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ON HOMOGENEOUS FUNCTIONS IN SECOND-ORDER FIELD THEORY

31.1 INTRODUCTION

In this paper we analyse properties of functions, which are invariant with respect to diffeomorphisms of their domains of definition, and give rise to variational functionals independent of parametrizations. This leads to the (higher-order) homogeneity concept, expressed by the well-known Zermelo conditions (see e.g. McKiernan [7], Kawaguchi [3], Matsyuk [6], Crampin and Saunders [1]); physical meaning of these conditions was studied by Kondo [4]. To this purpose we use an elementary version of the Ehresmann's theory of jets, differential groups and jet prolongations of curves and tangent spaces (see Grigore and Krupka [2], Krupka and Janyška [5], and references therein). On this basis, we introduce the positive homogeneity concept for functions, depending on mappings and their (partial) derivatives of first and second order.

Our main results consist in a characterization of second-order Lagrangians whose extremals are set-solutions, and in the study of a class of variational second-order partial differential equations. We show that variational systems with positive homogeneous Lagrangians are defined by positive homogeneous functions, and vice versa. Variational foundations of Finsler geometry, where fundamental functions satisfies the positive homogeneity condition, could be extended by means of the presented theory.

This paper extends the work Urban and Krupka [8] to mappings between Euclidean spaces (*field theory*). Throughout the paper the proofs are omitted.

31.2 REGULAR VELOCITIES OVER EUCLIDEAN SPACES

In this section we consider the mappings $\gamma : \mathbf{R}^n \to \mathbf{R}^{m+n}$, $n \geq 2$, $m \geq 1$, between Euclidean spaces, which associate a space endowed with a finite-dimensional Lie group action, the reparametrization of mappings. The canonical coordinates on $\mathbf{R}^{m+n}$ are denoted by y^K, $K = 1, 2, \dots, m + n$.

By an n-velocity of order 2 at a point $y \in \mathbf{R}^{m+n}$ we mean a 2-jet $J_0^2\gamma$ with source $0 \in \mathbf{R}^n$ and target $\gamma(0) = y$. If γ is a representative of $J_0^2\gamma$, then the equivalence class $J_0^2\gamma$ is identified with the ordered collection of numbers $(y^K(J_0^2\gamma), y_j^K(J_0^2\gamma), y_{jk}^K(J_0^2\gamma))$, where $1 \leq j \leq k \leq n$, defined by the derivatives of the mapping $(x^i) \to y^K\gamma(x^i)$ at the origin $0 \in \mathbf{R}^n$, i.e.

$$\begin{aligned} &y^K(J_0^2\gamma) = y^K(\gamma(0)), \quad y_j^K(J_0^2\gamma) = D_j(y^K\gamma)(0), \\ &y_{jk}^K(J_0^2\gamma) = D_kD_j(y^K\gamma)(0). \end{aligned} \tag{31.1}$$

The set of n-velocities of order 2 at $y \in \mathbf{R}^{m+n}$ is the jet space $J^2_{(0,y)}(\mathbf{R}^n, \mathbf{R}^{m+n})$, and we denote

$$T_n^2\mathbf{R}^{m+n} = \bigcup_{y\in\mathbf{R}^{m+n}} J^2_{(0,y)}(\mathbf{R}^n, \mathbf{R}^{m+n}).$$

The *canonical jet projection* $\tau^{2,0} : T_n^2\mathbf{R}^{m+n} \to \mathbf{R}^{m+n}$, is a surjective mapping defined by $\tau^{2,0}(J_0^2\gamma) = \gamma(0)$. We consider the set $T_n^2\mathbf{R}^{m+n}$ of velocities of order 2 with standard geometric structures. The functions (y^K, y_j^K, y_{jk}^K), defined by (31.1), are the (global) *canonical coordinates* on $T_n^2\mathbf{R}^{m+n}$. The canonical trivialization of $T_n^2\mathbf{R}^{m+n}$ is the mapping

$$T_n^2\mathbf{R}^{m+n} \ni J_0^2\gamma \mapsto (\gamma(0), J_0^2\mathrm{tr}_{\psi\gamma(0)}\psi\gamma) \in \mathbf{R}^{m+n} \times J^2_{(0,0)}(\mathbf{R}^n, \mathbf{R}^{m+n}). \tag{31.2}$$

In particular, the mapping (31.2) shows that $T_n^2\mathbf{R}^{m+n}$ is a trivial vector bundle with base $\mathbf{R}^{m+n}$, projection $\tau^{2,0} : T_n^2\mathbf{R}^{m+n} \to \mathbf{R}^{m+n}$, and type fiber $J^2_{0,0}(\mathbf{R}^n, \mathbf{R}^{m+n})$. We call $T_n^2\mathbf{R}^{m+n}$ the *manifold of n-velocities* of order 2 over $\mathbf{R}^{m+n}$.

