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Internet digital content pricing and subscribers control

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Abstract

We use optimal control theory to determine the optimal rate of change in the subscription fee and the optimal ratio of ad space to the total web page space for a web content provider. An optimal solution is obtained using the maximum principle approach and the model predictive control approach. Numerical experiments show that it is preferable to use the first approach when the planning horizon is short and the second approach when the planning horizon is long.

Keywords: *maximum principle, model predictive control, optimal control, web content pricing*

1. Introduction

With the development of digital content like video platforms, social media, news platforms, and e-commerce, the number of online consumers increased exponentially in the last two decades. Consumers are more willing to consume digital content and pay for it. The Census Bureau of the Department of Commerce announced that the U.S. retail e-commerce sales for the first quarter of 2020 were \$160.3 billion (US Census Bureau [13]), an increase of 2.4 and 14.8% from the fourth quarter and the first quarter of 2019, respectively.

For digital content providers, advertising and subscription fees are the most popular income sources. In the US, the digital ad revenue neared \$125 billion in 2019 (Interactive Advertising Bureau [6]), up 16% compared to 2018 and almost 10 times the internet advertising revenues in 2005 (\$12.5 billion). Many websites adopt the US Census Bureau strategy of providing free services to maximize membership and advertising. For example, Google and Yahoo provide free search engines and email services, while YouTube offers free videos. The revenues of such sites are mainly dependent on advertising. According to the statistics data firm Statista [11], Google's ad revenue amounted to almost 134 billion US dollars in 2019 compared to 28 and 0.07 billion US dollars in the years 2010 and 2001, respectively. The company generates advertising revenue through its Google Ads platform, which enables advertisers to display ads, product listings, and service offerings across Google's extensive ad network (properties, partner sites, and

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apps) to web users. Some other digital content providers adopt the subscription fee for content access like news outlets (like New York Times and the Wall Street Journal) and video streaming outlets (like Netflix, Disney+, Amazon Prime Video). To entice consumers, these content providers may offer some limited content. For instance, Netflix offers one-month full access to its content while the New York Times offers four articles for all readers for free.

Online price strategies have been extensively studied in research. Digital pricing refers to using digitalized tools, processes, and algorithms to set prices for different products and services. Dynamic pricing is a form of digital pricing and it refers to offering products and services based on different prices which change based on market conditions. Victor et al. [14] have researched to study the factors that affect customer behavior in a dynamic pricing setting. An exploratory factor analysis (EFA) has been adopted in their research to determine the different measures such as shopping experience, dynamic pricing awareness, and privacy.

The relationship between dynamic pricing and revenue maximization in tourism has been analyzed by Abrate et al. [1]. A hedonic revenue model is proposed and applied to a large sample of observations, and results found that higher hotel revenues can be achieved by higher price dynamics. Moreover, Sun et al. [12] have also proposed an optimal pricing model for car-hailing services. Different factors contributing to prices such as ride length, congestion, and rush hour variability were considered. The price and profit relationship has been analyzed and compared under different traffic conditions. Furthermore, Choi et al. [2] have used the mean-risk theory to see how the optimal pricing changes based on customers' risk attitude. Two scenarios were proposed; under the first one, heterogeneous customers who share the same risk attitude will cause the optimal service price to decrease. Whereas the optimal service pricing increases under the second scenario, where customers who have different attitudes towards risk - with the use of blockchain technology - are being provided with different prices.

Accordingly, adopting the best strategy leading to the maximum profit or revenues is of utmost importance for content providers. Among the researchers that investigated content provider strategies is Riggins [9]. He developed an analytical model to examine the monopolist's choice of content quality and price for a fee-based site and the content quality level for a sponsored free site. A reduction in ad revenues results in lower content quality on the free site but permits the seller to raise the fee charged on the fee-based site. In another paper (Fridgeirsdottir et al. [4]), the authors consider an online advertising setting in which a web publisher posts display ads on its website and charges using the cost-per-impression (CPM) pricing scheme while promising to deliver a certain number of impressions on the ads posted. The authors formulate the problem as a queueing system and show that the optimal price to charge per impression can increase the number of impressions made of each ad, which is in contrast with the quantity discount commonly offered in practice. Also, Halbheer et al. [5] propose an analytical framework to study the optimal content strategy for online publishers including a paid content strategy, a sampling strategy, and a free content strategy. The authors found that it can be optimal to use sampling to reduce high prior expectations and content demand and also not to reveal high content quality through free samples.

As content providers have different strategy options to choose from, generating income from ads and subscriptions at the same time is adopted by many content providers. However, content providers are faced with many constraints like ad space, content quality, and optimal pricing. One of the research works that investigated a number of options, the subscription price, and the amount of advertising that should be offered is Prasad et al. [8]. The authors covered also heterogeneous consumers that can pay a higher price and view fewer ads or pay a lower price with more ads and found that the optimal strategy (with exceptions) is to charge a subscription price and have ads but offer options to consumers.

While the former work considers a static model, many other authors analyzed the heterogeneous ads-subscription-fee problem in a dynamic setup using optimal control theory. For instance, Dewan et al. [3] model the problem of balancing content and advertising for free websites where content is costly but increases visitors traffic, whereas ads generate revenues but decrease traffic. The paper shows that it might be optimal for the website to initially have negative cash flows from having fewer advertisements and investing in content that will be compensated for by future profits. In another work, Kumar and Sethi [7] develop a dynamic pricing and advertising model for web content providers with a hybrid revenue model based on a combination of subscription fees and advertising revenues. The work is a profit maximization model based on optimal control theory where the control variables are the ad space ratio and the subscription fee rate. Also, the work presents several analytical and numerical results based on the obtained solution and investigates the system parameters' impact (like the advertisement revenue rate, natural growth rate, content's utility factor, etc.) on the obtained solution through sensitivity analysis.

