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Free Vibrations of Column Subjected to Euler's Load with Consideration of Timoshenko's Theory

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Abstract

In this paper the single-rod cantilever column subjected to compressive Euler's load is investigated. The boundary problem has been formulated on the basis of Hamilton's principle and Timoshenko's theory. Numerical simulations of characteristic curves have been plotted on the plane external load-vibration frequency for different magnitudes of slenderness factor of the system. The results of numerical calculations of Timoshenko's beam are compared to the ones obtained from mathematical Bernoulli-Euler's model. The comparison of the results of characteristic curves calculated by means of Timoshenko's theory and Bernoulli-Euler's model are done for first three vibration frequencies.

Keywords: column, Timoshenko's theory, Bernoulli-Euler's theory, free vibrations, kinetic criterion of stability, divergence instability, characteristic curves

1. Introduction

The results of numerical calculations of supporting systems subjected to external loads of various types are often presented in the form of characteristic curves (see [5-8]). By means of these curves the relation between vibration frequency and external load which changes from zero up to the critical load can be observed. In the case when the system is subjected to a non-conservative load (it is destroyed by the vibrations of increasing amplitude - flutter type of instability) the critical force can be only determined on the basis of the characteristic curves (kinetic stability criterion) (see [5, 7, 8]). The supporting systems (columns) are generally characterized by great slenderness factor. For systems with great slenderness in order to formulate the boundary problem it is sufficient to apply the theory of Bernoulli - Euler. With the decrease of slenderness factor magnitude the noticeable effect of shear potential energy and cross-section rotational kinetic energy on the characteristic curves can be observed (see [1-4]). An influence of these two components is taken into account in the theory of Timoshenko's beam.

The study on the influence of non-dilatational strain and rotational inertia on the critical flutter loading have been performed by Kounadis and Katsikadelis (comp. [3]). They have studied the different types of supports and column shapes by means of variable slenderness magnitude of the considered system. Stability of columns subjected to the follower force with consideration of Timoshenko's theory has been presented by Nemat-Nasser in [4]. It has been concluded that, at lower magnitudes of slenderness factor associated with shear force and rotational inertia of the cross-section a significant effect of these parameters on the critical load can be observed (destabilizing effect). In considerations of Namat-Nasser the material of the rod was Kelvin's type.

In the paper [1] Glabisz solved the vibrations problem of the column with consideration of Timoshenko's theory. The areas of instability of the cantilever column loaded by independently of one another conservative and non-conservative force have been presented.

The main purpose of this paper is to study an influence of non-dilatational strain (shear effect) and rotational inertia of cross-section on characteristic curves (curves plotted on the plane external load - vibration frequency).

2. Boundary problem formulation on the basis of Hamilton's principle

The investigated system has been presented in the figure 1. The cantilever column is subjected to compressive load (force *P*) with constant line of action (Euler's load). The investigated system is considered as a single-rod column.

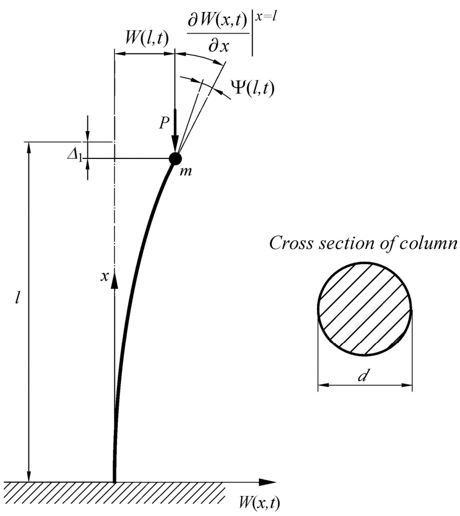

Figure 1. Considered column subjected to Euler's load

In this paper the boundary problem of natural vibrations has been formulated on the basis of Hamilton's principle:

$$\delta\int_{t_1}^{t_2}(T-V)=0 \tag{1}$$

The kinetic energy of the column is expressed as follows:

$$T = \frac{1}{2}(\rho A)\int_0^l \left[\frac{\partial W(x,t)}{\partial t}\right]^2 dx + \frac{1}{2}(\rho J)\int_0^l \left[\frac{\partial \Psi(x,t)}{\partial t}\right]^2 dx + \frac{1}{2} m \left[\left.\frac{\partial W(x,t)}{\partial t}\right|^{x=l}\right]^2 \tag{2}$$

