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PROCESSOR SHARING QUEUEING SYSTEMS WITH NON-HOMOGENEOUS CUSTOMERS

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Abstract. We investigate processor sharing queueing systems with non-homogeneous customers having some random space requirements. Such systems have been used to model and solve various practical problems occurring in the design of computer or communicating systems. The above non-homogenity means that each customer (independently of others) has some random space requirement and his length (or amount of work for his service) generally depends on the space requirement. In real systems, a total sum of space requirements of customers presenting in the system is limited by some constant value (memory capacity) $V > 0$. We estimate loss characteristcs for such a system using queueing models with unlimited memory space.

1. Introduction

Egalitarian processor sharing (EPS) systems are used for modeling of computer and communicating networks [1]. Presently, they are applicable to situations where a common resource is shared by a varying number of concurrent users [2] (for example, to WEB-servers modeling [3]).

The EPS discipline was first introduced by Kleinrock [4] as a limiting case for modeling time sharing systems. The aim of the paper is to analyze classical and non-classical EPS systems. First, we shall analyze the classical EPS system notated by $M/G/1-EPS$. All the customers present in the classical $M/G/1-EPS$ system are served simultaneously. If there are $n > 1$ customers in the system at an arbitrary instant, then all of them are served at this instant n times slowly than in the case of $n = 1$.

Later on, the customer length means the amount of work necessary for customer's service, i.e. the service time under condition that there are no other customers in the system during his presence in it. Analogously, the residual length of the customer means his residual service time after some time instant under the same condition (see [2]).

We introduce the following additional assumption for the classical $M/G/1-EPS$ system. Assume that each customer is characterized by some non-negative random capacity. This random variable can be interpreted as a part of system's memory space used by the customer during his presence in the system. A total sum of customer capacities $\sigma(t)$ in the system at arbitrary time t is referred as the total customers capacity.

The random value $\sigma(t)$ can be limited by some constant value V $(0 < V < \infty)$, which is called the memory volume of the system. In this case we have a non-classical processor sharing system that will be notated by $M/G/1(V)-EPS$.

The purpose of the paper is

1) to obtain the non-stationary and stationary distribution of total customers capacity in the system $M/G/1-EPS$;

2) to determine some estimations of loss characteristics for systems $M/G/1(V)-EPS$ with limited memory space $(V < \infty)$ based on the model with unlimited one;

3) to compare processor sharing systems $M/G/1(V) - EPS$ and $M/G/1-EPS$ from the viewpoint of estimation of loss characteristics.

2. Classical processor sharing system

In this section we investigate the classical system $M/G/1-EPS$. Denote by $\eta(t)$ the number of customers present in the system at time t and by $\xi_i^*(t)$ the residual length of the ith customer at this time, $i=\overline{1,\eta(t)}$. Let

$$F(x,t)=\mathbf{P}\{\zeta < x, \xi < t\}$$

be the joint distribution function of the customer capacity ζ and his length ξ (we assume that customer's capacity and his length do not depend on his arrival time and on characteristics of other customers). Then $L(x)=F(x,\infty)$ and $B(t)=F(\infty,t)$ are the distribution functions of the random variables ζ and ξ, respectively. Let a be an arrival rate of entrance flow of customers,

$$\alpha(s,q)=\int_0^\infty\int_0^\infty e^{-sx-qt}dF(x,t)$$

be the double Laplace-Stieltjes transform (with respect to x and t) of the distribution function $F(x,t)$, $\varphi(s)=\alpha(s,0)$, and $\beta(q)=\alpha(0,q)$ be the Laplace-Stieltjes transform of the distributin functions $L(x)$ and $B(t)$, respectively.

