Archives of Materials Science and Engineering

Volume 113
Issue 1
January 2022
Pages 13-18

International Scientific Journal published monthly by the World Academy of Materials and Manufacturing Engineering

DOI: 10.5604/01.3001.0015.6967

Construction of conical axoids on the basis of congruent spherical ellipses

T. Kresan [a], S. Pylypaka [a], Z. Ruzhylo [b], I. Rogovskii [c], O. Trokhaniak [b,*]

[a] Department of Descriptive Geometry, Computer Graphics and Design, Faculty of Construction and Design, National University of Life and Environmental Sciences of Ukraine, Kyiv, Ukraine

[b] Department of Reliability of Equipment, Faculty of Construction and Design, National University of Life and Environmental Sciences of Ukraine, Kyiv, Ukraine

[c] Department of Technical Service and Engineering Management named after M.P. Momotenka, Computer Graphics and Design, Mechanical and Technological Faculty, National University of Life and Environmental Sciences of Ukraine, Kyiv, Ukraine

* Corresponding e-mail address: klendii_o@ukr.net

ORCID identifier: https://orcid.org/0000-0002-4671-5824 (O.T.)

ABSTRACT

Purpose: To carry out the transition from a cylindrical gear in which the centroids are congruent ellipses with centres of rotation in the foci, to a bevel gear on the basic of congruent spherical ellipses.

Design/methodology/approach: Congruent ellipses with centres of rotation in the foci serve as centroids for the design of cylindrical gears with non-circular wheels. The article analytically shows that the analogues of ellipses on the plane - congruent spherical ellipses are the basis for the construction of the axoids of the corresponding bevel gears. An analogue of the centre-to-centre distance for ellipses in the plane is the angle between the axes of rotation of conical axoids.

Findings: Based on the equality of the arcs of ellipses, the dependence of the angle of rotation of one axoid on the angle of rotation of another is found. Graphs of this dependence for separate cases are given. It is shown under what conditions the axes of axoids intersect at right angle. The parametric equations of spherical ellipses and corresponding axoids are given. They were used to construct spherical ellipses and corresponding conical axoids for different cases. For gears with right angle between the axes, separate positions of the axoids with different angles of their rotation around their axes are constructed.

Practical implications: Spherical ellipses are directing curves for the construction of the corresponding conical axoids.

Originality/value: The paper shows that congruent spherical ellipses act as centroids for the design of axoids of bevel gears. They roll one by one without sliding, rotating around axes that intersect in the centre of the sphere. To design such gears, it is important to know the interdependence between the geometric parameters, especially for common gears with a right angle between the axes.

Keywords: Spherical ellipse, Conical axoid, Angle of rotation dependence, Bevel gear, Rolling of axoids

Reference to this paper should be given in the following way:
T. Kresan, S. Pylypaka, Z. Ruzhylo, I. Rogovskii, O. Trokhaniak, Construction of conical axoids on the basis of congruent spherical ellipses, Archives of Materials Science and Engineering 113/1 (2022) 13-18. DOI: https://doi.org/10.5604/01.3001.0015.6967

METHODOLOGY OF RESEARCH, ANALYSIS AND MODELLING

1. Introduction

The construction of spherical ellipses takes place according to the same algorithm as on the plane with the difference that the distances on the surface of the sphere are measured by angles. It is known that congruent ellipses with centres of rotation in the foci serve as centroids of cylindrical gears with non-circular wheels [1].

It is known from the literature that the properties of the spherical ellipse were studied by L. Euler's student and assistant, a Russian mathematician of Swiss descent, L.I. Fuss [2]. He found out that a spherical ellipse under certain circumstances turns into a circle of large diameter of the sphere. It should be noted that the usage of the surface of the sphere is widely used in solving geometric problems [3,4]. It is associated with the concept of spherical mapping of surfaces, spherical curves, spherical indicatrix; it is the basis of a separate section of trigonometry – spherical. The formulas of spherical trigonometry play an important role in the study of the kinematics of spherical figures.