Furthermore, some other research works tried to inject the viewer experience and perspective into the analysis. As an example, Xu and Duan [15] investigate how an online content provider decides the optimal subscription price and the advertising space allocation considering the reference price effect (the price that a purchaser announces that it is willing to pay for a good or service). The viewers in this paper are classified into two groups: subscribers that pay a subscription fee and can view all the content, and non-subscribers that can view only a fraction of the content. The paper concludes that when the viewer's sensitivity to advertising is relatively small, the provider should adopt a subscription-support model rather than a hybrid business model, and should reduce the advertising space when viewers pay more attention to the viewer experience.

We consider in this paper a web content provider that generates revenue from subscription fees and ads. The model is of the tracking type and consists in determining the optimal rate of change in the subscription fee and the optimal ratio of ad space to the total web page space. Two optimal control techniques are used for the sake of comparison. The maximum principle is the approach that is the most used to tackle dynamic optimal control problems that arise in operations research and management science [10]. The model predictive control approach is seldom used, although it can be superior to the first approach.

Following this introduction, the notation is introduced and the model is described through its dynamics. Section 3 implements the maximum principle approach while Section 4 implements the model predictive control approach. Numerical illustrations are carried out in Section 5, and Section 6 concludes the paper.

2. Model dynamics and notation

Consider a web content provider that generates revenue from subscription fees and ads. Let $H > 0$ denote the length of the planning horizon. The state of the system is described by $x(t)$ – the number of subscribers at time t and $p(t)$ – the subscription fee at time t. Control of the system occurs through $u(t)$ – the rate of change in the subscription fee at time t and $a(t)$ – the ratio of ad space to the total web page space at time t.

Following Kumar and Sethi [7], the system evolves according to the dynamics:

$$\frac{d}{dt}x(t) = \eta + \pi\big(1 - a(t)\big) - \psi a(t) - \phi u(t), \quad x(0) = x_0 \tag{1}$$

$$\frac{d}{dt}p(t) = u(t), \quad p(0) = p_0 \tag{2}$$

where x_0 and p_0 are the initial number of subscribers and initial subscription fee, respectively, η is the natural growth rate, π is the utility factor of content, ψ is the disutility factor of ads, and ϕ is the sensitivity to the subscription fee.

Assume the system is of the tracking type, that is the firm set goals for the state variables: $\hat{x}(t)$ is the target number of subscribers at time t and $\hat{p}(t)$ is the target subscription fee at time t. These functions can be constant or dynamic. They can be obtained, for example, through benchmarking. Corresponding to the target state variables are the target control variables: $\hat{u}(t)$ is the target rate of change in the subscription fee at time t and $\hat{a}(t)$ is the target ratio of ad space to the total web page space at time t. Since the target variables satisfy the state equations, we can readily obtain the target control variables in terms of the target state variables

$$\hat{u}(t) = \frac{d}{dt}\hat{p}(t) \tag{3}$$

$$\hat{a}(t) = \frac{1}{\pi + \psi}\left(\eta + \pi - \phi\frac{d}{dt}\hat{p}(t) - \frac{d}{dt}\hat{x}(t)\right) \tag{4}$$

One more notation we introduce is that of the shift operator Δ defined for any function f, by

$$\Delta f(t) = f(t) - \hat{f}(t)$$

As we will see it turns out that it is more convenient to build the model in terms of this operator. Subtracting (3) from (1) and (4) from (2)

$$\frac{d}{dt}\Delta x(t) = -\big(\pi + \psi\big)\Delta a(t) - \phi\Delta u(t) \tag{5}$$

$$\frac{d}{dt}\Delta p(t) = \Delta u(t) \tag{6}$$

Now, since the problem is to find the optimal state and control variables, we need to introduce some performance index. The objective function that is naturally used in many works is either the total profit to maximize or the total cost to minimize. The other objective function that is used, especially when the system is of the tracking type, is the total deviation to minimize [10].

Concerning the solution approach, in the case of dynamic systems, the most widely used is the maximum principle (MP) technique. Another approach, as efficient, albeit less used, is the model predictive control (MPC) technique. We are adopting in this paper the total deviation objective function. In the next section, we are employing the MP technique to obtain the optimal state and control variables. In the following section, the MPC technique is used. A comparison between the techniques is carried out in the numerical example section.

3. Maximum principle approach

In this approach, given the target state variables $\hat{x}(t)$ and $\hat{p}(t)$ and the initial state values x_0 and p_0, we are interested in obtaining the optimal state variables $x^*(t)$ and $p^*(t)$ and the optimal control variables $u^*(t)$ and $a^*(t)$ during the planning horizon $[0, H]$. To define the objective function, we introduce for $i = 1, 2$ the penalties:
q_i – penalty incurred when the ith state variable deviates from its target,
r_i – penalty incurred when the ith control variable deviates from its target,
c_i – final state penalty for the ith state variable.