$D(x,t) = \mathbf{P}\{\sigma(t) < x\}$ is the distribution function of total customers capacity at time t,

$$\delta(s,t) = \mathbf{E}e^{-s\sigma(t)} = \int_0^\infty e^{-sx} d_x D(x,t)$$

is the Laplace-Stieltjes transform of the function $D(x,t)$ with respect to x,

$$\overline{\delta}(s,q) = \int_0^\infty e^{-qt}\mathbf{E}e^{-s\sigma(t)}dt = \int_0^\infty e^{-qt}\delta(s,t)dt$$

is the Laplace transform of the function $\delta(s,t)$ with respect to t.

The mixed $(i+j)$th moments of the random variables ζ and ξ (if they exist) take the form:

$$\alpha_{i\,j} = \mathbf{E}(\zeta^i\xi^j) = (-1)^{i+j}\frac{\partial^{i+j}}{\partial s^i \partial q^j}\alpha(s,q)\Big|_{s=0,q=0}.$$

Assume that customers in the considered system at an arbitrary time t are numerated as random; i.e. if the number of customers is k, then there are $k!$ ways to enumerate them, and each enumeration can be chosen with the same probability $1/k!$.

One can easily show that the system under consideration is described by the Markov process

$$(\eta(t), \xi_i^*(t), i = \overline{1, \eta(t)}), \tag{1}$$

where components $\xi_i^*(t)$ are absent if $\eta(t) = 0$. In this case we also have $\sigma(t) = 0$.

In what follows, to simplify the notation, we denote $Y_k = (y_1, \ldots, y_k)$. Sometimes in the case $k = 1$, instead of Y_1 we write y_1 or the value that this component takes, and in the case $k = 2$, instead of Y_2 we write (y_1, y_2) or their values. In other words, we sometimes specify vectors of small dimensions by indicating their components. We also use the notation $(y_1, \ldots, y_k, u) = (Y_k, u)$.

We characterize the process (1) by functions with the following probabilistic sense:

$$P_0(t) = \mathbf{P}\{\eta(t) = 0\}; \tag{2}$$

$$\Theta_k(Y_k, t) = \mathbf{P}\{\eta(t) = k, \xi_j^*(t) < y_j, j = \overline{1,k}\},\ k = 1, 2, \ldots; \tag{3}$$

$$P_k(t) = \mathbf{P}\{\eta(t) = k\} = \Theta_k(\infty_k, t),\ k = 1, 2, \ldots, \tag{4}$$

where $\infty_k = (\infty, \ldots, \infty)$ is a k-component vector.

Note that the functions $\Theta_k(Y_k, t)$ are symmetric with respect to permutations of components of the vector Y_k due to our random enumeration of customers in the system.

Let us determine the function $\overline{\delta}(s,q)$ under zero initial condition $\eta(0) = \sigma(0) = 0$.

Denote by $\overline{p}_0(q) = \int_0^\infty e^{-qt}P_0(t)dt$ and $\overline{\theta}_k(Y_k,q) = \int_0^\infty e^{-qt}\Theta_k(Y_k,t)dt$ the Laplace transforms with respect to t of the functions $P_0(t)$ and $\Theta_k(Y_k,t)$, respectively. It is known (see [2]) that

$$\overline{p}_0(q) = [q + a - a\pi(q)]^{-1} \tag{5}$$

under zero initial condition, where $\pi(q)$ is the Laplace-Stieltjes transform of the busy period distribution function for the system under consideration. Note [2] that $\pi(q)$ is a unique solution of the functional equation $\pi(q) = \beta(q + a - a\pi(q))$ such that $|\pi(q)| \le 1$.