Problems on kinematics, in particular the rolling of geometric figures one by one, are also associated with curves [5,6]. The formation of flat curves according to the given kinematic parameters is considered in [7]. Spherical ellipses are analogous to flat centroids of non-circular wheels, the simulation of which is considered in [8-18].

2. Description of the approach, work methodology, materials for research

A spherical ellipse is constructed by analogy with an ellipse in a plane: the sum of the distances from any point of the ellipse to its foci is the constant value. The peculiarity of a spherical ellipse is that distances are not measured in linear quantities, but in angular ones. In Figure 1a shows a spherical ellipse in which the first focus is the point of intersection of the axis OZ with the surface of the sphere, and the second F_1 is located on the meridian at a distance FF_1 from the first. The distance FF_1 on the surface of the sphere can be measured by the angle θ (Fig. 1b), which in the curvilinear coordinates of the sphere is called latitude. The distances from any point of the ellipse to its foci are arcs of circles of large diameter, so they are also measured by angles. In Figure 1a for clarity the point A is taken on the ellipse, the distances to which from the foci F and F_1 coincide with the arcs of curvilinear coordinates: the distance FA is latitude, and the distance F_1A is longitude. The sum of these angles ψ for any point of the ellipse is a constant value and is analogous to the major axis of the ellipse (Fig. 1b).

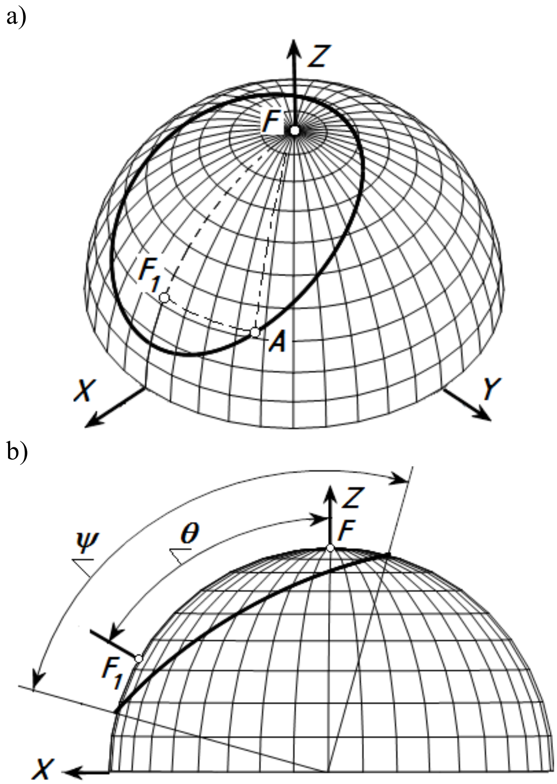

Fig. 1. Graphic illustrations and notation of angles to the construction of a spherical ellipse: a) image of a spherical ellipse with foci F and F_1 and point A on it; b) side view

Let`s denote the latitude by ε and the longitude by γ. The dependence between these curvilinear coordinates in the form $\varepsilon=\varepsilon(\gamma)$ defines a spherical line on the sphere. The authors of the article found the following dependence for a spherical ellipse in which one focus is located on the OZ axis:

$$\varepsilon = \text{arctg}\left(\frac{\cos\theta - \cos\psi}{\sin\psi - \cos\gamma\sin\theta}\right) \quad (1)$$

Equation (1) is the internal equation of a spherical ellipse in the curvilinear coordinates of the sphere.

To construct it, we use the parametric equations of a sphere of radius *R*:

$$X = R\sin\varepsilon\cos\gamma\,; \quad Y = R\sin\varepsilon\sin\gamma\,; \quad Z = R\cos\varepsilon \quad (2)$$

