309-321.

[5] X. Y. Jiang, M. Y. Xu, *The time fractional heat conduction equation in the general orthogonal curvilinear coordinate and the cylindrical coordinate system*, Physica A 389, No 17 (2010), 3368-3374.

[6] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.

[7] F. Mainardi, *The fundamental solutions for the fractional diffusion-wave equation*, Appl. Math. Lett. **9** (1996), 23-28.

[8] F. Mainardi, *Fractional relaxation-oscillation and fractional diffusion-wave phenomena*, Chaos, Solitons Fractals **7** (1996), 1461-1477.

[9] B. N. Narahari Achar, J. W. Hanneken, *Fractional radial diffusion in a cylinder*, J. Mol. Liq. **114** (2004), 147-151.

[10] N. Özdemir, D. Karadeniz, *Fractional diffusion-wave problem in cylindrical coordinates*, Phys. Lett. A **372** (2008), 5968-5972.

[11] N. Özdemir, D. Karadeniz, B. B. Iskender, *Fractional, optimal control problem of a distributed system in cylindrical coordinates*, Phys. Lett. A **373** (2009), 221-226.

[12] N. Özdemir, O. P. Agrawal, D. Karadeniz, B. B. Iskender, *Fractional optimal control problem of an axis-symmetric diffusion-wave propagation*, Phys. Scr. T **136** (2009), 014024.

[13] I. Podlubny, *Fractional Differential Equations*. Academic Press, New York, 1999.

[14] Y. Z. Povstenko, *Fractional heat conduction equation and associated thermal stresses*, J. Thermal Stresses **28** (2005), 83-102.

[15] Y. Z. Povstenko, *Two-dimensional axisymmetric stresses exerted by instantaneous pulses and sources of diffusion in an infinite space in a case of time-fractional diffusion equation*, Int. J. Solids Structures **44** (2007), 2324-2348.

[16] Y. Z. Povstenko, *Fractional radial diffusion in a cylinder*, J. Mol. Liq. **137** (2008), 46-50.

[17] Y. Povstenko, *Analysis of fundamental solutions to fractional diffusion-wave equation in polar coordinates*, Sci. Issues Jan Długosz Univ. Częstochowa, Mathematics **XIV** (2009), 97-104.

[18] Y. Povstenko, *Axisymmetric solutions to the Cauchy problem for time-fractional diffusion equation in a circle*, Sci. Issues Jan Długosz Univ. Częstochowa, Mathematics **XV** (2010), 109-117.

[19] Y. Povstenko, *Non-axisymmetric solutions to time-fractional diffusion-wave equation in an infinite cylinder*, Fract. Calc. Appl. Anal. **14** (2011), 418-435.

[20] Y. Povstenko, *Solutions to time-fractional diffusion-wave equation in cylindrical coordinates*, Adv. Difference Eqs **2011** (2011), Article ID 930297, 14.

[21] Y. Povstenko, *Different kinds of boundary conditions for time-fractional heat conduction equation*, Scientific Issues, Jan Długosz University, Mathematics XVI, (2011), 61-66.

[22] Y. Povstenko, *The Neumann boundary problem for axisymmetric fractional heat conduction equation in a solid with cylindrical hole and associated thermal stresses*, Meccanica 47, (2012), 23-29.

[23] Y. Povstenko, *Time-fractional radial heat conduction in a cylinder and associated thermal stresses*, Arch. Appl. Mech. 82, (2012), 345-362.

[24] H. Qi, J. Liu, *Time-fractional radial diffusion in hollow geometries*, Meccanica 45, (2010), 577-583.

[25] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*. Gordon and Breach, Amsterdam, 1993.

[26] W. R. Schneider, W. Wyss, *Fractional diffusion and wave equations*, J. Math. Phys. 30, (1989), 134-144.

[27] I. N. Sneddon, *The Use of Integral Transforms*. McGraw-Hill, New York, 1972.

[28] W. Wyss, *The fractional diffusion equation*, J. Math. Phys. 27, (1986), 2782-2785.

Received: July 2, 2013

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