Lemma 1. *Under zero initial condition, the functions $\overline{\theta}_k(Y_k,q)$, where $k = 1,2,\dots$, have the following form:*

$$\overline{\theta}_k(Y_k,q) = \overline{p}_0(q)\prod_{i=1}^{k}\int_0^{y_i}[q + a - aB(u)]du.$$

Proof. Using the method of auxiliary variables [5] and taking into account the symmetric property of the functions $\Theta_k(Y_k,t)$, we can write out partial differential equations for functions (3):

$$\frac{\partial\Theta_1(y,t)}{\partial t} - \frac{\partial\Theta_1(y,t)}{\partial y} + \left.\frac{\partial\Theta_1(y,t)}{\partial y}\right|_{y=0} = aP_0(t)B(y) - a\Theta_1(y,t) + \\ + \left.\frac{\partial\Theta_2(y,u,t)}{\partial u}\right|_{u=0}; \tag{6}$$

$$\frac{\partial\Theta_k(Y_k,t)}{\partial t} - \frac{\partial\Theta_k(Y_k,t)}{\partial y_k} + \left.\frac{\partial\Theta_k(Y_k,t)}{\partial y_k}\right|_{y_k=0} = a\Theta_{k-1}(Y_{k-1},t)B(y_k) - \\ - a\Theta_k(Y_k,t) + \left.\frac{\partial\Theta_{k+1}((Y_k,u),t)}{\partial u}\right|_{u=0}, \; k = 2,3,\dots. \tag{7}$$

Passing to Laplace transform in the equations (6), (7), we obtain

$$-\frac{\partial\overline{\theta}_1(y,q)}{\partial y} = a\overline{p}_0(q)B(y) - (q+a)\overline{\theta}_1(y,q) - \left.\frac{\partial\overline{\theta}_1(y,q)}{\partial y}\right|_{y=0} + \\ + \left.\frac{\partial\overline{\theta}_2(y,u,q)}{\partial u}\right|_{u=0}; \tag{8}$$

$$-\frac{\partial\overline{\theta}_k(Y_k,q)}{\partial y_k} = a\overline{\theta}_{k-1}(Y_{k-1},q)B(y_k) - (q+a)\overline{\theta}_k(Y_k,q) -$$

$$-\frac{\partial\overline{\theta}_k(Y_k,q)}{\partial y_k}\bigg|_{y_k=0}+\frac{\partial\overline{\theta}_{k+1}((Y_k,u),q)}{\partial u}\bigg|_{u=0},\ k=2,3,\dots. \tag{9}$$

By direct substitution, we can prove that the solution of Eqs. (8) and (9) has the form

$$\overline{\theta}_k(Y_k,q)=C(q)\prod_{i=1}^{k}\int_0^{y_i}[q+a-aB(u)]du, \tag{10}$$

where $C(q)$ is some function that can be determined if we substitute the relation (10) into Eq. (8). Then, we have $C(q)=\overline{p}_0(q)$.

The lemma is proved.

Let $\beta_i=\mathbf{E}\xi^i=(-1)^i\beta^{(i)}(0)$ be the ith moment of the customer length, $i=1,2,\dots$.

Corollary 1. *If $\rho=a\beta_1<1$, then the limits $\theta_k(Y_k)=\lim\limits_{t\to\infty}\Theta_k(Y_k,t)$, $k=1,2,\dots$, exist being independent of initial condition and have the form:*

$$\theta_k(Y_k)=(1-\rho)a^k\prod_{i=1}^{k}\int_0^{y_i}[1-B(u)]du.$$

Proof. If $\rho<1$, then the process (1) is regenerative with points of regeneration coinciding with epochs of termination of busy periods. It follows from the theory of regenerative processes [6] that the limit $\lim\limits_{t\to\infty}\Theta_k(Y_k,t)=$ $=\theta_k(Y_k)$ exists and

$$\theta_k(Y_k)=\lim_{q\to 0}q\overline{\theta}_k(Y_k,q)=(1-\rho)a^k\prod_{i=1}^{k}\int_0^{y_i}[1-B(u)]du.$$

