WAVES IN ELASTIC MATERIALS AND STRUCTURES

ZBIGNIEW WESOŁOWSKI

The mathematical description of wave propagation in elastic materials and structures is presented. Specifically, the focus is on the water as a transmission medium for acoustic waves. The following issues are discussed: discontinuity, reflection, and propagation speed, among others.

1. LINEAR AND NONLINEAR ELASTICITY

The Cartesian coordinate system is used. The displacement vector u_i is a function of position (x_1, x_2, x_3) and time t (cf. [1],[8])

$$u_i = u_i(x_1,x_2,x_3,t). \tag{1}$$

The second order deformation tensor ε_{ij} is given by the expression:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2}$$

A comma placed after index k denotes differentiation with respect to the coordinate x_k. Elastic energy σ stored in a unit of volume is a function of deformation tensor

$$\sigma = \sigma\left(\varepsilon_{ij}\right). \tag{3}$$

The stress tensor τ_{ij} equals the derivative of the energy with respect to the deformation:

$$\tau_{ij} = \rho\frac{\partial\sigma}{\partial\varepsilon_{ij}} = c_{ijkl}u_{k,l}. \tag{4}$$

There exist at most (for crystals of triclinic symmetry) 21 independent elastic functions $c_{ijkl}(\varepsilon_{\kappa l})$, corresponding to 21 elastic constants c_{ijkl} in linear material. Crystals with cubic symmetry are characterized by 3 elastic constants. For isotropic linear elastic material for which

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{5}$$

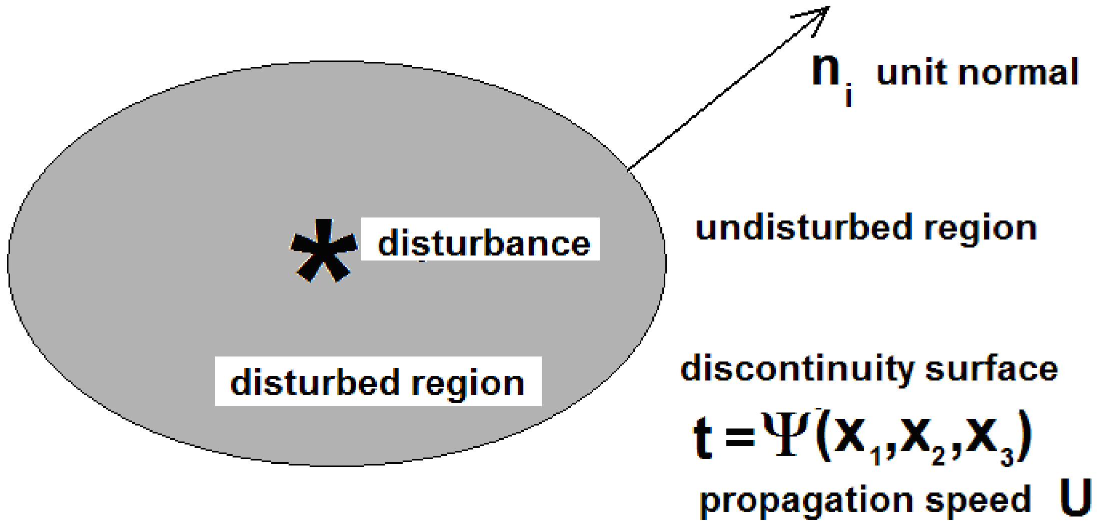
there exist only two independent elastic constants λ and μ, called Lame constants. Nonlinear, second order differential equation of motion reads:

$$(c_{ijkl}u_{k,l})_{,j} = \rho\frac{\partial^2 u_i}{\partial t^2}\,. \tag{6}$$

2. WAVE

There exist precise definitions of all particular waves, e.g., longitudinal, torsional, sinusoidal, shock, standing, solitary. A general definition of a wave does not exist.

When confronted with the notion of *wave,* engineers try to describe it using the sine function. In contrast the mathematicians, geologists try to describe it using generalized functions, specifically by the Dirac and the Heaviside functions (cf. [2], [4], [5], [8]).