The potential energy is equal to potential energy of bending and shear and compression caused by external load:

$$V = \frac{1}{2}(EJ)\int_0^l \left[\frac{\partial^2 \Psi(x,t)}{\partial x^2}\right]^2 dx + \frac{1}{2}(AG\kappa)\int_0^l \left[\frac{\partial W(x,t)}{\partial x} - \Psi(x,t)\right]^2 dx + \\ -\frac{1}{2} P \int_0^l \left[\frac{\partial W(x,t)}{\partial x}\right]^2 dx \tag{3}$$

where: $W(x,t)$ – deflection of the section, $\Psi(x,t)$ – rotation angle of the section, E – Young modulus, G – Kirchhoff modulus, A – cross-section area, J – axial geometrical moment of inertia of the column's section, κ - the shear coefficient which depends on section's shape (for circular cross-section $\kappa = 0.91$), ρ - density of the material.

Introducing the kinetic and potential energies (2 and 3) into Hamilton's principle allows one to obtain the two differential equations:

$$(EJ)\frac{\partial^2 \Psi(x,t)}{\partial x^2} + AG\kappa\left[\frac{\partial W(x,t)}{\partial x} - \Psi(x,t)\right] - (\rho J)\frac{\partial^2 \Psi(x,t)}{\partial t^2} = 0 \tag{4}$$

$$AG\kappa\left[\frac{\partial^2 W(x,t)}{\partial x^2} - \frac{\partial \Psi(x,t)}{\partial x}\right] - P\frac{\partial^2 W(x,t)}{\partial x^2} - (\rho A)\frac{\partial^2 W(x,t)}{\partial t^2} = 0 \tag{5}$$

Performing mathematical operation and seperating space and time variables $W(x,t) = (Y(x)\cos(\omega t)$; $\Psi(x,t) = \psi(x)\cos(\omega t)$ (where: ω – natural vibration frequency) leads to differential equations in the form:

$$y^{IV}(\xi) + \Gamma y^{II}(\xi) - \Phi y(\xi) = 0 \tag{6}$$

$$\psi^{IV}(\xi) + \Gamma \psi^{II}(\xi) - \Phi \psi(\xi) = 0 \tag{7}$$

where:

$$\Gamma = \frac{\Theta^2\left(-\Omega^2[\phi+1] - \Theta^2\phi\lambda\right) + \Omega^2\lambda}{\Theta^2\left(\lambda - \Theta^2\phi\right)} \,;\, \Phi = -\frac{\Omega^2\left(\Theta^4\phi - \Omega^2\right)}{\Theta^2\left(\lambda - \Theta^2\phi\right)} \tag{8}$$

The non-dimensional parameters ξ, $y(\xi)$, λ, Θ, ϕ and Ω are expressed as follows:

$$\xi = \frac{x}{l},\ y(\xi) = \frac{Y(x)}{l},\ \lambda = \frac{Pl^2}{(EJ)},\ \Theta^2 = \frac{Al^2}{J}, \tag{9a-d}$$

$$\phi = \frac{\kappa G}{E}\,,\ \Omega^2 = \frac{(\rho A) l^4 \omega^2}{(EJ)}\,,\ \zeta_m = \frac{m}{\rho A l} \tag{10a-c}$$

The introduction of geometrical boundary conditions into Hamilton's principle

$$y(0) = \psi(0) = 0 \tag{11a,b}$$

allows one to obtain the natural ones:

$$\Theta^2 \phi\left[\psi(1) - y^I(\xi)\Big|^{\xi=1}\right] + \lambda y^I(\xi)\Big|^{\xi=1} + \zeta_m \Omega^2 y(1) = 0\,,\ \psi^I(\xi)\Big|^{\xi=1} = 0 \tag{12a,b}$$

2. Solution of the boundary problem

The solution of differential equations (6) and (7) depends on relation between Γ and Φ. The three different types of solutions are presented in the form:

- solution A - if ($\Gamma > 0$ and $\Gamma/2 < (\Gamma^2/4+\Phi)^{0.5}$) or ($\Gamma < 0$ and ($\Gamma/2 + (\Gamma^2/4+\Phi)^{0.5}$) > 0):

$$y(\xi) = B_{A1}\cosh(\alpha_A \xi) + B_{A2}\sinh(\alpha_A \xi) + B_{A3}\cos(\beta_A \xi) + B_{A4}\sin(\beta_A \xi) \tag{13}$$

$$\psi(\xi) = C_{A1}\cosh(\alpha_A \xi) + C_{A2}\sinh(\alpha_A \xi) + C_{A3}\cos(\beta_A \xi) + C_{A4}\sin(\beta_A \xi) \tag{14}$$

where:

$$\alpha_A = \sqrt{-\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} + \Phi}}\,,\ \beta_A = \sqrt{\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} + \Phi}} \tag{15a,b}$$