Corollary 2. *Let $\overline{p}_k(q)$ be the Laplace transform of the function $P_k(t)$, $k=0,1,\dots$, under zero initial condition. Then we have*

$$\overline{p}_k(q)=\frac{a^k(1-\pi(q))^k}{(q+a-a\pi(q))^{k+1}}.$$

Proof. It is obvious that $\overline{p}_k(q)=\overline{\theta}_k(\infty_k,q)$. Let us prove the equality

$$\int_0^{\infty}(q+a-aB(y))dy=\frac{a(1-\pi(q))}{q+a-a\pi(q)}. \tag{11}$$

It follows from the normalization condition written in terms of Laplace transforms that

$$\overline{p}_0(q)+\sum_{k=1}^{\infty}\overline{\theta}_k(\infty_k,q)=1/q,$$

whence, taking into account the result of lemma 1, we obtain:

$$1+\sum_{k=1}^{\infty}\left[\int_0^{\infty}(q+a-aB(y))dy\right]^k=\frac{1}{q}[q+a-a\pi(q)].$$

From the last relation we have (11). Now, the statement of the corollary follows from formulae (5) and (10).

From corollary 1 we can obtain the known relation for the stationary distribution $\{p_k\}$ of the number of customers in the system ($\rho=a\beta_1<1$)[2]:

$$p_k=\theta_k(\infty_k)=(1-\rho)\rho^k,\ k=0,1,\dots.$$

Let $\chi(t)$ be the capacity of a customer being on service at the time t and $\xi^*(t)$ be the residual length of this customer at the time t. We shall use the notation $E_y(x)=\mathbf{P}\{\chi(t)<x|\,\xi^*(t)=y\}$. It is known [7] that the Laplace–Stieltjes transform of the conditional distribution function $E_y(x)$ has the form:

$$e_y(s)=[1-B(y)]^{-1}\int_{x=0}^{\infty}e^{-sx}\int_{u=y}^{\infty}dF(x,u). \tag{12}$$

We introduce the notation

$$d_{Y_k}\Theta_k(Y_k,t)=\mathbf{P}\{\eta(t)=k,\xi_i^*(t)\in[y_i,y_i+dy_i),i=\overline{1,k}\}=$$
$$=\frac{\partial^k\Theta_k(Y_k,t)}{\partial y_1\dots\partial y_k}dy_1\dots dy_k.$$

Later on, we use the notation $\overset{k}{\underset{i=1}{*}}R_i(x)$ for Stieltjes convolution of distribution functions $R_i(x)$, $i=1,2,\dots$, $R_i(x)=0$, if $x\le 0$.

Theorem 1. *For zero initial condition, the function* $\overline{\delta}(s,q)$ *is determined by the relation*

$$\overline{\delta}(s,q)=\{[q+a-a\pi(q)][1-I(s,q)]\}^{-1},$$

where

$$I(s,q)=\int_0^{\infty}(q+a-aB(y))e_y(s)dy$$

and $e_y(s)$ *is determined by relation* (12).

Proof. The distribution function $D(x,t)$ can be represented as

$$D(x,t)=P_0(t)+$$
$$+\sum_{k=1}^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\mathbf{P}\{\sigma(t)<x|\,\eta(t)=k,\xi_i^*(t)=y_i,i=\overline{1,k}\}d_{Y_k}\Theta_k(Y_k,t).$$

From the random enumeration of components of the vector Y_k it is obvious that

$$\mathbf{P}\{\sigma(t) < x |\, \eta(t) = k, \xi_i^*(t) = y_i, i = \overline{1,k}\} = \mathop{*}_{i=1}^{k} E_{y_i}(x).$$

Then we get:

$$D(x,t) = P_0(t) + \sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \mathop{*}_{i=1}^{k} E_{y_i}(x) d_{Y_k} \Theta_k(Y_k, t).$$

Passing in the last relation to Laplace–Stieltjes transform with respect to x, we have:

$$\delta(s,t) = P_0(t) + \sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{k} e_{y_i}(s) d_{Y_k} \Theta_k(Y_k, t).$$