Recall now the concept of the differential group of $\mathbf{R}^n$. Consider the manifold $J^2_{(0,0)}(\mathbf{R}^n, \mathbf{R}^n)$ of 2-jets with source and target at the origin $0 \in \mathbf{R}^n$. Let α^i denotes the i-th component of the mapping $\alpha : \mathbf{R}^n \to \mathbf{R}^n$ for which $\alpha(0) = 0$. For every $J_0^2\alpha \in J^2_{(0,0)}(\mathbf{R}^n, \mathbf{R}^n)$, we define real-valued functions a_j^i and a_{jk}^i on $J^r_{0,0}(\mathbf{R}^n, \mathbf{R}^n)$ by

$$a_j^i(J_0^2\alpha) = D_j\alpha^i(0), \quad a_{jk}^i(J_0^2\alpha) = D_kD_j\alpha^i(0), \tag{31.3}$$

where $1 \leq j \leq k \leq n$. These functions are called the *canonical coordinates*, and constitute a global chart on $J^2_{(0,0)}(\mathbf{R}^n, \mathbf{R}^n)$. The *invertible* jets form a subset of $J^2_{(0,0)}(\mathbf{R}^n, \mathbf{R}^n)$, consisting of the 2-jets $J_0^2\alpha$ represented by *diffeomorphisms* α, which we denote by

$$L_n^2 = \mathrm{Imm}\, J^2_{(0,0)}(\mathbf{R}^n, \mathbf{R}^n) = \{J_0^2\alpha \in J^2_{(0,0)}(\mathbf{R}^n, \mathbf{R}^n) \,|\, \det(a_j^i(J_0^2\alpha)) \neq 0\}.$$

Clearly, L_n^2 is dense and open subset in $J^2_{(0,0)}(\mathbf{R}^n, \mathbf{R}^n)$. Restricting the functions a_j^i, a_{jk}^i (31.3) to the domain L_n^2, we obtain the canonical global coordinates on L_n^2. A mapping

$$L_n^2 \times L_n^2 \ni (J_0^2\alpha, J_0^2\beta) \mapsto J_0^2\alpha \circ J_0^2\beta = J_0^2(\alpha \circ \beta) \in L_n^2, \tag{31.4}$$

defined by the composition of jets, represents a group multiplication on L_n^2. The set L_n^2 with the group multiplication (31.4) has a Lie group structure, and it is called the *second differential group* of $\mathbf{R}^n$.

The manifold $T^2_n\mathbf{R}^{m+n}$ is endowed with the right action of the differential group L^2_n given by composition of jets,

$$T^2_n\mathbf{R}^{m+n} \times L^2_n \ni (J^2_0\gamma, J^2_0\alpha) \mapsto J^2_0\gamma \circ J^2_0\alpha = J^2_0(\gamma \circ \alpha) \in T^2_n\mathbf{R}^{m+n}, \tag{31.5}$$

In order to describe this action in canonical coordinates, we denote

$$\bar{y}^K(J^r_0\gamma) = y^K(J^r_0(\gamma \circ \alpha)), \quad \bar{y}^K_j(J^r_0\gamma) = y^K_j(J^r_0(\gamma \circ \alpha)), \quad \bar{y}^K_{jk}(J^r_0\gamma) = y^K_{jk}(J^r_0(\gamma \circ \alpha)).$$

Lemma 31.1. *The group action* (31.5) *of the differential group* L^2_n *onto* $T^2_n\mathbf{R}^{m+n}$ *is expressed by the equations*

$$\bar{y}^K = y^K, \quad \bar{y}^K_j = y^K_p a^p_j, \quad \bar{y}^K_{jk} = y^K_{pq} a^p_j a^q_k + y^K_p a^p_{jk}. \tag{31.6}$$