The problem is to minimize the objective function

$$\begin{aligned} J =& \frac{1}{2}\int_0^H \big(q_1\Delta x(t)^2 + q_2\Delta p(t)^2 + r_1\Delta u(t)^2 + r_2\Delta a(t)^2\big)dt \\ &+ \frac{c_1}{2}\Delta x(H)^2 + \frac{c_2}{2}\Delta p(H)^2 \end{aligned} \tag{7}$$

subject to the state equations (5), (6). In the MP approach [10], an adjoint function $\lambda_i(t)$ is associated with the ith state equation, and the Hamiltonian function is

$$\begin{aligned} \mathcal{H} =& -\frac{1}{2}\big(q_1\Delta x(t)^2 + q_2\Delta p(t)^2 + r_1\Delta u(t)^2 + r_2\Delta a(t)^2\big) \\ &- \lambda_1(t)\big(\pi + \psi)\Delta a(t) + \phi\Delta u(t)\big) + \lambda_2(t)\Delta u(t) \end{aligned}$$

To write the model using matrix notation, introduce the state, control, and adjoint vectors

$$X(t) = \begin{bmatrix} \Delta x(t) \\ \Delta p(t) \end{bmatrix}, \quad U(t) = \begin{bmatrix} \Delta u(t) \\ \Delta a(x) \end{bmatrix}, \quad \text{and} \quad \Lambda(t) = \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix}$$

respectively, and let

$$B = \begin{bmatrix} -\phi & -(\pi+\psi) \\ 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, \quad R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$$

The problem becomes:

$$\min J = \frac{1}{2}\int_0^H \big(\|X(t)\|_Q^2 + \|U(t)\|_R^2\big)dt + \frac{1}{2}\|X(t)\|_C^2$$

subject to

$$\frac{d}{dt}X(t) = BU(t)$$

where $\|x\|_A = \sqrt{x^\top Ax}$ and the Hamiltonian is written concisely as:

$$\mathcal{H} = -\frac{1}{2}\big(\|X(t)\|_Q^2 + \|U(t)\|_R^2\big) + \Lambda(t)^\top BU(t)$$

The necessary optimality conditions yield the two-point boundary value problem:

$$\frac{d}{dt}Z(t) = \Phi Z(t), \quad X(0) = X_0, \quad \Lambda(T) = CX(T) \tag{8}$$

where

$$Z(t) = \begin{bmatrix} X(t) \\ \Lambda(t) \end{bmatrix} \quad \text{and} \quad \Phi = \begin{bmatrix} 0 & BR^{-1}B^\top \\ Q & 0 \end{bmatrix}$$

The solution to (8) is given by

$$Z(t) = \varphi(t)Z(0) \tag{9}$$

where $\varphi(t) = \mathrm{e}^{\Phi t}$.

Partitioning the matrix $\varphi(t)$ as follows:

$$\varphi(t) = \begin{bmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_3(t) & \varphi_4(t) \end{bmatrix}$$

we readily obtain the optimal state variables

$$\begin{bmatrix} \Delta x^*(t) \\ \Delta p^*(t) \end{bmatrix} = \big(\varphi_1(t) + \varphi_2(t)\xi(T)\big) \begin{bmatrix} \Delta x(0) \\ \Delta p(0) \end{bmatrix}$$

and the optimal control variables

$$\begin{bmatrix} \Delta u^*(t) \\ \Delta a^*(x) \end{bmatrix} = R^{-1}B^\top \big(\varphi_3(t) + \varphi_4(t)\xi(T)\big) \begin{bmatrix} \Delta x(0) \\ \Delta p(0) \end{bmatrix}$$

where $\xi(T) = \big(C\varphi_2(T) - \varphi_4(T)]^{-1}[\varphi_3(T) - C\varphi_1(T)\big)$.

4. Model predictive control approach

As mentioned earlier, MP seeks, at time $t = 0$, the optimal variables on the planning horizon time interval $[0, H]$. In contrast, MPC seeks, at any time $t_0 \in [0, H]$, the optimal variables on the prediction horizon time interval $[t_0, t_0 + T]$. Here $T > 0$ and $T \ll H$. Initially, $t_0 = 0$ and given x_0 and p_0 the optimal solution is found on $[0, T]$. Then, the final values $x(T)$ and $p(T)$ become the initial values, and the optimal solution is found on $[T, 2T]$. This process is repeated until time H is reached.

Thus, let t_0 in $[0, H]$. The MPC problem is to minimize the objective function

$$\begin{aligned} J =& \frac{1}{2} \int_{t_0}^{t_0+T} \big(q_1 \Delta x(t)^2 + q_2 \Delta p(t)^2 + r_1 \Delta u(t)^2 + r_2 \Delta a(t)^2\big) dt \\ &+ \frac{c_1}{2} \Delta x(t_0 + T)^2 + \frac{c_2}{2} \Delta p(t_0 + T)^2 \end{aligned} \tag{10}$$

subject to the state equations (5), (6). Dividing the prediction horizon $[t_0, t_0 + T]$ into m subintervals of equal length $h = \dfrac{T}{m}$, the trapezoid formula is used to calculate the integral in the objective function.

Some lengthy calculations yield

$$J = x_0(t) - x_1(t)\Delta u(t) - x_2(t)\Delta a(t) + x_3\Delta u(t)^2 + x_4\Delta a(t)^2 + x_5\Delta u(t)\Delta a(t)$$

$$+ \frac{hr_1}{2}\sum_{i=1}^{m-1}\Delta u(t+ih)^2 + \frac{hr_2}{2}\sum_{i=1}^{m-1}\Delta a(t+ih)^2$$

where $x_0(t)$ is independent of the control variables,

$$x_1(t) = \left(\alpha h^2 q_1\phi + \left(\frac{hq_1}{2} + c_1\right) hm\phi\right)\Delta x(t) - \left([\alpha h^2 q_2 + \left(\frac{hq_2}{2} + c_2\right) hm\right)\Delta p(t)$$