Passing to Laplace transform with respect to t, we obtain:

$$\overline{\delta}(s,q) = \overline{p}_0(q) + \sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{k} e_{y_i}(s) d_{Y_k} \overline{\theta}_k(Y_k, q),$$

where $d_{Y_k}\overline{\theta}_k(Y_k,q) = \overline{p}_0(q) \prod_{i=1}^{k} [q + a - aB(y_i)] dy_i$ (it follows from Eq. (8) and the relation $C(q) = \overline{p}_0(q)$). Then we get:

$$\overline{\delta}(s,q) = \overline{p}_0(q) + \overline{p}_0(q) \sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{k} e_{y_i}(s)[q + a - aB(y_i)] dy_i =$$

$$= \overline{p}_0(q) \left\{ 1 + \sum_{k=1}^{\infty} \left[\int_0^{\infty} (q + a - aB(y)) e_y(s) dy \right]^k \right\} =$$

$$= \overline{p}_0(q) \left[1 + \sum_{k=1}^{\infty} (I(s,q))^k \right], \qquad (13)$$

where

$$I(s,q) = \int_0^{\infty} (q + a - aB(y)) e_y(s) dy.$$

Now, the statement of the theorem follows from formula (13).

Corollary 3. *If the random variables ζ and ξ are independent, we obtain:*

$$\overline{\delta}(s,q) = [q + a(1 - \pi(q))(1 - \varphi(s))]^{-1}. \qquad (14)$$

Proof. In this case, taking into account Equation (12) and the relation $F(x,t) = L(x)B(t)$, we have that

$$I(s,q) = \int_{y=0}^{\infty} \frac{q+a-aB(y)}{1-B(y)} \int_{x=0}^{\infty} e^{-sx} \int_{u=y}^{\infty} dF(x,u) =$$

$$= \varphi(s) \int_0^{\infty} [q+a-aB(y)]dy = \frac{a\varphi(s)(1-\pi(q))}{q+a-a\pi(q)},$$

whence relation (14) follows.

Corollary 4. *Under zero initial condition, the Laplace transform $g(s,q)$ with respect to t of generation function $P(z,t) = \sum_{k=0}^{\infty} P_k(t)z^k$, $|z| \le 1$, of the customers number in the system at time t has the following form:*

$$g(z,q) = \int_0^{\infty} e^{-qt} P(z,t)dt = [q + a(1-z)(1-\pi(q))]^{-1}. \tag{15}$$

Proof. It follows from corollary 1 that in the case when the customer length does not depend on his capacity and the capacity is equal to 1, we have $\varphi(s) = e^{-s}$ and

$$\overline{\delta}(s,q) = [q + a(1-\pi(q))(1-e^{-s})]^{-1} =$$

$$= \int_0^{\infty} e^{-qt} \mathbf{E} e^{-s\sigma(t)} dt = \int_0^{\infty} e^{-qt} P(e^{-s}, t)dt,$$

whence Eq. (15) follows if we substitute e^{-s} by z.

Corollary 5. *Let $\rho = a\beta_1 < 1$. Then stationary mode exists. The Laplace–Stieltjes transform $\delta(s)$ of the stationary distribution function $D(x) = \lim_{t\to\infty} D(x,t)$ of customers total capacity has the form:*

$$\delta(s) = \frac{1-\rho}{1 + a\alpha'_q(s,q)|_{q=0}}. \tag{16}$$

Note that relation (16) was first obtained by Sengupta [8].

Proof. It follows from the theory of regenerative processes [6] that the limit $\delta(s) = \lim_{t\to\infty} \delta(s,t)$ exists and

$$\delta(s) = \lim_{q\to 0} q\overline{\delta}(s,q) = (1-\rho) \lim_{q\to 0} [1 - I(s,q)]^{-1},$$

where, as it follows from theorem 1,

$$\lim_{q\to 0} I(s,q) = a\int_0^\infty [1-B(y)]e_y(s)dy =$$

$$= a\int_{x=0}^\infty \int_{u=0}^\infty ue^{-sx}dF(x,u) = -a\alpha'_q(s,q)|_{q=0},$$

whence the statement of the corollary follows.

