CURVATURE

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Abstract. The problem discussed in this paper concerns the curvature of the surface of our globe. The measure of the globe curvature will be determined by a relative growth of the relative change of linear distance between two geodesics which are parallel at their origin but converge when approaching to the Pole. A sphere and an ellipsoid of revolution will be taken as models for further consideration.

Key Words: curvature, non-Euclidean geometry.

1. Introduction

About 300 B.C., Euclid of Alexandria wrote a treatise in thirteen books called the *Elements* [1, p.3]. In the logical development of any branch of mathematics, any definition of a concept or relation involves other concepts or relations. Therefore the only way to avoid a vicious circle is to allow certain *primitive* concepts and relations (usually as few as possible) to remain undefined. Similarly, the proof of each proposition uses other propositions, called *postulates* or *axioms* must remain unproved. Euclid did not specify his primitive concepts and relations, but was content to give definitions in terms of ideas that would be familiar to everybody. [1, p.4]. Thus we have… a *Euclidean* geometry, in which the following axiom will be valid [1, p.288]:
The Euclidean Axiom. *For some point A and some line r, not through A, there is not more than one line through A, in the plane Ar, not meeting r.*

H.S.M. Coxeter [1]

It has been time, when it was believed that the Fifth Euclidean Axiom can be derived from the other postulates or axioms of the Euclidean geometry. The other mathematicians, such as N. I. Lobachevsky, C. F. Gauss and J. Bólyai dealt with this problem, but it was only Gauss who first admitted the necessity of recognition this postulate to be an axiom. Gaussian research work on differential geometry of surfaces brought him not only to the *Theorema Egregium* on curvature, which had earlier been defined by O. Rodrigues, but also led Gauss to understanding various principal, geometrical measures existing in a given surface.

Every sufficiently small portion of a geodesic is the shortest path on the surface connecting the end-points of the portion….All the intrinsic properties of a surface (such as Gaussian curvature) can be determined by drawing geodesics and measuring their arc lengths…. The entire course of a geodesic is determined if one of its points and its direction at this point is given…. The straightest lines may also be characterized by the geometric requirement that the osculating plane of the curve is to contain the normal to the surface at every point of the surface.

Hilbert and Cohn-Vossen [2]

The curves so determined are called geodesics. Since geodesics are the curves of shortest lengths, the geodesics on a sphere are the great circles, and the geodesics in a plane are the straight lines [1, p. 371]. In particular, geodesics on the surface can be considered to be equivalent to straight lines on surfaces. According to Gauss, geodesics do not always comply with the Fifth Euclidean Axiom.

There are various kinds of geometries, which can be considered on surfaces. In general, Euclidean geometry will not be included into these considerations. The character of geometry applied to a certain surface depends on the choice of a point at which it will be examined and the point neighborhood. B. Riemann was a scientist who first put attention to the fact that geometry may be implied on various types of surfaces.

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2. Gaussian Curvature

C.F. Gauss introduced the notion of curvature κ. When κ is constantly zero then the curve is a straight line. "Curvature" κ measures the rate at which any non-straight curve tends to depart from its tangent [1. p. 322]. The curvature may be determined also on a surface, the normal curvature κ attains at least one maximum and one minimum values. These values are called the *principal curvatures*, the positions of **t** in which they occur are called the *principal directions*, and the curves whose direction is always principal are called the *lines of curvature* [1, p.352]. The curvature of the plane is equal to zero. Curvature κ of a surface can be determined in the following way. We consider a geodesic triangle for which the sum of interior angles is greater than 180°. Difference between the sum of interior angles and 180° is proportional to the area of a geodesic triangle. Curvature κ is the ratio, i.e. a coefficient of proportion , between these two values.

Many surfaces may have continuous variation in their curvature. In such cases we should limit the neighboring area and thus we obtain a ratio of curvature at a specific point.

Curvature κ of a surface implies the type of geometry used for further research. If curvature κ is constant and greater than zero then there are no straight lines and elliptic geometry will be applied (e.g. on the sphere). If curvature κ is equal to zero then Euclidean (parabolic) geometry will be considered. Again, if curvature κ is less than zero then we have to do with geometry in which through a given point more than one line parallel to a given line can be passing and this is a case of a hyperbolic geometry.

3. Geometry on the sphere surface

The idea of curvature can be illustrated based on two-parameter geometry on the sphere surface.

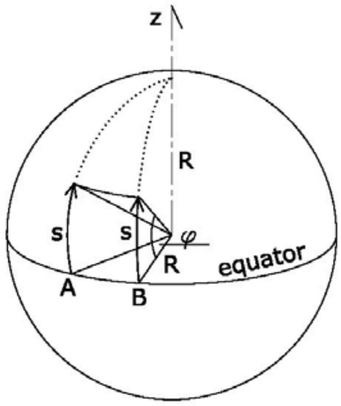

Figure 1. Sphere

In case of points with latitude not much different from the equator, it is sufficient to limit the equation to only two first elements in an expansion of an exponential series of a cosine function. We obtain the following expression for measuring the distance between the two given points A and B

$$\Delta x = (\Delta x)_o \left(1 - \varphi^2 / 2\right) \qquad (1)$$

where φ – is a point latitude.