$$x_2(t) = x_2\Delta x(t) \quad \text{with} \quad x_2 = \left(\alpha h^2 q_1(\pi+\psi) + \left(\frac{hq_1}{2} + c_1\right) hm(\pi+\psi)\right)\Delta x(t)$$

$$x_3 = \frac{hr_1}{4} + \frac{\beta h^3 q_1\phi^2}{2} + \frac{\beta h^3 q_2}{2} + \left(\frac{hq_1}{4} + \frac{c_1}{2}\right) h^2m^2\phi^2 + \left(\frac{hq_2}{4} + \frac{c_2}{2}\right) h^2m^2$$

$$x_4 = \frac{hr_2}{4} + \frac{\beta h^3 q_1}{2}(\pi+\psi)^2 + \left(\frac{hq_1}{4} + \frac{c_1}{2}\right) h^2m^2(\pi+\psi)^2$$

$$x_5 = \beta h^3 q_1\phi(\pi+\psi) + \left(\frac{hq_1}{2} + c_1\right) h^2m^2\phi(\pi+\psi)$$

and

$$\alpha = \frac{m(m-1)}{2}, \quad \beta = \frac{m(m-1)(2m-1)}{6}$$

The necessary optimality conditions allow us to write the control variables as

$$\Delta u(t) = \frac{2x_4\big(y_1\Delta x(t) - y_2\Delta p(t)\big) - x_2x_5\Delta x(t)}{4x_3x_4 - x_5^2}$$

$$\Delta a(t) = \frac{2x_2x_3\Delta x(t) - x_5\big(y_1\Delta x(t) - y_2\Delta p(t)\big)}{4x_3x_4 - x_5^2}$$

where

$$y_1 = \alpha h^2 q_1\phi + \left(\frac{hq_1}{2} + c_1\right) hm\phi \quad \text{and} \quad y_2 = \alpha h^2 q_2 + \left(\frac{hq_2}{2} + c_2\right) hm.$$

Therefore,

$$\Delta u(t) = \xi_1\Delta x(t) + \xi_2\Delta p(t) \tag{11}$$

$$\Delta a(t) = \xi_3\Delta x(t) + \xi_4\Delta p(t) \tag{12}$$

where

$$\xi_1 = \frac{2x_4y_1 - x_2x_5}{4x_3x_4 - x_5^2}, \qquad \xi_2 = -\frac{2x_4y_2}{4x_3x_4 - x_5^2}$$

$$\xi_3 = \frac{2x_2x_3 - x_5y_1}{4x_3x_4 - x_5^2}, \qquad \xi_4 = \frac{x_5y_2}{4x_3x_4 - x_5^2}$$

Inserting the previous explicit forms of $\Delta u(t)$ and $\Delta a(t)$ in the state equations (5), (6) results in the following differential system of two equations with two variables $\Delta x(t)$ and $\Delta p(t)$:

$$\frac{d}{dt}\Delta x(t) = a_{11}\Delta x(t) + a_{12}\Delta p(t), \quad \frac{d}{dt}\Delta p(t) = a_{21}\Delta x(t) + a_{22}\Delta p(t)$$

where

$$a_{11} = -\big((\pi+\psi)\xi_3 + \phi\xi_1\big), \qquad a_{12} = -\big((\pi+\psi)\xi_4 + \phi\xi_2\big)$$
$$a_{21} = \xi_1, \qquad a_{22} = \xi_2$$

This is a homogeneous system of two equations with constant coefficients that are easily solved using standard methods. Using for example the elimination method, we obtain a second-order linear homogeneous equation:

$$\frac{d^2}{dt^2}\Delta x(t) - (a_{11}+a_{22})\,\frac{d}{dt}\Delta x(t) + (a_{11}a_{22} - a_{12}a_{21})\,\Delta x(t) = 0$$

We can construct the solution of the second-order linear equation if we know the roots of the characteristic equation:

$$\lambda^2 - (a_{11}+a_{22})\,\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

1. Discriminant of the characteristic quadratic equation $D > 0$. Then the roots of the characteristic equation λ_1 and λ_2 are real and distinct. In this case, the general solution is given by the following function

$$\Delta x^*(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

where C_1 and C_2 are arbitrary real numbers that are obtained using the initial conditions.

2. Discriminant of the characteristic quadratic equation $D = 0$. Then there exists one repeated root λ of order 2. The general solution of the differential equation has the form

$$\Delta x^*(t) = \big(C_1 t + C_2\big)e^{\lambda t}$$

3. Discriminant of the characteristic quadratic equation $D < 0$. Such an equation has complex roots $\lambda_1 = \alpha_1 + i\beta_1$, $\lambda_2 = \alpha_2 - i\beta_2$. Note that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. The general solution is written as

$$\Delta x^*(t) = e^{\alpha_1 t}\big(C_1\cos(\beta_1 t) + C_2\sin(\beta_1 t)\big)$$

Once the first optimal state variable $\Delta x^*(t)$ is determined, the second optimal state variable $\Delta p^*(t)$ can be found from the first equation of the differential system

$$\Delta p^*(t) = \frac{1}{a_{12}}\left(\frac{d}{dt}\Delta x^*(t) - a_{11}\Delta x^*(t)\right)$$

Finally, substitute the solutions $\Delta x(t)$ and $\Delta p(t)$ from (11)- to obtain the optimal control variables

$$\Delta u^*(t) = \xi_1\Delta x^*(t) + \xi_2\Delta p^*(t), \quad \Delta a^*(t) = \xi_3\Delta x^*(t) + \xi_4\Delta p^*(t)$$

5. Numerical examples

Numerical examples are presented in this section to illustrate the results obtained and to compare the effectiveness of the two proposed solution approaches.