The latitude ε and the longitude γ are independent variables of the equations of sphere (2). On substituting the dependence (1) into (2), these equations become dependent on only one variable – the longitude γ. In this case, they no longer describe the surface of the sphere, and the line on it (spherical ellipse). In Figure 2a by equations (2) and (1) a spherical ellipse is constructed at $R=1$, $\theta=45°$, $\psi=60°$ (denoted by the number *1*). Bearing in mind that in this case equations (2) are not equations of the surface but they are equations of the line, we pass from the notation of the equations in capital letters to the notation in lowercase. For further research, we need to find the derivative of the length of the arc *s* of the spherical ellipse. To do this, we find the derivatives of equations (2) taking into account that $\varepsilon=\varepsilon(\gamma)$:

$$x' = -R\sin\varepsilon\sin\gamma + R\varepsilon'\cos\varepsilon\cos\gamma\,; \quad y' = R\sin\varepsilon\cos\gamma + R\varepsilon'\cos\varepsilon\sin\gamma\,; \quad z' = -R\varepsilon'\sin\varepsilon. \quad (3)$$

According to the known formula we find:

$$s' = \sqrt{x'^2 + y'^2 + z'^2} = R\sqrt{\sin^2\varepsilon + \varepsilon'^2}. \quad (4)$$

In order that the ellipses could be in contact, the second congruent ellipse must be rotated relative to the first around the *OY* axis by an angle ψ (Fig. 2b). When rotating around the axes passing through the foci of the ellipses, they should roll without sliding one by one and maintain point contact. Let`s substantiate it analytically.

When the ellipses roll, the point of contact will change its position, so the angle ε of the first ellipse will also change. Its value can be found by formula (1) depending on the angle of rotation γ. But the second ellipse for each point of contact will have its own angle ε_1. The sum of these angles must be constant: $\varepsilon+\varepsilon_1=\psi$. From here we find the expression for the angle $\varepsilon_1=\psi-\varepsilon$. When rotating the first ellipse about the axis *OZ* by the angle γ, we can find the coordinate ε of the point of contact by formula (1). The corresponding coordinate on the second ellipse is also known: $\varepsilon_1=\psi-\varepsilon$. But the second coordinate φ, which corresponds to the angle γ of the first ellipse, is unknown. It can be found based on the fact that rolling occurs without sliding. In this case, the lengths of the arcs s and s_1 of the first and second ellipses must be equal. To find the length of the arc of the second ellipse, we need to have its equation. The second curve (which should be a congruent ellipse) we`ll also search by equations (2), where $\varepsilon_1=\psi-\varepsilon$, and the angle $\varphi$ will correspond to the angle $\gamma$, and also we`ll make it dependent on γ, that is $\varphi=\varphi(\gamma)$.

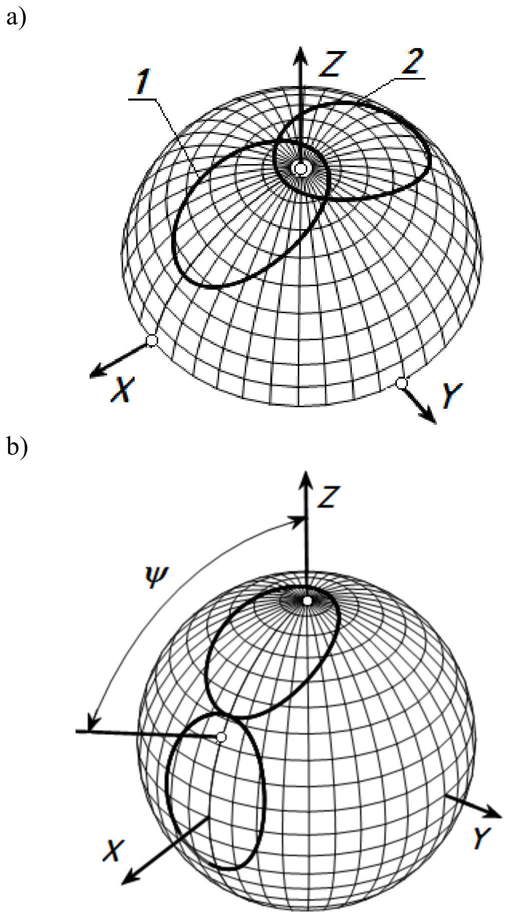

Fig. 2. Spherical ellipses on the surface of the sphere: a) the initial position of two ellipses; b) position of ellipses with point contact