Any differentiable mapping γ with values in $\mathbf{R}^{m+n}$, defined on an open subset $U \subset \mathbf{R}^n$, associates the mapping

$$U \ni x \mapsto T^2_n\gamma(x) = J^2_0(\gamma \circ \mathrm{tr}_{-x}) \in T^2_n\mathbf{R}^{m+n}, \tag{31.7}$$

called the (second) *jet prolongation* of the mapping γ. For every isomorphism $\mu : V \to U$ of open subsets of $\mathbf{R}^n$, and every point $z \in V$, $J^2_0(\mathrm{tr}_{\mu(z)} \circ \mu \circ \mathrm{tr}_{-z})$ belongs to the differential group L^2_n. Denote

$$\mu_z = \mathrm{tr}_{\mu(z)} \circ \mu \circ \mathrm{tr}_{-z}, \quad \mu^{(2)}(z) = J^2_0\mu_z. \tag{31.8}$$

In the following lemma we describe the formula for reparametrizations of the domain of definition of $T^2_n\gamma$.

Lemma 31.2. *For any diffeomorphism* $\mu : V \to U$ *of open subsets of* $\mathbf{R}^n$*, the jet prolongation* $T^2_n\gamma$ *of a mapping* $\gamma : U \to \mathbf{R}^{m+n}$ *satisfies*

$$T^2(\gamma \circ \mu)(z) = T^2\gamma(\mu(z)) \circ \mu^{(2)}(z). \tag{31.9}$$

We restrict our attention to mappings which are *immersions* (i.e. their tangent mappings are injective). $J^2_0\gamma \in T^2_n\mathbf{R}^{m+n}$ is called *regular*, if every representative of $J^2_0\gamma$ is an immersion at $0 \in \mathbf{R}^n$. The set $\mathrm{Imm}\, T^2_n\mathbf{R}^{m+n}$ of regular velocities form an open subset of $T^2_n\mathbf{R}^{m+n}$, and with the open submanifold structure $\mathrm{Imm}\, T^2_n\mathbf{R}^{m+n}$ is called the *manifold of regular n-velocities of order* 2 *over* $\mathbf{R}^{m+n}$. Restricting the canonical coordinates (y^K, y^K_j, y^K_{jk}), we get the canonical charts on $\mathrm{Imm}\, T^2_n\mathbf{R}^{m+n}$, induced by the canonical atlas of $T^2_n\mathbf{R}^{m+n}$.

The manifold of regular velocities $\mathrm{Imm}\, T^2_n\mathbf{R}^{m+n}$ is also endowed with another smooth structure. Denote by $(i) = (i_1, i_2, \ldots, i_n)$ an increasing n-subsequence of the sequence $(1, 2, \ldots, m+n)$, and by (σ) the complementary increasing subsequence. From the definition of an immersion it follows that for every regular velocity $J^2_0\gamma$ there exists an n-subsequence (i) of $(1, 2, \ldots, m+n)$ such that

$$\det y^i_j(J^1_0\zeta) = \det(D_j(y^i\zeta)(0)) \neq 0,$$

where $i \in (i)$, $1 \leq j \leq n$. Put

$$W_n^{2(i)} = \{J_0^2\gamma \in \operatorname{Imm} T_n^2\mathbf{R}^{m+n} \mid \det(D_j(y^i\zeta)(0)) \neq 0\}, \tag{31.10}$$

for every n-subsequence (i) of $(1, 2, \dots, m+n)$. Clearly, $W_n^{2(i)}$ is an open subset of $\operatorname{Imm} T_n^2\mathbf{R}^{m+n}$, and $\operatorname{Imm} T_n^2\mathbf{R}^{m+n}$ is covered by the sets $W_n^{2(i)}$, where (i) runs through all n-subsequences of $(1, 2, \dots, m+n)$. We introduce the adapted coordinates $\chi_n^{2(i)} = (w^i, w^\sigma, w^i_j, w^\sigma_i, w^i_{jk}, w^\sigma_{ip})$ on $W_n^{2(i)} \subset \operatorname{Imm} T_n^2\mathbf{R}^{m+n}$, which arise from the canonical coordinates and their derivatives in the following way:

$$\begin{aligned} &y^i = w^i, \quad y^\sigma = w^\sigma, \quad y^i_j = w^i_j, \quad y^\sigma_j = y^i_j w^\sigma_i, \\ &y^i_{jk} = w^i_{jk}, \quad y^\sigma_{jk} = y^i_{jk} w^\sigma_i + y^i_j y^p_k w^\sigma_{ip}. \end{aligned} \tag{31.11}$$

It is easy to observe that the coordinates w^i, w^σ, w^σ_i, w^σ_{ip}, are L^2_n-invariant. The chart $(W_n^{2(i)}, \chi_n^{2(i)})$ is called the (i)-*subordinate chart*, *adapted* to the canonical group action of L^2_n on $\operatorname{Imm} T_n^2\mathbf{R}^{m+n}$.

