Derivation of Simplified Formulas for the Approximate Calculation of the Meridian Convergence and Scale Error in Křovák Projection

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Summary

Obligatory geodetic reference system for the implementation of geodetic activities in the Czech and Slovak Republic is a coordinate system called the "Súradnicový systém Jednotnej trigonometrickej siete katastrálnej" (S-JTSK), whose mathematical basis is Křovák projection. Points upon the topographic surface are projected on the Bessel reference ellipsoid and grid coordinates in the projection plane are obtained by four successive transformations. Since the mathematical basis of projection is mathematically quite complicated, the relationships to quantify the length distortion and meridian convergence are also quite complex and they are cumbersome for fieldwork. The article presents simplified approximate equations for the length distortion and meridian convergence, which are suitable for use in the terrain or in underground mining conditions where the use of a programmable portable computer technique can be restricted owing to security or other reasons.

Keywords: meridian convergence, length distortion, Křovák projection

Introduction

Czechoslovak Republic after its establishment in 1918, inherited the fragmented and inconsistent surveying materials on its own territory that have been made at different times for different parts of the country, in different quality and in many geodetic coordinate systems. These legitimate and natural needs led to the creation of new, appropriate cartographic projection for property administration of real estate and management of technical and economic activities. [2, 4, 6, 13]. Křovák's projection, which is used officially since 1932, best satisfy requirements of the newly created contour of the national territory, as well as other technical requirements.

Mathematical basis of Křovák's projection

Křovák's projection is conformal. Calculation of grid coordinates (X, Y) from geographical coordinates (φ, λ) that located on Bessel reference ellipsoid consists of four subsequent transformations.

Conformal projection of Bessel ellipsoid to Gauss sphere (φλ→U,V)

The cartographic coordinates located on Bessel ellipsoid, in the first step, are conformal projected on the Gauss sphere by equations [2, 6].

$$\tan\left(\frac{U}{2}+45°\right)=\frac{1}{k}\left[\tan\left(\frac{\varphi}{2}+45°\right)\left(\frac{1-e\sin\varphi}{1+e\sin\varphi}\right)^{e/2}\right]^{\alpha} \quad (1a)$$

$$V=\alpha\,\lambda_{Ferro} \quad (1b)$$

where: φ, is latitude and λ_{Ferro} is longitude on Bessel reference ellipsoid oriented east of a prime meridian of *Ferro* located on island of the Canaries. Output coordinates *U, V* are latitude and longitude on Gauss sphere with a radius *R*=6 380 703.610 5 *m.* The base parallel of latitude $\varphi_0 = 49°30'$ was chosen for the whole territory of the former Czechoslovakia and the following constants were determined from it: $k = 1.003\ 419\ 164$, $\alpha = 1.000\ 597\ 498\ 372$, $e = 0.0816968303$, $U_0 = 49°27'35.846\ 25$ [2, 3, 6, 11].

Projection is related to the Bessel reference ellipsoid (1841) with the prime meridian of *Ferro*, located 17°40′ [7] west of the prime meridian of *Greenwich*

$$\lambda_{Greenwich}=\lambda_{Ferro}-17°40'. \quad (2)$$

Bessel reference ellipsoid is defined by the length of the semi–major axis $a = 6\ 377\ 397{,}155$ *m* and ellipsoidal flattening *f* = 1:299.152 8128.

Transformation from geographic to cartographic coordinates on Gauss sphere (U,V→S,D)

Geographic coordinates are transformed into cartographic coordinates (Fig. 1) according to the following relationship [2, 6]:

$$\sin S=\cos a\sin U+\sin a\cos U\cos\left(V_Q-V\right) \quad (3a)$$

$$\sin D=\sin\left(V_Q-V\right)\cos U\sec S \quad (3b)$$

where S is cartographic latitude and D is cartographic longitude on the Gauss sphere applied to cartographic pole Q:

$\varphi_Q = 59°45'27''$, $\lambda_Q = 42°30'00''$,
$U_Q = 59°42'42.69689''$, $V_Q = 42°31'41725''$
and $a = (90° - U_Q) = 30°17'17.30311''$