- solution B - if ($\Gamma > 0$ and $\Gamma/2 > (\Gamma^2/4+\Phi)^{0.5}$):

$$y(\xi) = B_{B1}\cos(\beta_{B1}\xi) + B_{B2}\sin(\beta_{B1}\xi) + B_{B3}\cos(\beta_{B2}\xi) + B_{B4}\sin(\beta_{B2}\xi) \tag{16}$$

$$\psi(\xi) = C_{B1}\cos(\beta_{B1}\xi) + C_{B2}\sin(\beta_{B1}\xi) + C_{B3}\cos(\beta_{B2}\xi) + C_{B4}\sin(\beta_{B2}\xi) \tag{17}$$

where:

$$\beta_{B1} = \sqrt{\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} + \Phi}}\,,\ \beta_{B2} = \sqrt{\frac{\Gamma}{2} - \sqrt{\frac{\Gamma^2}{4} + \Phi}} \tag{18a,b}$$

- solution C - if ($\Gamma < 0$ and ($\Gamma/2 + (\Gamma^2/4+\Phi)^{0.5}$) < 0):

$$y(\xi) = B_{C1}\cosh(\alpha_{C1}\xi) + B_{C2}\sinh(\alpha_{C1}\xi) + B_{C3}\cosh(\alpha_{C2}\xi) + B_{C4}\sinh(\alpha_{C2}\xi) \tag{19}$$

$$\psi(\xi) = C_{C1}\cosh(\alpha_{C1}\xi) + C_{C2}\sinh(\alpha_{C1}\xi) + C_{C3}\cosh(\alpha_{C2}\xi) + C_{C4}\sinh(\alpha_{C2}\xi) \tag{20}$$

where:

$$\alpha_{C1} = \sqrt{-\frac{\Gamma}{2} - \sqrt{\frac{\Gamma^2}{4} + \Phi}}\,,\ \alpha_{C2} = \sqrt{-\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} + \Phi}} \tag{21a,b}$$

The constants of integration of solutions $\psi(\xi)$ depend on constants of integration of solutions $y(\xi)$. Constants of integration C_{Ai}, C_{Bi}, C_{Ci} are expressed as follows:

- solution A:

$$C_{A1} = B_{A2}\frac{\alpha_A^2\left(\phi\Theta^2-\lambda\right)+\Omega^2}{\alpha_A\phi\Theta^2},\ C_{A2} = B_{A1}\frac{\alpha_A^2\left(\phi\Theta^2-\lambda\right)+\Omega^2}{\alpha_A\phi\Theta^2} \tag{22a,b}$$

$$C_{A3} = B_{A4}\frac{\beta_A^2\left(\phi\Theta^2-\lambda\right)-\Omega^2}{\beta_A\phi\Theta^2},\ C_{A4} = B_{A3}\frac{-\beta_A^2\left(\phi\Theta^2-\lambda\right)+\Omega^2}{\beta_A\phi\Theta^2} \tag{22c,d}$$

- solution B:

$$C_{B1} = B_{B2}\frac{\beta_{B1}^2\left(\phi\Theta^2-\lambda\right)-\Omega^2}{\beta_{B1}\phi\Theta^2},\ C_{B2} = B_{B1}\frac{-\beta_{B1}^2\left(\phi\Theta^2-\lambda\right)+\Omega^2}{\beta_{B1}\phi\Theta^2} \tag{23a,b}$$

$$C_{B3} = B_{B4}\frac{\beta_{B2}^2\left(\phi\Theta^2-\lambda\right)-\Omega^2}{\beta_{B2}\phi\Theta^2},\ C_{B4} = B_{B3}\frac{-\beta_{B2}^2\left(\phi\Theta^2-\lambda\right)+\Omega^2}{\beta_{B2}\phi\Theta^2} \tag{23c,d}$$

- solution C:

$$C_{C1} = B_{C2}\frac{\alpha_{C1}^2\left(\phi\Theta^2-\lambda\right)+\Omega^2}{\alpha_{C1}\phi\Theta^2},\ C_{C2} = B_{C1}\frac{\alpha_{C1}^2\left(\phi\Theta^2-\lambda\right)+\Omega^2}{\alpha_{C1}\phi\Theta^2} \tag{24a,b}$$

$$C_{C3} = B_{C4}\frac{\alpha_{C2}^2\left(\phi\Theta^2-\lambda\right)+\Omega^2}{\alpha_{C2}\phi\Theta^2},\ C_{C4} = B_{C3}\frac{\alpha_{C2}^2\left(\phi\Theta^2-\lambda\right)+\Omega^2}{\alpha_{C2}\phi\Theta^2} \tag{24c,d}$$