31.3 HOMOGENEOUS FUNCTIONS AND THE EULER–ZERMELO THEOREM

Consider the manifold of regular velocities $\operatorname{Imm} T_n^2\mathbf{R}^{m+n}$, endowed with the canonical coordinates y^K, y^K_j, y^K_{jk}, and the differential group L^2_n with the canonical coordinates a^i_j, a^i_{jk}.

A function $F : \operatorname{Imm} T_n^2\mathbf{R}^{m+n} \to \mathbf{R}$ is called *positive homogeneous* in the variables y^K, y^K_j, y^K_{jk} (of degree 1), or just *positive homogeneous*, if

$$F(J_0^2\gamma \circ J_0^2\alpha) = \det D\alpha \cdot F(J_0^2\gamma) \tag{31.12}$$

for all $J_0^2\gamma \in \operatorname{Imm} T_n^2\mathbf{R}^{m+n}$ and all $J_0^2\alpha \in L^2_n$ such that $\det D\alpha > 0$.

Note that the elements of the differential group L^2_n from this definition are represented by *orientation-preserving* diffeomorphisms. Equivalently, condition (31.12) can be expressed as

$$F(\bar{y}^K, \bar{y}^K_j, \bar{y}^K_{jk}) = \det(a^i_j) \cdot F(y^K, y^K_j, y^K_{jk})$$

for all points $(y^K, y^K_j, y^K_{jk}) \in \operatorname{Imm} T_n^2\mathbf{R}^{m+n}$ (see (31.6)), and all real numbers a^i_j, a^i_{jk} such that $\det(a^i_j) > 0$.

Let $F : \operatorname{Imm} T_n^2\mathbf{R}^{m+n} \to \mathbf{R}$ be a function, and denote by ω_0 the global volume element of $\mathbf{R}^n$, i.e. $\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$. Let $\gamma : U \to T_n^2\mathbf{R}^{m+n}$ be an immersion defined on an open subset U of $\mathbf{R}^n$, and $T_n^2\gamma$ the jet prolongation of γ (cf. (31.7)). Any compact subset S of U associates with F the integral

$$F_S(\gamma) = \int_S (F \circ T_n^2\gamma)(x)\, \omega_0. \tag{31.13}$$

Our main theorem is a criterion of parameter-invariance of the integral (31.13), and positive homogeneity of the corresponding function.

Theorem 31.1. (Euler-Zermelo) *Let* $F : \operatorname{Imm} T^2_n \mathbf{R}^{m+n} \to \mathbf{R}$ *be a differentiable function. The following conditions are equivalent:*

(a) *The function* F *is positive homogeneous.*

(b) *The integral* $F_S(\gamma)$ (31.13) *is parameter-invariant.*

(c) *The function* F *satisfies the conditions*

$$\frac{\partial F}{\partial y^K_i} y^K_j + 2\frac{\partial F}{\partial y^K_{jk}} y^K_{ik} = \delta^i_j F, \quad \frac{\partial F}{\partial y^K_{jk}} y^K_i = 0, \tag{31.14}$$

where $1 \leq i, j, k \leq n$*, and* $j \leq k$*.*

The proof is based, roughly speaking, on differentiating the positive homogeneity condition (31.12) on one side, and in particular on the use of the adapted coordinates $(w^i, w^\sigma, w^i_j, w^\sigma_i, w^i_{jk}, w^\sigma_{ip})$ (31.11) on open subsets $W^{2(i)}_n$ of $\operatorname{Imm} T^2_n \mathbf{R}^{m+n}$ with respect to the differential group L^2_n on the other side. For regular curves ($n = 1$), the proof can be found in Urban and Krupka [8].

The conditions (31.14) are the well-known *Zermelo conditions*, which generalize the standard Euler theorem on homogeneous functions. In the case of first order velocities, $r = 1$, the Zermelo conditions for functions of several variables were studied by McKiernan [7]. Recently, Crampin and Saunders [1] applied the Zermelo conditions within the homogeneous variational theory on infinite-order frame bundles.

31.4 HOMOGENEOUS VARIATIONAL EQUATIONS

We give a note on a class of second-order variational partial differential equations with a positive homogeneous Lagrangian.