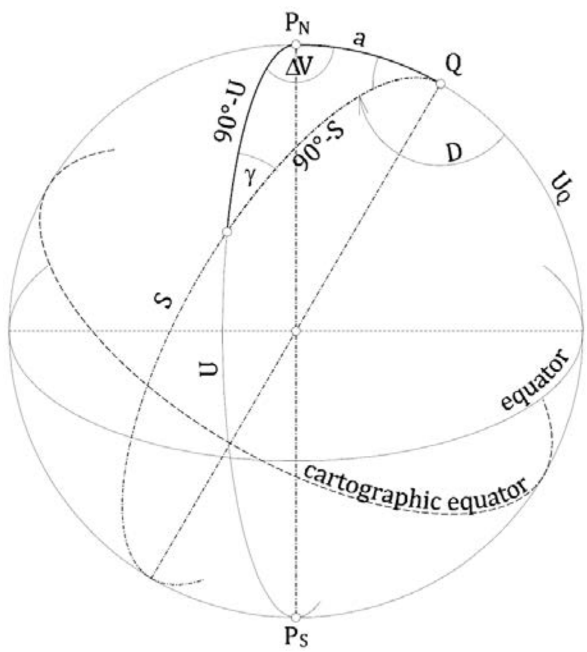

Fig. 1. Transformation of geographic coordinates U,V into cartographic coordinates S,D

Rys. 1. Transformacja koordynat geograficznych U, V na koordynaty kartograficzne S, D

Conformal projection of Gauss sphere to the oblique tangent cone (S, D→ρ, ε)

Surface of Gauss sphere is conformally projected on the cone located in general position. The apex of the cone V is about 131 km away from the cartographic pole Q along the cone axis (Fig. 2). The radius of the Gaussian sphere is multiplied by a coefficient 0.999. This result in a favorable improvement of length distortion at the edge of the mapped area and at the same time projection doesn't lose its conformity.

Polar coordinates (ρ, ε) are calculated from the cartographic coordinates (S, D) on the surface of rolled out cone by equations [2, 6]

$$\rho = \rho_0 \left(\frac{\tan \frac{S_0}{2} + 45°}{\tan \frac{S}{2} + 45°} \right)^n , \tag{4a}$$

$$\varepsilon = n\,D , \tag{4b}$$

in which the cartographic constants take values for undistorted cartographic parallel of latitude $S_0 = 78°30'$:

$$\rho_0 = 0.9999 R \cot S_0 = 1\,298\,039.0046\ m , \tag{5}$$

$$n = \sin S_0 = 0.979\,924\,704\,621 . \tag{6}$$

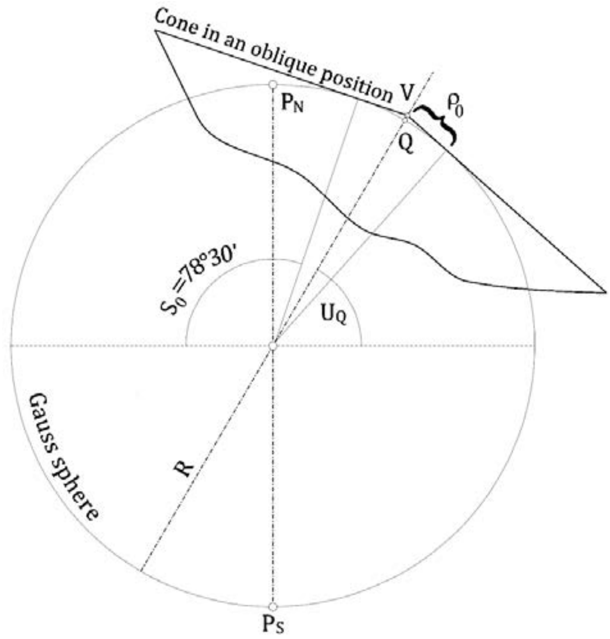

Fig. 2. Conformal projection of Gauss Sphere to the oblique tangent cone

Rys. 2. Projekcja formalna ze sfery Gaussa do stycznej do stożka skośnego

Transformation of the oblique cone to the S-JTSK grid plane (ρε→X, Y)

Transformation of polar coordinates to grid plane coordinates is the last step of Křovák projection. The projection of the prime cartographic meridian ($42°30'$ east of Ferro) in a southerly direction represents the axis X. Cartographical meridians are shown as a beam of straight lines centered at the apex of the cone V. In this way, the entire territory of Czechoslovakia was placed in the first quadrant, thus ensuring only positive grid coordinates.