The function of three space variables x_1, x_2, x_3:

$$t=\Psi(x_1,x_2,x_3), \tag{7}$$

defines the moving discontinuity surface. At time t, it separates the disturbed region from the undisturbed region. The regions change their position and shape. The unit normal to the discontinuity surface is

$$\mathbf{n}_i = \frac{\partial \Psi}{\partial \mathbf{x}_i}\left[\left(\frac{\partial \Psi}{\partial \mathbf{x}_1}\right)^2 + \left(\frac{\partial \Psi}{\partial \mathbf{x}_2}\right)^2 + \left(\frac{\partial \Psi}{\partial \mathbf{x}_3}\right)^2\right]^{-\frac{1}{2}} \tag{8}$$

and its propagation speed U is given by the relation

$$U = \left[\left(\frac{\partial \Psi}{\partial x_1}\right)^2 + \left(\frac{\partial \Psi}{\partial x_2}\right)^2 + \left(\frac{\partial \Psi}{\partial x_3}\right)^2\right]^{-\frac{1}{2}}. \tag{9}$$

3. DISCONTINUITY WAVE

Assume that the quantity $H=g(x_1, x_2, x_3, t)$ (for example stress, derivative of temperature or integral of the density) is continuous on the surface $\Psi(x_1,x_2,x_3)$. The derivatives of H may be discontinuous on Ψ. Two vertical lines denote the jump, i.e., the difference between the value of the derivative of H on the front side of Ψ and the the derivative of H on the rear side of Ψ. The jump of the derivatives of H satisfies the relations

$$\left\|\frac{\partial H}{\partial x_i}\right\| = An_i, \quad \left\|\frac{\partial H}{\partial t}\right\| = -AU. \tag{10}$$

4. WEAK DISCONTINUITY WAVE

For a weak discontinuity wave all ten derivatives of the displacement u_k (cf. [3],[8]):

$$H = \frac{\partial u_k}{\partial x_l} \text{ oraz } H = \frac{\partial u_k}{\partial t} \tag{11}$$

possess no jump on the discontinuity surface. The higher derivatives, i.e.,

$$\frac{\partial^2 u_k}{\partial x_l \partial x_m}, \frac{\partial^2 u_k}{\partial x_l \partial t}, \frac{\partial^2 u_k}{\partial t^2} \tag{12}$$

may be discontinuous. The above relations lead to the conclusion, that these 27+9+3=39 jumps must have the form

$$\left\| \frac{\partial u_i}{\partial x_k \partial x_r} \right\| = A_i n_k n_r, \left\| \frac{\partial u_i}{\partial x_k \partial t} \right\| = -A_i U n_k, \left\| \frac{\partial^2 u_i}{\partial t^2} \right\| = A_i U^2 . \tag{13}$$

In this situation, the coefficients of the equation of motion are continuous. Therefore, the following equation must be satisfied:

$$c_{ijkl} n_j n_l A_k = \rho U^2 A_i \tag{14}$$

The acoustic tensor is defined as

$$Q_{ik} = c_{ijkl} n_j n_l \tag{15}$$

from which is obtained the propagation condition

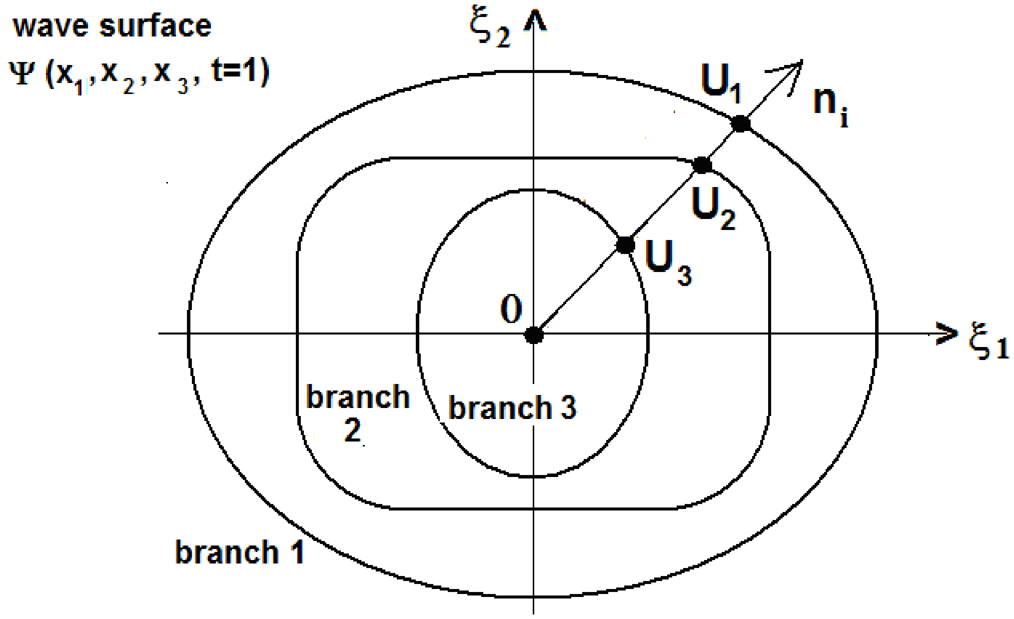
$$(Q_{ik} - \rho U^2 \delta_{ik}) A_k = 0 , \tag{16}$$