Let $F : \operatorname{Imm} T^1_n \mathbb{R}^{m+n} \to \mathbb{R}$ be a function, and $\gamma : U \to \mathbb{R}^{m+n}$ be an immersion defined on an open subset U of $\mathbb{R}^n$. Any compact subset S of U associates with F the integral $F_S(\gamma)$, (31.13), and the variational functional

$$\gamma \to F_S(\gamma) = \int_S (F \circ T^1_n \gamma)(x)\omega_0. \tag{31.15}$$

The equations for extremals of (31.15) are the *Euler–Lagrange equations* $E_K(F) = 0$, where $E_K(F)$ are real-valued functions on $\operatorname{Imm} T^2_n \mathbb{R}^{m+n}$,

$$E_K(F) = \frac{\partial F}{\partial y^K} - d_j \frac{\partial F}{\partial y^K_j} = \frac{\partial F}{\partial y^K} - \frac{\partial^2 F}{\partial y^L \partial y^K_j} y^L_j - \frac{\partial^2 F}{\partial y^L_s \partial y^K_j} y^L_{js}, \tag{31.16}$$

called the *Euler–Lagrange expressions* of F. In (31.16), $d_j f$ denotes the j-th *formal derivative* of $f = f(y^K, y^K_j)$ with respect to the canonical coordinates y^K.

Consider now a system of second-order differential equations,

$$\varepsilon_M(y^K, y^K_j, y^K_{jk}) = 0, \tag{31.17}$$

$1 \leq K, M \leq m+n$, where the functions ε_M are defined on $\operatorname{Imm} T_n^2\mathbb{R}^{m+n}$. We say that (31.17) is variational, if there exists a real function $F : \operatorname{Imm} T_n^1\mathbb{R}^{m+n} \to \mathbb{R}$ such that $\varepsilon_M = E_M(F)$, i.e. if (31.17) coincide with equations for extremals of a certain variational functional (31.15).

We conclude this paper by describing an important property of variational systems of equations on $\operatorname{Imm} T_n^2\mathbb{R}^{m+n}$, defined by positive homogeneous functions.

Theorem 31.2. *Suppose the system of second-order equations,* $\varepsilon_M(y^K, y^K_j, y^K_{jk}) = 0$, (31.17) *is variational. The following conditions are equivalent:*

(a) *The functions* ε_M *are positive homogeneous.*

(b) *The system* (31.17) *has a positive homogeneous Lagrangian.*

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ON HOMOGENEOUS FUNCTIONS IN SECOND-ORDER FIELD THEORY

Abstract: *The classical concept of a homogeneous function is introduced and extended within the theory of differential groups, known in the theory of differential invariants. Invariance under reparametrizations of solutions of partial differential equations is studied. On this basis the well-known generalizations of the Euler theorem are obtained (the Zermelo conditions). The positive homogeneity concept is then applied to second-order variational equations in field theory.*

Keywords: *Lagrangian, Euler-Lagrange equations, Zermelo conditions, jet, differential group, field theory.*

O HOMOGENNÍCH FUNKCÍCH V TEORII POLE DRUHÉHO ŘÁDU

Abstrakt: *Standardní koncept homogenní funkce je zaveden a zobecněn pomocí užití diferenciálních grup, známých v teorii diferenciálních invariantů. Studujeme invarianci vzhledem k reparametrizacím integrálních křivek parciálních diferenciálních rovnic. Na základě tohoto přístupu obdržíme známé zobecnění Eulerova teorému, tzv. Zermelovy podmínky. Koncept pozitivní homogenity aplikujeme na variační rovnice druhého řádu v teorii pole.*

Klíčová slova: *Lagrangián, Eulerovy-Lagrangeovy rovnice, Zermelovy podmínky, jet, diferenciální grupa, teorie pole*

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Mgr. Ján BRAJERČÍK, Ph.D.,
University of Prešov
Department of Physics, Mathematics and Techniques
17. novembra 1, 081 16, Prešov, Slovak Republic
tel.: +421 51 7570316, e-mail: jan.brajercik@unipo.sk

RNDr. Zbyněk URBAN, Ph.D.,
VŠB – Technical University of Ostrava
Department of Mathematics and Descriptive Geometry
17. listopadu 15, 708 33, Ostrava, Czech Republic
tel.: +420 597 324 152, e-mail: zbynek.urban@vsb.cz