$$X = \rho \cos \varepsilon , \tag{7a}$$

$$Y = \rho \sin \varepsilon . \tag{7b}$$

Length distortion

The value of length distortion is given in conformal projection of Bessel ellipsoid on the Gaussian sphere by the relationship [2, 6]

$$m = \frac{\alpha R \cos U}{N \cos \varphi} \tag{9}$$

and these values (9) are insignificant for the whole territory of the former Czechoslovakia ($\varphi_N = 51°03.5'$, $\varphi_S = 47°43.9'$). Symbol N denotes the transverse radius of curvature of Bessel ellipsoid. Therefore, the length distortion m is calculated in Křovák projection as a ratio of length element on map to the corresponding length element on the Gauss sphere with radius R [2]

$$m = \frac{n\rho}{R\cos S} \tag{10}$$

or using power series [2, 6, 9, 10, 14, 15]

$$m = 0.9999 + 0.00012282\Delta\rho^2 + \\ - 0.00000315\Delta\rho^3 + 0.00000018\Delta\rho^4 \tag{11}$$

where $\Delta\rho = \rho - \rho_0$ is entered into equation (10) in hundreds of kilometers [2, 6].

As results from relations (9) and (10) the relations for m are not directly functions of grid coordinates X and Y in the coordinate system S-JTSK (Fig. 3). Thus its calculation using a calculator in the fieldwork is quite tedious and inefficient. Geodetic surveying works are considerably prolonged without portable computers and relevant software [12, 14]. Therefore the aim derives a simplified relationship that length distortion can be calculated with sufficient accuracy directly from the grid coordinates.

As it's reflected in the relationships (9) and (10), the value of m is a function of distance $\rho = \sqrt{X^2 + Y^2}$ of point to the beginning of the coordinate system V.

$$m = g(\rho) = f(X, Y) \tag{12}$$

For the derivation of equation (12) the cartographic meridian is formed, on which nodal points are selected in increments $\Delta\rho = 5\,km$ (Fig. 4). Here

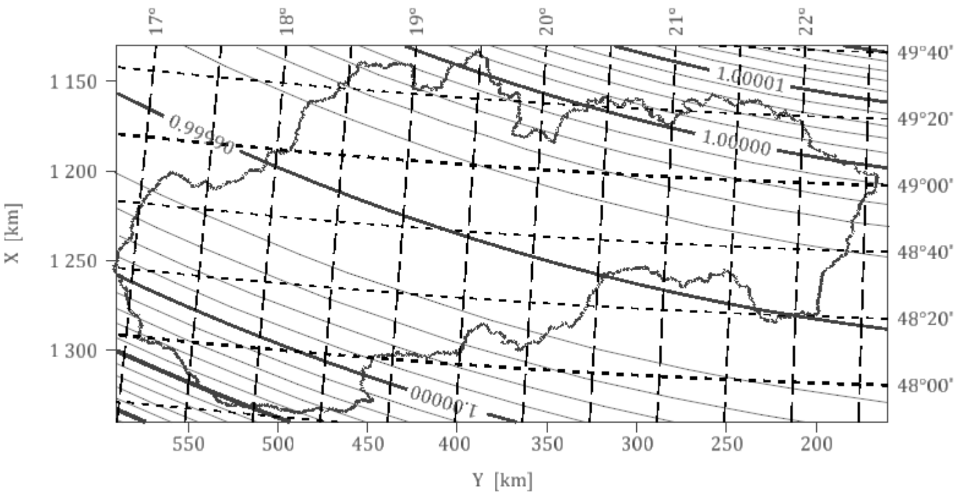

Fig. 3. Graphical representation of the length distortion in territory of Slovakia