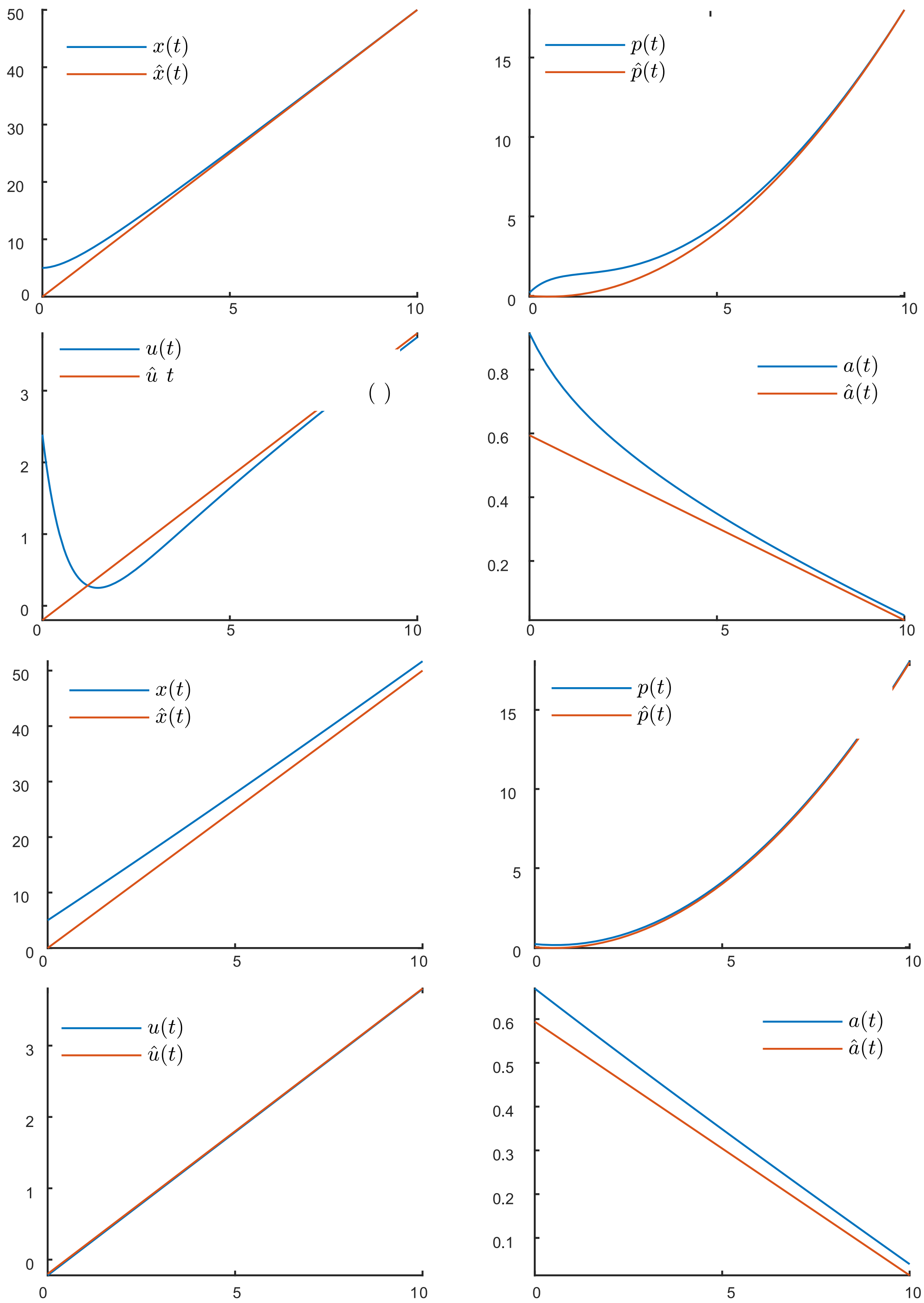

Figure 1. MP solution (two upper rows) and MPC solution (two lower rowst)

In the subsequent numerical experiments, the base parameter values are as follows: $H = 10$, $T = 10$, $m = 100$, $t_0 = 0$, $\eta = 5$, $\pi = 3.9$, $\psi = 3$, $\phi = 1$, $x_0 = 5$, $p_0 = 0.2$, $q_1 = 0.01$, $q_2 = 0.01$, $r_1 = 0.01$,

$r_2 = 1.8$, $c_1 = 5$, and $c_2 = 10$. Also, the state variables targets are $\hat{x}(t) = 5t$ and $\hat{p}(t) = 0.2t(t-1)$. According to Figure 1, both methods show convergence: state and control variables converge toward their respective targets.

Sensitivity analysis. Sensitivity analysis is a tool that can greatly help in the decision-making process. We conducted a large number of experiments to assess the impact of the system parameters on the optimal solution and on the optimal objective function value. In the experiment, we vary one or more of the parameter values (which are clearly specified) and the rest of the parameters are set at their base values.

Effect of H on the objective function of both approaches: The length of the planning horizon is the parameter that has the largest impact on the choice of the approach used. The results below show that the two methods perform differently for different H ranges when compared to each other. First, Figure 2 shows that for small values of H, the MP objective function is smaller than the one corresponding to MPC for $H = [5, 16]$. For example, for $H = 10$, the MP and the MPC objective functions are equal to 0.21 and 7.66, respectively.