According to (2) the parametric equations of the second curve are written:

$$x_1 = R\sin(\psi-\varepsilon)\cos\phi\,; \quad y_1 = R\sin(\psi-\varepsilon)\sin\phi\,; \quad z_1 = R\cos(\psi-\varepsilon). \quad (5)$$

Let`s find the derivatives of equations (5), having in mind that $\varepsilon=\varepsilon(\gamma)$ and $\varphi=\varphi(\gamma)$:

$$x_1' = R[\varepsilon'\cos(\psi-\varepsilon)\cos\phi + \phi'\sin(\psi-\varepsilon)\sin\phi]; \quad y_1' = -R[\varepsilon'\cos(\psi-\varepsilon)\sin\phi - \phi'\sin(\psi-\varepsilon)\cos\phi]; \quad z_1' = R\varepsilon'\sin(\psi-\varepsilon). \quad (6)$$

Similarly (4) we find the derivative of the arc of the curve (5):

$$s_1' = \sqrt{x_1'^2 + y_1'^2 + z_1'^2} = R\sqrt{\phi'^2 \sin^2(\psi - \varepsilon) + \varepsilon'^2}. \qquad (7)$$

Based on the equality of the arcs of the curves, which depend on the common angle γ, their derivatives will also be equal. We equate (4) and (7) and solve the obtained equation with respect to φ'. After trigonometric transformations we obtain the following result:

$$\frac{d\phi}{d\gamma} = \frac{\varepsilon}{\sin\psi - \varepsilon\cos\psi}. \qquad (8)$$

Substitute in (8) the expression of the angle ε from (1). The result of integrating the obtained dependence $\varphi'=\varphi'(\gamma)$ using the software product "Mathematica" is the following result:

$$\phi = -2\mathrm{arctg}\frac{2\sin^2\left(\frac{\theta+\psi}{2}\right)\mathrm{tg}\frac{\gamma}{2}}{\cos\theta - \cos\psi}. \qquad (9)$$

Substituting the dependence (9) into (5) gives the parametric equations of the curve, which should roll on a spherical ellipse without sliding. According to these equations in Figure 2a constructed the curve at $R=1$, $\theta=45°$, $\psi=60°$ (denoted by the number *2*). The figure shows that the obtained curve *2* is symmetric to the spherical ellipse. If it is rotated by *180°* about the *OZ* axis, it completely coincides with curve *1* that is both curves are congruent. So, the equations of curves *1* and *2* describe the same spherical ellipse. The difference is that in one case, when constructing the point moves clockwise on the ellipse, and in the second – counter clockwise.

3. Results and discussion

Based on the obtained analytical results, we construct separate positions of pairs of ellipses with point contact, which roll one by one, simultaneously rotating about their axes. The angle ψ between the axes of rotation is assumed to be equal to *90°*, as it is usually accepted in mechanisms. In this case, one axis of rotation is the axis *OZ*, and the other – the axis *OY*. In Figure 3,a the initial position of the ellipses at $\theta=60°$ is shown. The area of the meridian between the axes of rotation for clarity is represented by a dashed line.