Rys. 3. Graficzne przedstawienie zniekształcenia długości w terenie Słowacji

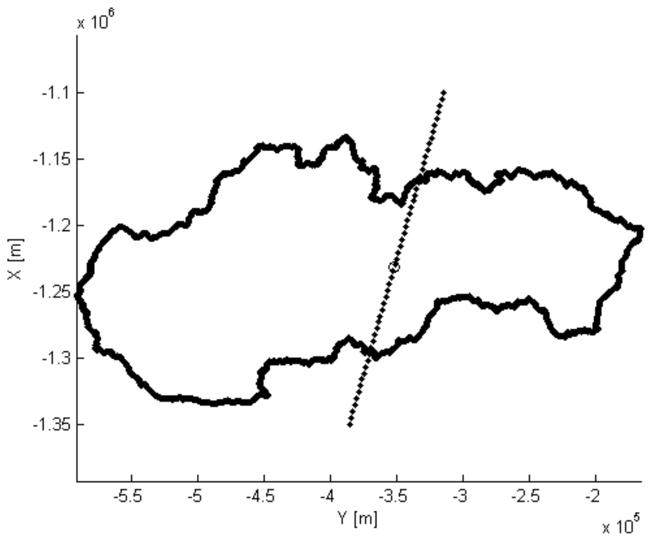

Fig. 4. Nodal points onto a specific cartographic meridian

Rys. 4. Punkty węzłowe dla południka kartograficznego was created 53 nodal points (Fig. 4) in the interval X∈<1100-1350 km> and Y∈<314-386 km>, where values were determined according to equation (10). As seen in Fig. 5, the approximate function for the m can be determined by a polynomial of the second degree [8]

$$m = 1.254621 \cdot 10^{-14}\left(X^2 + Y^2\right) + \\ - 3.259913 \cdot 10^{-8}\sqrt{X^2 + Y^2} + 1.02107493 \tag{13}$$

Differences of length distortion given by relations (10) and (13) $\Delta m = m_{(10)} - m$ are shown in Fig. 6 and in Tab. 1.

The accuracy of the regression model can be characterized by the root mean square error

$$RMSE = \sqrt{\frac{\sum_i^n e_i^2}{n}} = 1.216 \cdot 10^{-6}, \tag{14}$$

which is estimated from grid of $n = 970$ points with constant increment $\Delta X = \Delta Y = 10\ km$, $X \in (1130 - 1340\ km$, $Y \in (160 - 590\ km$, $e_i = m_{i(10)} - m_{i(13)}$.