Introducing solutions $y(\xi)$ and $\psi(\xi)$ into boundary conditions one obtains:

$$\left[a_{ij}\right]col\left\{B_{i1},B_{i2},B_{i3},B_{i4}\right\}=0,\ i \equiv \text{A or B or C} \tag{25}$$

The determinant of the matrix of coefficients equated to zero is a equation from which the natural vibration frequency can be computed for given system's parameters:

$$\left|a_{ij}\right|=0 \tag{26}$$

3. Results of numerical calculations

In the Figures 2-4 the change of $\Lambda_{\omega i}$ parameter have been presented (where i stands for natural vibration frequencies, i = 1, 2, 3) in relation to external load of the system λ. By means of $\Lambda_{\omega i}$ parameter the comparison of natural vibration frequencies computed on the basis of Bernoulli - Euler's model $\omega_{B\text{-}E}$ and Timoshenko's theory ω_T are presented. The $\Lambda_{\omega i}$ parameter is expressed as follows:

$$\Lambda_{\omega i}=\frac{\omega_{Ti}-\omega_{B-Ei}}{\omega_{B-Ei}}100\% \tag{27}$$

The investigated system with circular cross-section is made of duraluminium. The numerical calculations were performed for different slenderness parameter λ_S magnitude ($\lambda_S = 300, 250, 200, 150, 100, 50$). The slenderness parameter λ_S is expressed as follows:

$$\lambda_S = \frac{\mu_b l}{\sqrt{\frac{J}{A}}} \tag{28}$$

where: buckling factor for investigated system is $\mu_b = 2$.

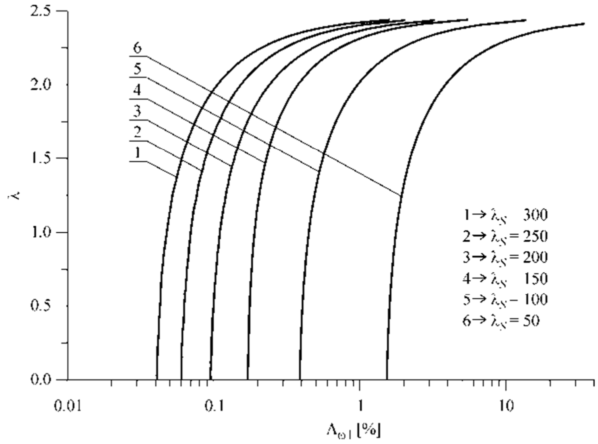

Figure 2. External load parameter λ in relation to parameter of free vibration frequency $\Lambda_{\omega 1}$

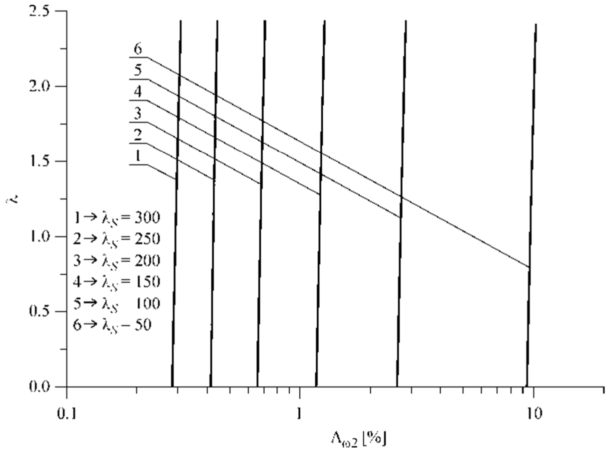

Figure 3. External load parameter λ in relation to parameter of free vibration frequency $\Lambda_{\omega 2}$

On the basis of the performed numerical simulations it can be concluded that the greatest change in Λ_ω parameter in relation to external load λ appears for first natural vibration frequency. In this case an increase of external load magnitude results in increase of difference in magnitudes of natural vibration frequencies calculated with Bernoulli - Euler's model and Timoshenko's theory. The slenderness factor has also an influence on Λ_ω parameter. While taking into account second and third natural vibration frequencies the change in Λ_ω parameter is inconsiderable with the increasing magnitude of external load.