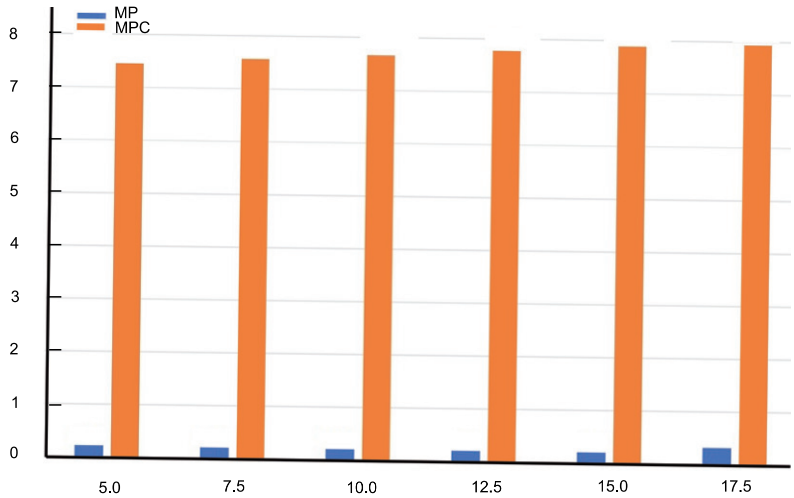

Figure 2. Optimal objective function value corresponding to both methods

However, MPC starts performing better when H exceeds 17, due to the MP objective function value increasing exponentially with H, as shown in Table 1. For instance, for $H = 20$, the MP and the MPC objective functions are equal to $22.42E + 07$ and 8.11, respectively.

Table 1. MP and MPC optimal objective function value for different values of H

H	17	17.5	18	19	20	30	40	50
MP	14.0572	15.4447	1.05E+04	1.19E+05	2.42E+07	6.60E+29	3.83E+42	4.97E+55
MPC	7.981	8.0036	8.0263	8.0715	8.1168	8.5713	9.0263	9.4807

Further sensitivity analyses are carried out next to see how some of the parameters affect the optimal objective function value and the optimal state variables. The sensitivity analyses are done using both approaches to make sure the results are consistent.

5.1. Maximum principle analysis

- Impact of the content's utility factor π: As can be seen from Table 2, the optimal objective function J^* decreases when π becomes higher.

Table 2. Impact of π on J^* in MP case

π	1	5	9	13	17	21	25
J^*	0.3634	0.1870	0.1291	0.0991	0.0806	0.0679	0.0587

Figure 3 shows the little impact of π on the optimal number of subscribers and the optimal subscription fee.

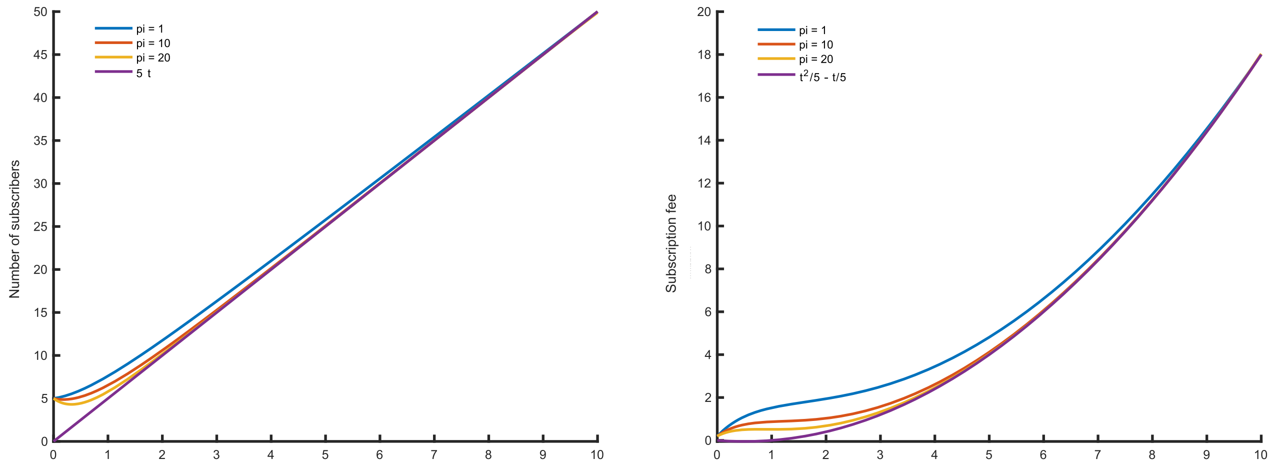

Figure 3. Impact of π on the number of subscribers (left) and on the subscription fee (right) – MP case

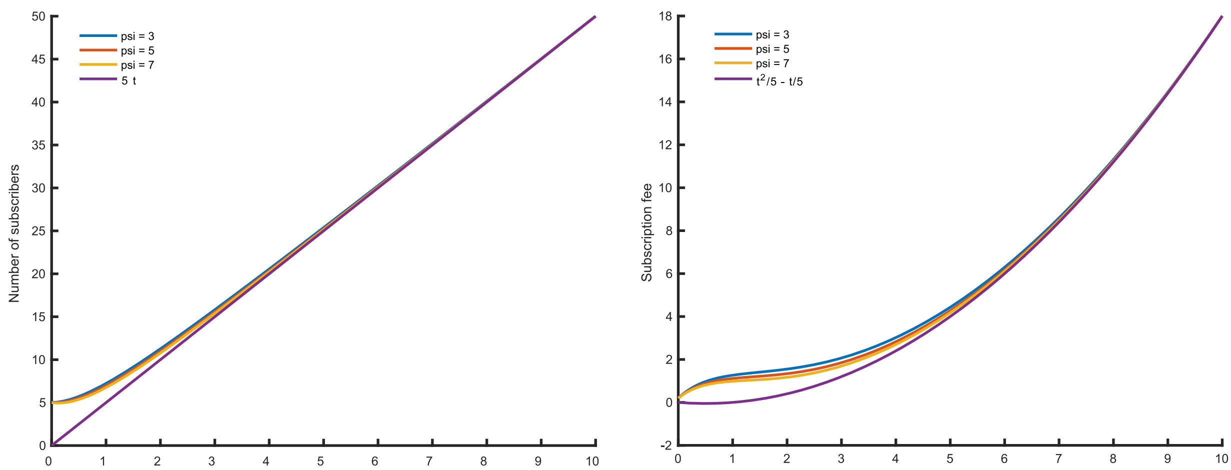

Figure 4. Impact of ψ on the number of subscribers (left) and on the subscription fee (right) – MP case