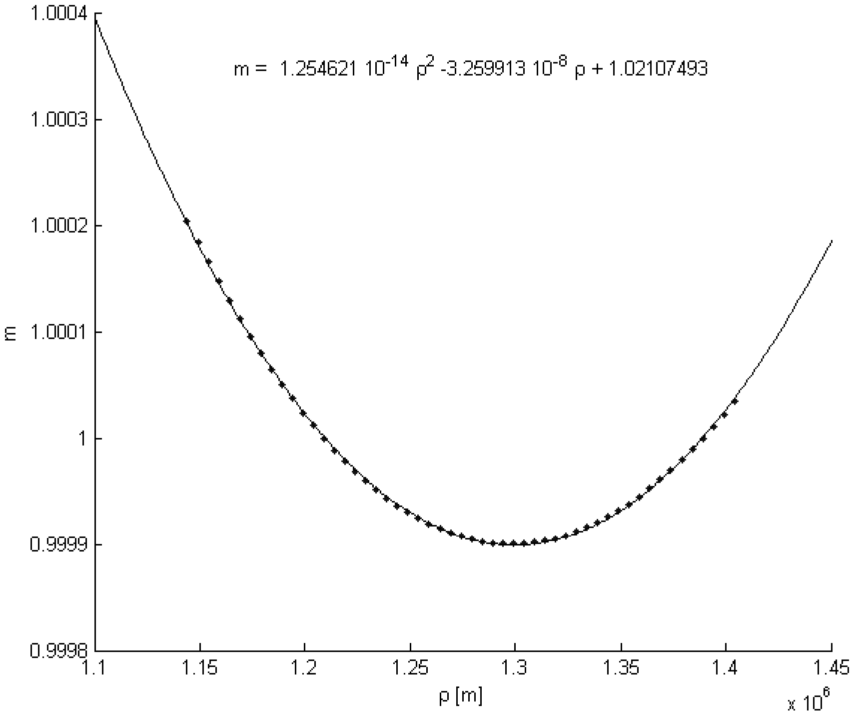

Fig. 5. Approximate function for the length distortion

Rys. 5. Przybliżona funkcja zniekształcenia długości

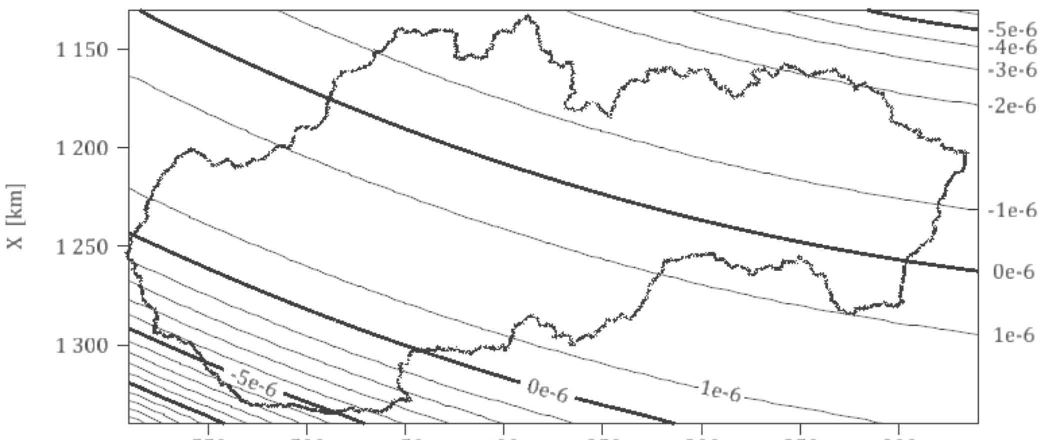

Fig. 6. Graphical representation of $\Delta m = m_{(10)} - m_{(13)}$

Rys. 6. Obraz graficzny $\Delta m = m_{(10)} - m_{(13)}$

Table 1. Values of the length distortions and influences of scale error

Tabela 1. Wartości zniekształceń długości i wpływ błędu skali

Point	m (10)	m (13)	Δm	$(m_{(10)}-1)\times 10^5$ [cm/km]	$(m_{(13)}-1)\times 10^5$ [cm/km]
1	1.000087573	1.000093162	–0.000005589	8.76	9.32
2	1.000037582	1.000040418	–0.000002836	3.76	4.04
3	0.999995267	0.999996130	–0.000000863	–0.47	–0.39
4	0.999960665	0.999960236	0.000000429	–3.93	–3.98
⋮	⋮	⋮	⋮	⋮	⋮
454	0.999900418	0.999899426	0.000000992	–9.96	–10.06
455	0.999902328	0.999901161	0.000001167	–9.77	–9.88
456	0.999906429	0.999905122	0.000001307	–9.36	–9.49
⋮	⋮	⋮	⋮	⋮	⋮
968	0.999916621	0.999917012	–0.000000391	–8.34	–8.30
969	0.999925894	0.999926585	–0.000000691	–7.41	–7.34
970	0.999931012	0.999931836	–0.000000824	–6.90	–6.82

Captions:

m The length distortion refers to equation (10) or (13)

$\Delta m = m_{(10)} - m_{(13)}$

$m - 1$ The influence of scale error

Meridian convergence

Meridian convergence C is angle, which is given by the cartographical projection of the meridian and parallel to the X axis (Fig. 7).

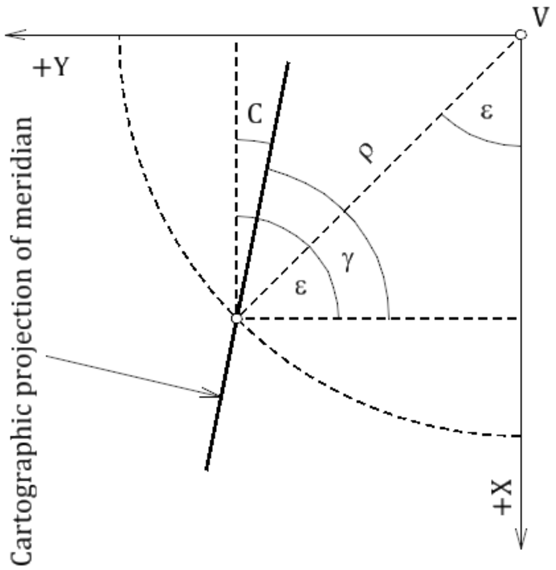

Fig. 7. Meridian convergence

Rys. 7. Meridian konwergencji

In the Křovák projection, the meridian convergence is calculated according the formula [2, 6]

$$C = \varepsilon - \gamma\,, \tag{15}$$

where:

$$\gamma = \sin a \sin D \sec U\text{ , or} \quad \text{(16a)}$$

$$\gamma = \sin a \sin \Delta V \sec S\,. \quad \text{(16a)}$$

Equation (15) is not directly function of grid coordinates X, Y, and therefore its calculation is tedious and impractical in a such cases as demanding staking-out works located in mountain localities or orientation measurements in condition of underground coal mines, where it is not possible to use portable computers from the security reasons [5]. As it's shown in Fig. 8, course of C inside of the area can be approximated by a system of parallel lines [1]. That is, the parameters a, b and c of the function (17) is necessary to determined

$$C = aY + b\frac{Y}{X} + c\,. \tag{17}$$

Least Squares estimation of unknown parameters is given by the solution of set of equations

$$\begin{pmatrix}\hat{a}\\ \hat{b}\\ \hat{c}\end{pmatrix} = \left(A^T A\right)^{-1} A^T C_{(15)}\,, \tag{18}$$

where:

vector $\underset{n,1}{C_{(15)}} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}$ consists of the values calculated

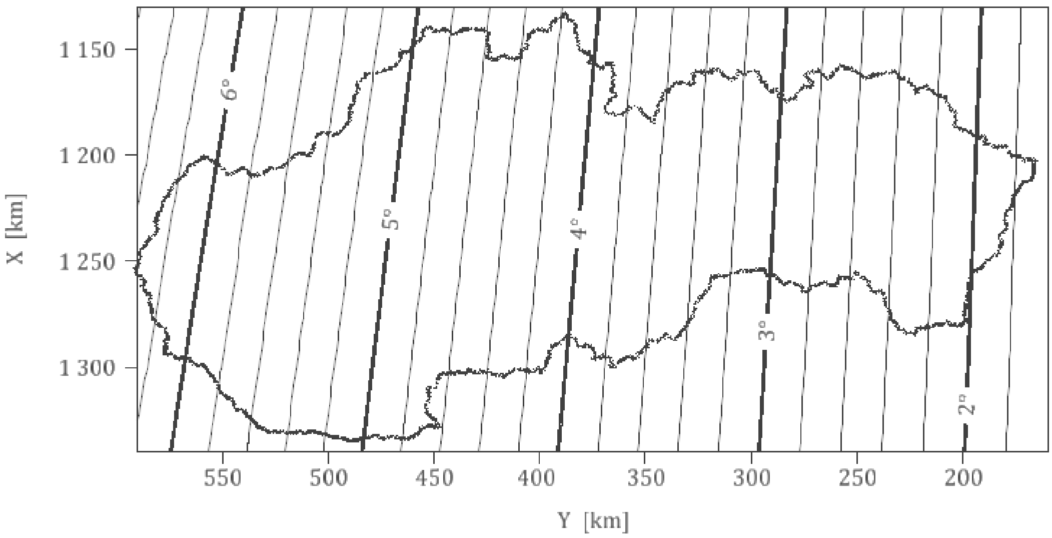

Fig. 8. Course of meridian convergence C

Rys. 8. Przebieg południka konwergencji C

according to equation (15) and $\underset{n,3}{A} = \begin{pmatrix} Y_1 & \frac{Y_1}{X_1} & 1 \\ Y_2 & \frac{Y_2}{X_2} & 1 \\ \vdots & \vdots & \vdots \\ Y_n & \frac{Y_n}{X_n} & 1 \end{pmatrix}$.

Estimation of the parameters $\hat{a}$, b and $\hat{c}$ from the equation (17) has been fixed from $n = 970$ points, which form a square grid with a regular spacing of 10×10 km.

Achieved accuracy $RMSE = 1.67368'$. Numerical quantification of the empirical relation (17) for the meridian convergence in radians and degrees is given the by equations (19) and (20).

$$C = 0.000128\,Y + 0.079164\frac{Y}{X} - 0.003181\,[rad] \quad (19)$$

$$C = 0.007355\,Y + 4.535749\frac{Y}{X} - 0.182248\,[°] \quad (20)$$

Graphical course of $\Delta C = C_{(15)} - C_{(20)}$ is in Table 2 and is shown in Fig. 9.

Conclusion

PDA, netbooks, laptops and other portable computing devices or computers that are directly integrated into the geodetic measuring instruments, are rapidly developed at current period and allow to solve numerically exacting tasks directly in the field. Their using leads to an increase in the effective and greater comfort of geodetic surveying fieldwork.