Corollary 6. *Let $\delta_1(t)$ be the first moment of the total customers capacity $\sigma(t)$ under zero initial condition, $\overline{\delta}_1(q)$ be the Laplace transform of the function $\delta_1(t)$. Then we have:*

$$\overline{\delta}_1(q) = \frac{a\alpha_{11} + q\int_0^\infty \int_0^\infty xS(t)dF(x,t)}{[q+a-a\pi(q)]\left[1-\rho-q\int_0^\infty S(t)dB(t)\right]^2},$$

where $S(t) = \int_0^t [1-B(y)]^{-1}dy.$

Let σ be a stationary total customers capacity ($\sigma(t) \Rightarrow \sigma$ in the sense of a weak convergence). The following known formulae [8]

$$\delta_1 = \mathbf{E}\sigma = -\delta'(0) = \frac{a\alpha_{11}}{1-\rho},\ \delta_2 = \mathbf{E}\sigma^2 = \delta''(0) = \frac{a\alpha_{21}}{1-\rho} + 2\delta_1^2 \qquad (17)$$

can be obtained from relation (16).

For some special cases we can get the distribution function $D(x)$ from formula (16). For example, consider the case when customer's capacity ζ and his length ξ are connected by the relation $\xi = c\zeta + \xi_1$, $c > 0$, where the random variables ζ and ξ_1 are independent (such dependence for customer's capacity and his length is true for many real information systems).

Denote by $\kappa_1 = \mathbf{E}\xi_1$ the first moment of the random variable ξ_1. In this case we have $\alpha(s,q) = \varphi(s+cq)\kappa(s)$, where $\kappa(s)$ is the Laplace–Stieltjes transform of the distribution function of the random variable ξ_1. Then relation (16) takes the following form:

$$\delta(s) = \frac{1-\rho}{1+a[c\varphi'(s)-\kappa_1\varphi(s)]}. \qquad (18)$$

Assume that customer capacity ζ has an exponential distribution with the parameter $f > 0$. Then from formula (18) we obtain:

$$\delta(s) = \frac{(1-\rho)(s+f)^2}{(s+f)^2 - \rho_1 f^2 - \rho_2 f(s+f)},$$

where $\rho_1 = ac/f$, $\rho_2 = a\kappa_1$, so that $\rho = a\beta_1 = \rho_1 + \rho_2$.

Now we can determine the inverse Laplace transform of $\delta(s)/s$, where $\delta(s)$ is defined by formula (18), and obtain the stationary distribution function $D(x)$:

$$D(x) = 1 - \frac{(1-\rho)e^{-fx}}{2b}\left[\frac{(\rho_2+b)^2 e^{(\rho_2+b)fx/2}}{2-\rho_2-b} - \frac{(\rho_2-b)^2 e^{(\rho_2-b)fx/2}}{2-\rho_2+b}\right], \quad (19)$$

where $b = \sqrt{\rho_2^2 + 4\rho_1}$.

3. Estimation of loss characteristics

The $M/G/1-EPS$ is a system without losing of customers ($V=\infty$). But with the help of this model we can estimate the memory capacity V in order to guarantee inexceeding of given loss probability.

Assume that we have a stationary queueing system Q_∞ with Poisson entrance flow without losses of customers. Let Q_V be a stationary system that differs from Q_∞ only with the fact that its total capacity is limited by the constant value V. We denote by $D(x)$ the distribution function of total customers capacity for the system Q_∞ and by $D_V(x)$ the distribution function of this random value for the system Q_V.

Theorem 2. *The inequality $D(x) \leq D_V(x)$ takes place for all $x > 0$.*

Proof of the theorem can be found in [7].