- Impact of the natural growth rate η: Table 3 shows that η has no effect on the optimal objective function value J^*.

Table 3. Impact of η on J^* in MP case

η	5	10	20	30	40	50	100
J^*	0.2142	0.2142	0.2142	0.2142	0.2142	0.2142	0.2142

- Impact of disutility factor of ads ψ: As can be seen from Table 4, the objective function decreases when ψ becomes higher.

Table 4. Impact of ψ on J^* in MP case

ψ	3	5	7	9	11	13	15
J^*	0.2142	0.1696	0.1410	0.1208	0.1058	0.0942	0.0849

However, Figure 4 shows little impact of ψ on the optimal number of subscribers and the optimal subscription fee.

5.2. Model predictive control analysis

- Impact of the content's utility factor π: Table 5 shows that the optimal objective value decreases slightly when π increases, while Figure 5 shows no significant impact on the optimal number of subscribers and the optimal subscription fee.

Table 5. Impact of π on J^* in MPC case

π	1	5	9	13	17	21	25
J^*	7.7098	7.6627	7.6539	7.6509	7.6495	7.6487	7.6482

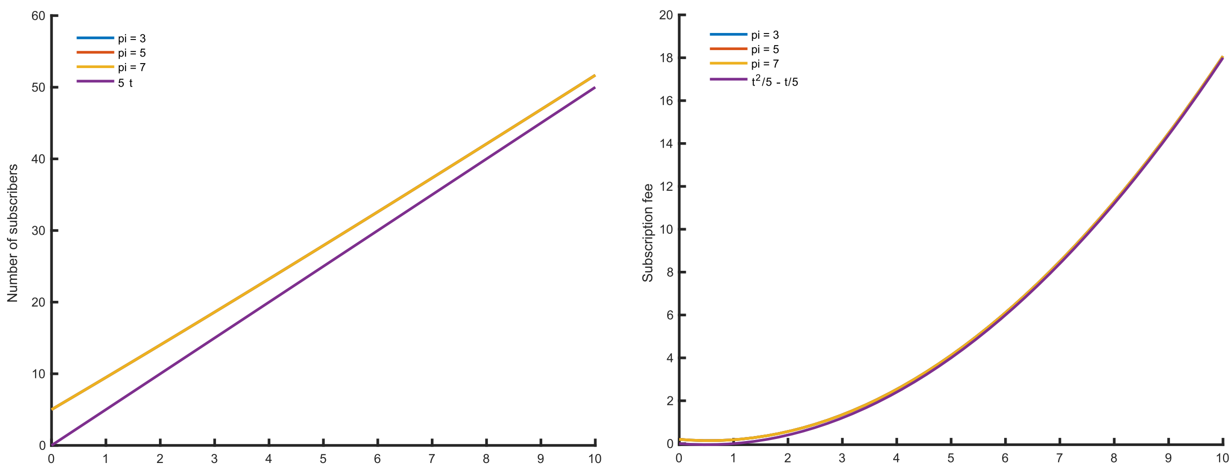

Figure 5. Impact of π on the number of subscribers (left) and on the subscription fee (right) – MPC case

- Impact of the natural growth rate η: Table 6 shows η does not affect the optimal objective value, while Figure 6 shows no significant impact on the optimal number of subscribers and the optimal subscription fee

Table 6. Impact of η on J^* in MPC case

η	5	10	20	30	40	50	100
J^*	7.6681	7.6681	7.6681	0.2142	0.2142	0.2142	0.2142

- Impact of disutility factor of ads ψ: Table 7 shows that the optimal objective value decreases slightly when ψ increases, while Figure 7 shows no significant impact on the optimal number of subscribers and the optimal subscription fee.