Table 2. Comparison of exact and approximated values of meridian convergence

Tabela 2. Porównanie dokładnych i przybliżonych wartości konwergencji południka

Point	$C_{(15)}$ [°]	$C_{(20)}$ [°]	ΔC [']
1	6.1345	6.1073	1.6301
2	6.1719	6.1389	1.9785
3	6.2104	6.1716	2.3282
⋮	⋮	⋮	⋮
331	3.2360	3.2599	-1.4340
332	3.2284	3.2503	-1.3153
333	3.2209	3.2410	-1.2021
⋮	⋮	⋮	⋮
833	1.6031	1.5361	4.0200
834	1.6493	1.6044	2.6938
969	5.9975	5.9570	2.4285
970	5.8752	5.8433	1.9112

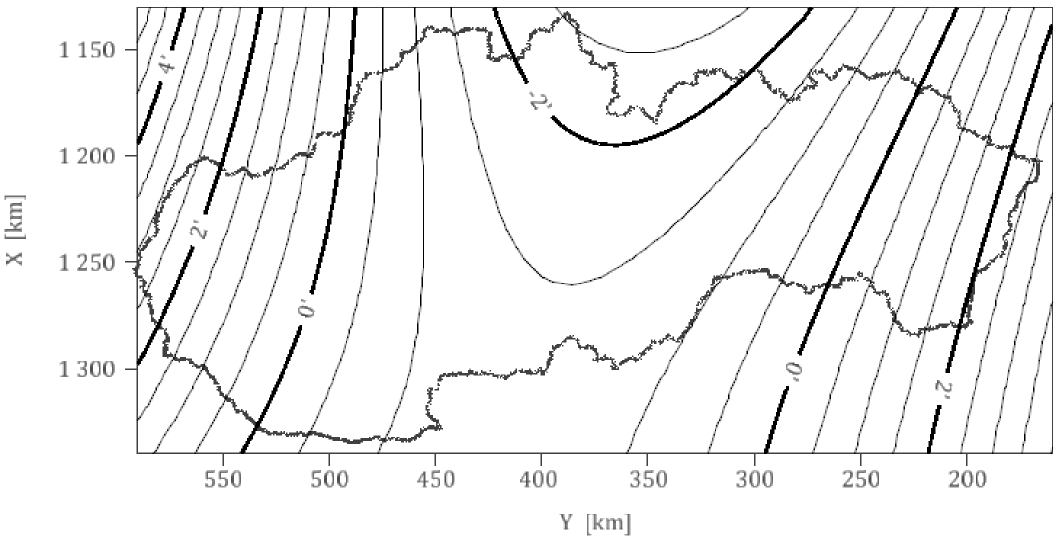

Fig. 9. Graphical course of ΔC

Rys. 9. Wykres ΔC

However, in practice, there are cases where the use of these devices is limited from different, primarily for safety reasons. For example, gaseous coal and lignite mines or petrochemical facilities and technological installations for the transport, storage and processing of oil and natural gas are typical examples of which may be encountered in practice. Exact relations for the quantifying of length distortion and meridian convergence in Křovák projection are quite difficult without the appropriate software and portable computers, with the result that geodetic surveying fieldwork are time-consuming. The paper presents a simplified relations derived for territory of the Slovak Republic, which are a directly functions of grid coordinates X, Y and in the case of compliance of relevant tolerances they can be simply used.

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Wyprowadzenie uproszczonych wzorów do obliczeń konwergencji południka i skali błędu w projekcji Křovák

Obowiązkowym systemem odniesienia dla prowadzenia czynności geodezyjnych w Czechach i na Słowacji jest system zwany „Súradnicový systém Jednotnej trigonometrickej siete katastrálnej" (S-JTSK), którego podstawą matematyczną jest część Křovák. Punkty na powierzchni topograficznej są rzutowane na elipsoidy odniesienia Bessela, a siatki współrzędnych na płaszczyźnie projekcji są uzyskiwane przez cztery kolejne przekształcenia. Ponieważ matematyczne podstawy projekcji są dość skomplikowane, zależności do oszacowania zniekształceń długości południka konwergencji są również dość skomplikowane i są one uciążliwe dla pracy w terenie. Artykuł przedstawia uproszczone przybliżone równania dla zniekształceń długości i konwergencji południka, które są odpowiednie do użycia w terenie lub w warunkach podziemnych, gdzie zastosowanie komputera przenośnego może być ograniczone przez względy bezpieczeństwa lub inne.

Słowa kluczowe: konwergencja południka, zniekształcenie długości, projekcja Křovák