It follows from theorem 2 that the loss probability P for the system Q_V satisfies the following inequality [7]:

$$P = 1 - \int_0^V D_V(V-x)dL(x) \leq 1 - \int_0^V D(V-x)dL(x) = P^*. \quad (20)$$

Thus, the value P^* is an upper estimation of loss probability for the system Q_V. If we choose V under condition that P^* is given so that the equality

$$\int_0^V D(V-x)dL(x) = 1 - P^*$$

is satisfied, then the real loss probability P does not exceed P^*. If only very rare losses are permitted in the system under consideration, the difference between the values P and P^* is inessential.

Note that the loss probability is not exhaustive characteristic of losses, because its value shows a part of lost customers, not a part of lost capacity or, in other words, information being lost. Really, it is obvious that customers having large capacity will be lost more often. Therefore, more objective losses estimation is the value

$$Q = 1 - \frac{1}{\varphi_1}\int_0^V xD_V(V-x)dL(x).$$

The value Q is the probability of losing a unit of customer capacity. The next inequality follows from theorem 2:

$$Q = 1 - \frac{1}{\varphi_1}\int_0^V xD_V(V-x)dL(x) \le 1 - \frac{1}{\varphi_1}\int_0^V xD(V-x)dL(x) = Q^*.$$

If only very rare losses are permitted in the system under consideration, the difference between the values Q and Q^* is inessential.

For example, in the case of the distribution function (19) we obtain:

$$P^* = \left\{1 - \frac{1-\rho}{b}\left[a_1\frac{1-e^{-(1-b_1)fV}}{b+\rho_2} + a_2\frac{1-e^{-(1-b_2)fV}}{b-\rho_2}\right]\right\}e^{-fV},$$

where $a_1 = \dfrac{(\rho_2+b)^2}{2-\rho_2-b}$, $a_2 = \dfrac{(\rho_2-b)^2}{2-\rho_2+b}$, $b_1 = -1 + \dfrac{\rho_2+b}{2}$, $b_2 = -1 + \dfrac{\rho_2-b}{2}$;

$$Q^* = \left\{1 + fV - \frac{2(1-\rho)}{b}\left[\frac{(a_1+a_2)fV}{8\rho_1} + \right.\right.$$

$$\left.\left. + a_1\frac{1-e^{-(1-b_1)fV}}{(b+\rho_2)^2} - a_2\frac{1-e^{-(1-b_2)fV}}{(b-\rho_2)^2}\right]\right\}e^{-fV}.$$

Note that in the most cases the calculation and estimation of the probability Q is very complicated. Therefore, we often must restrict ourselves to the calculation and estimation of the loss probability P.

If it is impossible to determine the form of the distribution function $D(x)$, we can estimate the value P^* by approximation of the function

$$\Phi(x) = \int_0^x D(x-u)dL(u)$$

being the distribution function of the sum of independent random variables σ and ζ, with the distribution function of the gamma distribution $\Phi^*(x) = \gamma(h, rx)/\Gamma(h)$, where $\gamma(h, rx) = \int_0^{hx} t^{h-1}e^{-t}dt$ is the incomplete gamma function, $\Gamma(h) = \gamma(h, \infty)$ is the gamma function. The parameters h and r of the approximate distribution should be chosen so that its first and second moments $f_1^* = h/r$ and $f_2^* = h(h+1)/r^2$ should be equal to the first and second moments of the distribution function $\Phi(x)$, respectively. It is obvious that these moments have the form

$$f_1 = \delta_1 + \varphi_1,\ f_2 = \delta_2 + \varphi_2 + 2\delta_1\varphi_1. \tag{21}$$

Thus, the parameters of the distribution function $\Phi^*(x)$ should be chosen as follows:

$$h = \frac{f_1^2}{f_2 - f_1^2}, \quad r = \frac{f_1}{f_2 - f_1^2},$$

where f_1 and f_2 can be calculated from relations (17), (21). Hence, we have the approximate formula

$$P^* \cong 1 - \Phi^*(V).$$

Note that in the case of not very small permissible loss probabilities, using the estimation P^* instead of P leads to unjustifiably surplus choice of the capacity volume V. Therefore, the direct analysis of processor sharing systems with limited memory space is very important.