The angle φ is equal to a ratio between the length of the meridian segment *s* and the globe radius R.

$$\varphi = s / R \tag{2}$$

Shortening of the initial distance $(\Delta x)_o$ will be expressed with the equation

$$(\Delta x)_o - (\Delta x) = (\Delta x)_o (\varphi^2 / 2) = (\Delta x)_o (s^2 / 2R^2) \tag{3}$$

The existence of this difference between the initial and the more distant lengths directly justifies application of geometry of curvilinear surfaces. The distance between two points A and B diminishes rapidly when we approach sphere Pole. Relative growth rate of the change of distance should be considered. To be more precise it will be the relative growth rate of the relative change of distance. The measure of the curvature corresponds to a linear measure between the two geodesics that are perfectly parallel at their origin.

Let us deal with the linear measure now. Let us consider a small, supplementary segment *ds* such as the overall distance now is *s* + *ds*, where *ds* denotes an infinitely small segment. In a new position the distance between the two geodesics will be again shorter

$$(\Delta x)_{new} = (\Delta x)_o - (\Delta x)_o (s + ds)^2 / 2R^2 \tag{4}$$

Let us now drop the small value of ds^2 in (4) and then we obtain

$$(\Delta x)_{new} = (\Delta x)_o - (\Delta x)_o (s^2 + 2sds) / 2R^2 \tag{5}$$

After transformation of (5) we receive

(relative change of distance) =

= (change of distance)/(growth of the linear measure) = $\dfrac{(\Delta x)_{new} - (\Delta x)}{ds} = -(\Delta x)_o s / R^2$ (6)

(relative growth of the relative change of distance) =

= (relative change of distance)/(distance from the point at which the relative growth of the linear measure is equal to zero) = $\dfrac{-(\Delta x)_o s / R^2}{s} = -(\Delta x)_o / R^2$ (7)

4. Geometry on the surface of an ellipsoid of revolution

Let us now consider the case of an ellipsoid of revolution. Each parallel of latitude (including an equator) is a circle lying in the plane perpendicular to the ellipsoid axis and having a radius $r = N \cos\varphi$, where *N* is a radius of curvature in a normal cross-section at a point with a given latitude φ.

Let us denote

$(\Delta x)_{new}$ - distance between points *A* and *B* as measured along a parallel

$\Delta\lambda = \lambda_2 - \lambda_1$ - difference between the longitudes at these points.

We have

$$(\Delta x)_{new} = r\Delta\lambda \tag{8}$$

and in consequence

$$(\Delta x)_{new} = N \cos\varphi (\lambda_2 - \lambda_1) \tag{9}$$

where *N* depends on a and e^2 , while a denotes the length of the ellipsoid axis, *e* – its eccentricity.

The formula

$$\Delta\lambda = \frac{\Delta x_{new}}{r} = \frac{\Delta x_{new}}{N \cos\varphi} \tag{10}$$

makes it possible to calculate the difference between longitudes of two point *A* and *B* based on the length of a parallel segment.

Spherical trigonometry finds application into solving geodesic problems whenever the measurements are taken in so vast areas of the globe surface that its curvature can not be discarded.

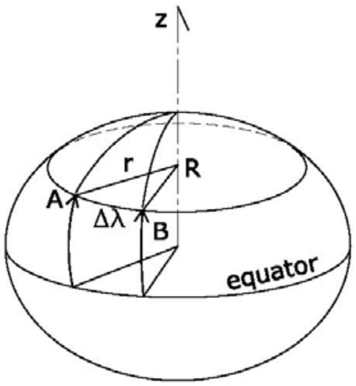

Figure 2. Ellipsoid of revolution

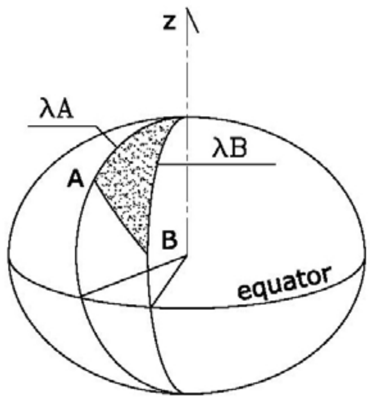

Figure 3: Spherical triangle *ZAB*

Distance *d* between two points *A* and *B* will be measured by the length of an arc segment lying on a great circle passing through these points. Let us consider a spherical triangle with vertices at points *Z* (the North Pole), *A* and *B*. The lengths of the sides *ZA* and *ZB* in triangle *ZAB* are respectively equal to $\pi/2-\varphi_A$ and $\pi/2-\varphi_B$, while the angle between points *A* and *B* equals $|\lambda_A-\lambda_B|$.

The arc measure *d* of the curvilinear segment *AB* is

$$\cos d = \cos(\pi/2-\varphi_A)\cos(\pi/2-\varphi_B)+\sin(\pi/2-\varphi_A)\sin(\pi/2-\varphi_B)\cos(\lambda_A-\lambda_B) \quad (11)$$

and

$$\cos d = \sin\varphi_A \sin\varphi_B + \cos\varphi_A \cos\varphi_B \cos(\lambda_A-\lambda_B) \quad (12)$$

while the questioned distance is equal to Rd,
where
R – denotes a radius of the globe,
d – denotes arc measure of the segment *AB.*

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KRZYWIZNA

Zagadnienie rozważane w niniejszej pracy dotyczy krzywizny na powierzchni kuli ziemskiej. Miarą krzywizny jest względny przyrost względnej miary odległości między dwiema geodezyjnymi początkowo idealnie równoległymi. Występujące zagadnienie porównano dla powierzchni kuli i dla elipsoidy obrotowej.