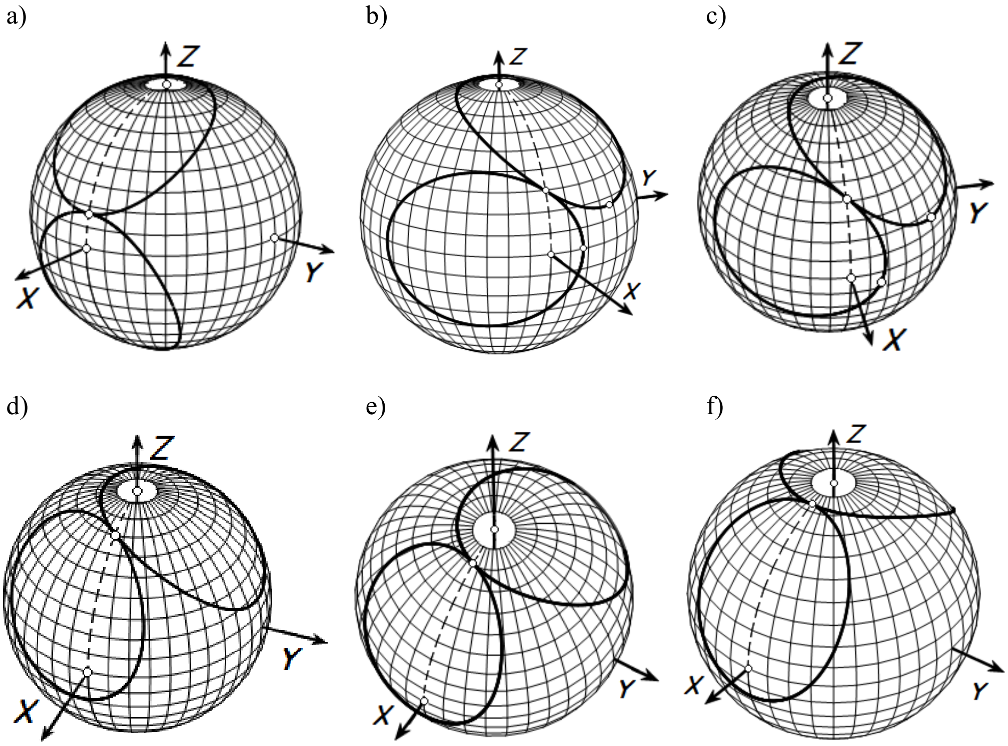

Fig. 3. Intermediate positions of spherical ellipses when they rotate around perpendicular axes (ψ=90°, θ=60°): a) γ=0°, φ=0°; b) γ=30°, φ=–1.5708; c) γ=45°, φ=–1.993;d) γ=90°, φ=–2.618; e) γ=135°, φ=–2.921; f) γ=180°, φ=–3.1416

We rotate the upper ellipse about the axis *OZ* by a given angle *γ*, the lower - about the axis *OZ* by the corresponding angle *φ*, the value of which is found by formula (9). For example, the angle *φ=–1.5708=–90°* is corresponding to the angle *γ=30°*. According to known formulas, we rotate each ellipse at its angle about its axis (Fig. 3b). This figure also shows the points that were in contact before turning. The distances from these points to the new point of contact must be equal, which is visually confirmed. The other figures show the intermediate positions of the ellipses after rotating them to a given angle *γ* and the found angle *φ*. In all cases, the point of contact is located on the meridian that connects the foci of the ellipses. So, when the spherical ellipses rotate about their *OZ* and *OX* axes, the ellipses move one by one without sliding, and their point of contact "floats" along the meridian that connects the foci of the ellipses.

Equation (9) cannot be solved with regard to the angle *γ* as *γ=γ(φ)*. But if we need to set the angle φ and find the angle γ corresponding to it, then this can be done approximately, using the graph *φ=φ(γ)* and then refine the result. In Figure 4 a graph *φ=φ(γ)* as a function of (9) at *ψ=90°* is shown, that is for the case when the axes of rotation intersect at right angle and for different values of the angle *θ*. It shows that at *θ=60°* the angle *φ=90°* corresponds to the angle *γ=30°*. The position of the ellipses for this case is shown in Figure 3b.

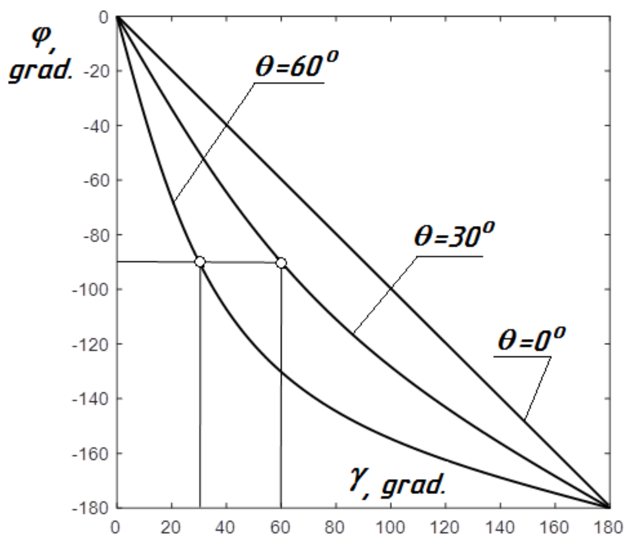

Fig. 4. Dependency graph *φ = φ (γ)* for *ψ* = 90° and different values of the angle *θ*