4. The case of limited total capacity

The system $M/G/1(V) - EPS$ with customers of different types was analyzed in detail in [9, 10]. We shall concider a special case of customers of the same type. Then, for stationary probabilities of number of customers present in the system we have:

$$p_0 = \left(\sum_{k=0}^{\infty} a^k A_*^{(k)}(V) \right)^{-1}, \; p_k = p_0 a^k A_*^{(k)}(V), \; k = 1, 2, \dots,$$

where $A_*^{(k)}(x)$ is a kth order Stieltjes convolution of the function

$$A(x) = \int_{u=0}^{x} \int_{t=0}^{\infty} u dF(u, t).$$

The loss probability has the form:

$$P = 1 - p_0 \left[L(V) - \sum_{k=1}^{\infty} a^k A_*^{(k)}(V) \right].$$

Assume additionally that customer capacity has an exponential distribution with parameter f, and let the customer length be proportional to his capacity ($\xi = c\zeta$, $c > 0$). Then, after some calculations we obtain

$$p_0 = \begin{cases} \dfrac{1-\rho}{1 - \sqrt{\rho} e^{-fV} \left[\sinh(\sqrt{\rho} fV) + \sqrt{\rho} \cosh(\sqrt{\rho} fV) \right]}, & \text{if } \rho \neq 1, \\[2ex] \dfrac{1 + e^{-2fV}}{1 + fV}, & \text{if } \rho = 1; \end{cases}$$

$$p_k = p_0\rho^k \left[1 - e^{-fV} \sum_{i=0}^{2k-1} \frac{(fV)^i}{i!}\right], \ k = 1, 2, \ldots ;$$

$$P = p_0 e^{-fV} \cosh(\sqrt{\rho} fV),$$

where $\rho = ac/f$.

Table 1: Probabilities P and Q for $\rho = 0.6$

V	P^*	Q^*	P	Q
0.0	1.00000	1.00000	1.00000	1.00000
0.2	0.92721	0.99569	0.81994	0.98269
0.4	0.86622	0.98366	0.67754	0.94034
0.6	0.81392	0.96529	0.56700	0.88482
0.8	0.76815	0.94194	0.48156	0.82409
1.0	0.72735	0.91487	0.41516	0.76311
2.0	0.56855	0.75562	0.23586	0.51290
3.0	0.45178	0.60242	0.15775	0.35596
4.0	0.35651	0.47628	0.11281	0.25640
5.0	0.28750	0.37679	0.08340	0.18993
6.0	0.22947	0.29888	0.06291	0.14330
7.0	0.18316	0.23763	0.04811	0.10963
8.0	0.14620	0.18925	0.03716	0.08464
10.0	0.09314	0.12034	0.02263	0.05165
15.0	0.03018	0.03896	0.00697	0.01589
20.0	0.00978	0.01262	0.00222	0.00512
30.0	0.00103	0.00133	0.00023	0.00054
40.0	0.00011	0.00014	0.00002	0.01589
50.0	0.00001	0.00002	0.00000	0.00001

Now we can compare the values P^* and P or Q^* and Q using analytical results and simulation. Table 1 presents the dependence of loss characteristics upon the memory capacity V. Here we assume that $\rho = 0.6$, the customer length is proportional to his capacity ($\xi = c\zeta$), where $c = 1$, and capacity ζ has an exponential distribution with parameter $f = 1$.

The values P^*, Q^*, P were obtained by calculation from the above relations, whereas the value Q was estimated by simulation. The table shows that estimators P^*, Q^* are not very precise, and we can use them for the case when the proper loss characteristics are near zero.

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