At smaller magnitudes of external load the slenderness factor has greater influence on difference between frequencies (second and third) computed with Bernoulli - Euler's model and Timoshenko's theory.

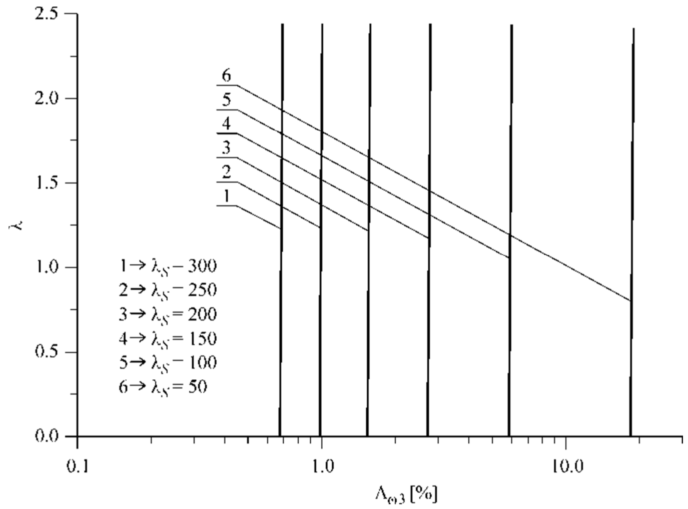

Figure 4. External load parameter λ in relation to parameter of free vibration frequency $\Lambda_{\omega 3}$

At the greatest slenderness λ_S = 300 and external force λ = 0 parameters $\Lambda_{\omega i}$ are as follows: $\Lambda_{\omega 1} \approx 0.04\%$, $\Lambda_{\omega 2} \approx 0.27\%$, $\Lambda_{\omega 3} \approx 0.67\%$. While λ_S = 50 (the lowest considered slenderness) and external force λ = 0 parameters $\Lambda_{\omega i}$ are:$\Lambda_{\omega 1} \approx 1.4\%$, $\Lambda_{\omega 2} \approx 9.4\%$, $\Lambda_{\omega 3} \approx 18\%$.

4. Conclusion

In this paper column subjected to a compressive Euler's load has been investigated. Comparison of results of numerical calculations of natural vibration frequencies obtained on the basis of two theories: Bernoulli - Euler (ω_{B-E}) and the Timoshenko (ω_T) have been performed. In order to demonstrate the differences in the frequencies of vibrations ω_{B-E} and ω_T the new parameter has been defined on the basis of which the percentage change in the first three natural frequencies can be presented for different magnitudes of the external load. The calculations are also concern on different values of slenderness parameter. The greatest differences in the two mathematical models (Euler - Bernoulli and Timoshenko) occurs at the third vibration frequency. The differences between the theories of Bernoulli - Euler and Timoshenko are increasing with greater magnitude of external load. In the case of the first characteristic curve corresponding to the first vibration frequency the differences are the greatest. For the second and third curves the change in magnitude of external load results in small difference between $\omega_{B\text{-}E}$ and ω_T frequencies.

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References

1. W. Grabisz, *O granicach obszarów stateczności pręta pod dwuparametrowym obciążeniem niepotencjalnym*, Mechanika Teoretyczna i stosowana, **1(29)** (1991) 173-188
2. L.Y. Jiang, Z. Yan, *Timoshenko beam model for static bending of nanowires with surface effects*, Physica E, **42** (2010) 2274-2279
3. A.N. Kounadis, J.T. *Katsikadelis, Shear and rotary inertia effect on Beck's columns*, J. Sound and Vib., **49**(2) (1976) 171-178
4. S. Nemat-Nasser, *Instability of a cantilever under a follower force according to Timoshenko beam theory*, J. Appl. Mech., **34** (1967) 484-485
5. C. Sundararajan, *Influence of an elastic end support on the vibration and stability of Beck's column*, Int. J. Mech. Sci., **18** (1976) 239-241.
6. L. Tomski, S. Uzny, *Free vibration and the stability of a geometrically non-linear column loaded by a follower force directed towards the positive pole*, International Journal of Solids and Structures, **45**(1) (2008) 87-112.
7. L. Tomski, S. Uzny, *The regions of flutter and divergence instability of a column subjected to Beck's generalized load, taking into account the torsional flexibility of the loaded end of the column*, Research Mechanics Communications, **38** (2011) 95-100.
8. L. Tomski, S. Uzny, Free *Vibrations and Stability of a New Slender System Subjected to a Conservative or Non-Conservative Load*, Journal of Engineering Mechanics-ASCE, **139**(8) (2013) 1133-1148