As the angle *θ*, decreases, that is the distance between the foci, the graph changes and becomes linear at *θ=0°*. As this takes place ellipses transform into circles. For example, at *θ=30°* the angle *γ= 60°*corresponds to the angle *φ=90°*, and the ellipses (Fig. 5a) are closer to the circle in comparison with the ellipses in Figure 3b. If ruled surfaces with vertices at the origin are drawn through spherical ellipses, we obtain conical axoids (Fig. 5b).

a) b)

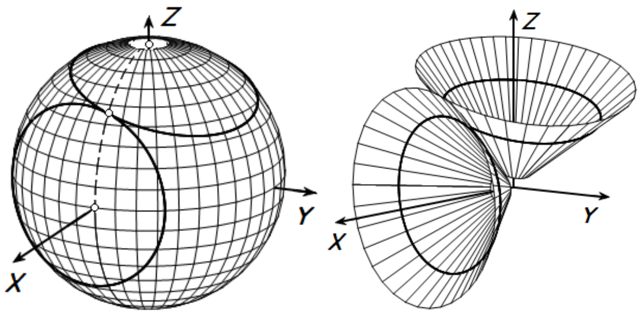

Fig. 5. Spherical ellipses and their corresponding conical axoids (*ψ*=90°, *θ*=30°): a) position of two ellipses at *φ*=90°, *γ*=60°; b) corresponding conical axoids passing through spherical ellipses

When rotating about fixed *OZ* and *OX* axes, they will roll one by one without sliding; herewith a straight line of their contact will pass through the point of contact of the ellipses. Figure 6 shows the relative position of axoids, sphere and ellipses for *ψ=90°* and *θ=60°*. They correspond to Figures 3a and 3b.

a) b)

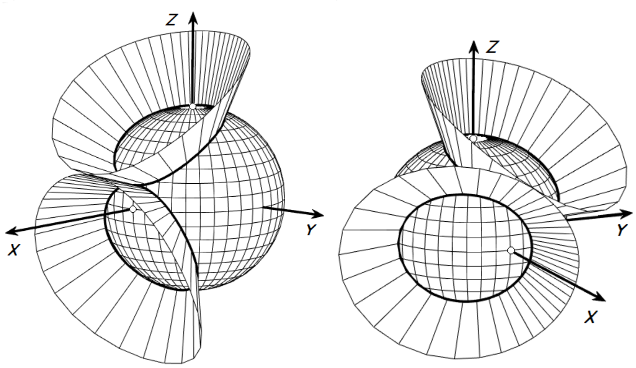

Fig. 6. Spherical ellipses and corresponding to them conical axoids (*ψ*=90°, *θ*=60°): a) position of two ellipses at *γ*=0°, *φ*=0°;b) position of two ellipses at *γ*=30°, *φ*=–90°

We obtain all possible variants of axoids at values of the angle *ψ* within *ψ=0...90°*. If we take the angle *ψ=100°*, we obtain axoids which axes intersect at an angle of *80°*, that is the same result as at *ψ=80°*.

4. Conclusions

The ellipses on the plane serve as centroids for the design of cylindrical gears with non-circular wheels, in which the axes of rotation pass through the foci of the ellipses. Spherical ellipses act as spherical centroids to design similar bevel gears. For spherical ellipses, the distances on the surface of the sphere are convenient to set in the angular dimension. An analogue of the major axis of a spherical ellipse is the angle ψ, which is the angle of intersection of the axes of rotation. At $\psi=90°$, the axes of rotation of spherical ellipses intersect at right angle regardless of the distance between the foci, that is the angle θ. As the angle θ decreases, the spherical ellipses approach the circles in shape and turn into them at $\theta=0°$. Spherical ellipses are directing curves for the construction of the corresponding conical axoids.