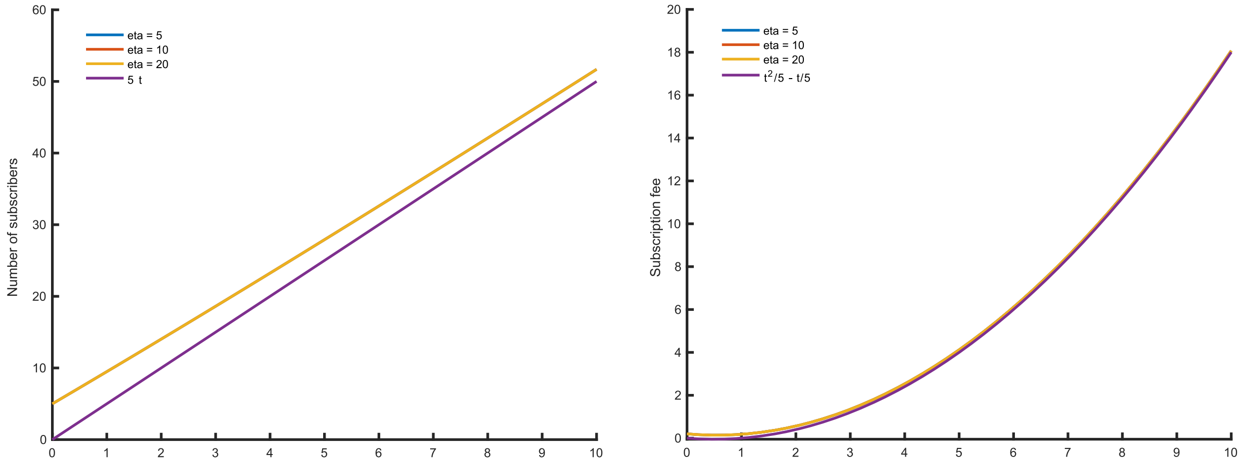

Figure 6. Impact of η on the number of subscribers (left) and on the subscription fee (right) – MPC case

Table 7. Impact of ψ on J^* in MPC case

ψ	3	5	7	9	11	13	15
J^*	7.6681	7.6596	7.6554	7.6530	7.6515	7.6505	7.6498

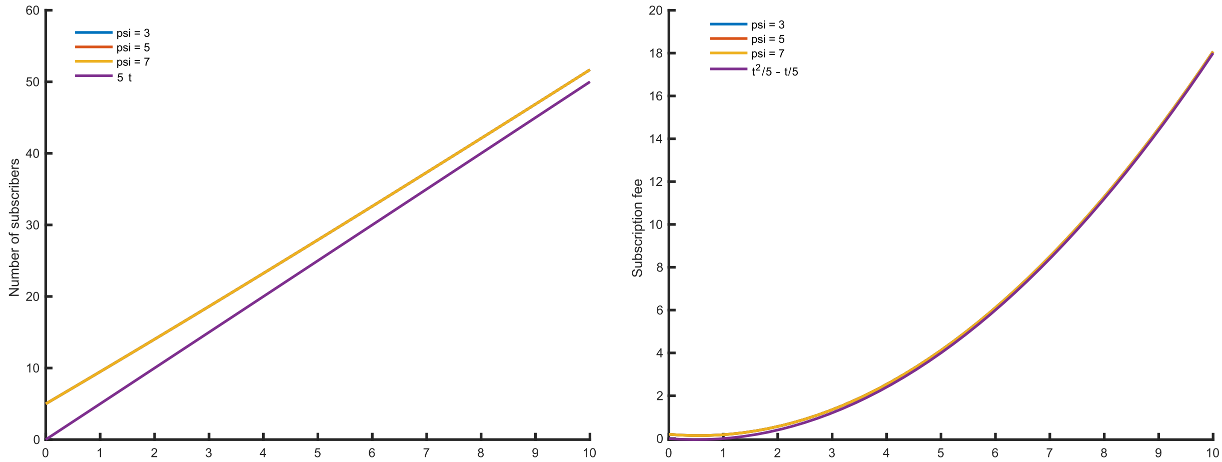

Figure 7. Impact of ψ on the number of subscribers (left) and on the subscription fee (right) – MPC case

5.3. Managerial implications

Both methods perform differently and deliver different objective function values depending on the length of the planning horizon H. In this case, the decision-maker should pick the method that delivers the lowest objective function values. This might also be true for the other system parameters (like π, η, ψ, etc.) and their assigned values. So, the decision maker is advised to pick the best method according to his/her system parameter values.

- Maximum principle analysis: Based on the sensitivity analysis conducted above, the objective function value changes when some of the system parameters change. For example, the objective function decreases when the content utility factor π becomes higher (the objective function is equal to 0.36 and 0.12 for the π values of 1 and 10, respectively). In this case, if the cost of increasing π from 1 to 10 through content improvement is less than the financial benefits of decreasing the objective function by 0.24, then the decision maker should invest in the content utility. The same recommendations can be applied to other parameters.

- Model predictive control analysis: From the sensitivity analysis conducted above, it can be seen that the objective function value changes for some system parameters change. For example, the objective function slightly decreases when the content utility factor π becomes higher (the objective function is equal to 7.67 and 7.65 for the π values of 3 and 7, respectively). In this case, if the cost of increasing π from 3 to 7 through content improvement is less than the financial benefits of decreasing the objective function by 0.02, then the decision maker should invest in the content utility. The same recommendations can be applied to other parameters.

6. Conclusion

A key decision by a web content provider is the balance between content and advertising. The optimal rate of change in the subscription fee and the optimal advertising space allocation need to be determined. An optimal control model treating this problem where the objective is to maximize profit was introduced by Kumar and Sethi [7]. Using the same dynamics but assuming a system of the tracking type, our objective function aims at reducing the gap between each variable and its target. Using two different optimal control techniques, we obtain the optimal solution analytically in explicit form.

While the above-mentioned models optimize the rate of change in the subscription fee and the ad space allocation in the presence of many assumptions, many other realistic ones may be added to the work. For example, the natural growth rate, the utility factor of content, the disutility factor of ads, and sensitivity to the subscription fee may be considered functions of time. Also, changing the system dynamics by offering viewers different levels of content access and the possibility of upgrades with different subscription fees. In light of these limitations, future work can include the following:

- offering different options to viewers such as limited content or lower quality for a lower subscription fee,
- assuming that some of the system parameters like the growth rate or disutility rate are dynamic,
- because of the competition of different digital content providers, the perception of the viewer of the quality of content or the reference price might change over time which will affect the viewer's sensitivity to the subscription fee (ϕ),
- using the maximum principle and the model predictive control approaches to find the optimal strategy decisions for a pure subscription fee or ad-based revenue websites.

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