References

[1] F.L. Litvin, Theory of gears, Second Edition, Nauka, 1968 (in Russian).

[2] V. Berezin, Spherical ellipse, Popular Science Physics and Mathematics Journal "Quantum" 2 (1978) 25.

[3] V.M. Gusyatin, A.E. Gromenko, R. V. Sorokin, Method of spherical interpolation of triangulated surfaces, Proceedings of the 2nd International Scientific Conference “Modern information systems. Problems and Development Trends”, 2007, 495-496.

[4] S.F. Pylypaka, I. Yu. Gryshchenko, T.A. Kresan, Modeling of strips of deployable surfaces tangent to the surface of the sphere, Applied Questions of Mathematical Modeling. Kherson: OLDI-PLUS 1 (2018) 81-88.

[5] G.A. Golub, V.V. Chuba, O.A. Marus, Determination of rolling radius of self-propelled machines' wheels, INMATEH - Agricultural Engineering 57/1 (2019) 81-90.

[6] T. Kresan, S. Pylypaka, Z. Ruzhylo. I. Rogovskii, O. Trokhaniak, External rolling of a polygon on closed curvilinear profile, Acta Polytechnica 60/4 (2020) 313-317. DOI: https://doi.org/10.14311/AP.2020.60.0313

[7] V. Bulgakov, S. Pilipaka, V. Adamchuk, J. Olt, Theory of motion of a material point along a plane curve with a constant pressure and velocity, Agronomy Research 12/3 (2014) 937-948.

[8] T.A. Kresan, S.F. Pylypaka, I.Yu. Gryshchenko, V.M. Babka, A special case of congruent centroids of non-circular wheels formed by arcs of a logarithmic spiral, Applied Geometry and Engineering Graphics 98 (2020) 84-93. DOI: https://doi.org/10.32347/0131-579x.2020.98.84-93

[9] B. Laczik, Design and Manufacturing of Non-Circular Gears by Given Transfer Function. Available online: http://www.hexagon.de/pdf/noncgear.pdf

[10] D. Mundo, G.A. Danneli, Use of Non-Circular Gears in Pressing Machine Driving Systems. Available online: http://www.wseas.us/e-library/conferences/udine2004/papers/483-172.pdf

[11] V.V. Kovregin, I.V. Molovik, Analytical description of centroids of non-circular gears, Proceedings of the Tavriya State Agrotechnological University. Applied Geometry and Engineering Graphics 49/4 (2011) 125-129.

[12] Ya.P. Legeta, Description and construction of coupled centroid of non-circular gears, Modern Problems of Modelling 3 (2014) 87-92.

[13] Ya.P. Legeta, O.V. Shoman, Geometric modeling of centroids of non-circular gears by transfer function, Geometric Modeling and Information Technology. Scientific Journal of the Mykolaiv National University named after V.O. Sukhomlinsky 2 (2016) 59-63.

[14] A.P. Padalko, N.A. Padalko, Tooth gear with non-circular wheel, Theory of Mechanisms and Machines 11/2 (2013) 89-96.

[15] A.N. Sobolev, A.Ya. Nekrasov, M.O. Arbuzov, Modeling of mechanical gears with non-circular gears, Bulletin of Moscow State University "Stankin" 40/1 (2017) 48-51.

[16] F. Taş, K. İlarslan, A New Approach to Design the Ruled Surface, International Journal of Geometric Methods in Modern Physics 16/6 (2019) 1950093. DOI: https://doi.org/10.1142/S0219887819500932

[17] Ö.G. Yıldız, M. Akyiğit, M. Tosun, On the trajectory ruled surfaces of framed base curves in the Euclidean space, Mathematical Methods in the Applied Sciences 44/9 (2021) 7463-7470. DOI: https://doi.org/10.1002/mma.6267

[18] J. Wallner, Ruled surfaces and developable surfaces. Available online: http://www.geometrie.tugraz.at/wallner/kurs.pdf