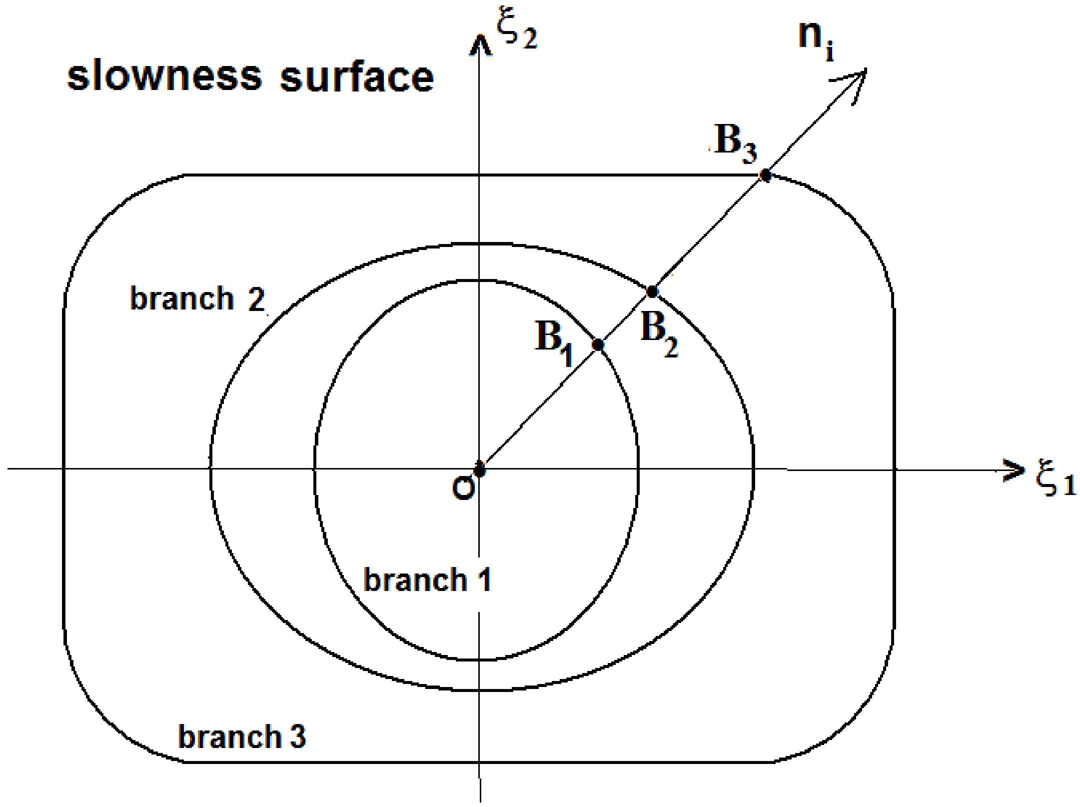
where δ_{ik} is the Kronecker delta. The amplitude is a proper vector, and the product $\rho\ U^2$ is the corresponding proper value of the acoustic tensor. Since the acceleration is discontinuous on ψ this wave is frequently called an acceleration wave. If A_k is parallel to n_k the wave is longitudinal; if A_k is perpendicular to n_k the wave is transverse. A typical wave is neither longitudinal nor transverse. The 6-th order algebraic equation

$$det(Q_{ik} - \rho U^2 \delta_{ik}) = 0 . \tag{17}$$

is satisfied.

Therefore, for each propagation direction (n_1,n_2,n_3) there exist 6 possible propagation speeds U_1 , - U_1 , U_2 , - U_2 , U_3 ,- U_3 and 6 corresponding amplitudes.

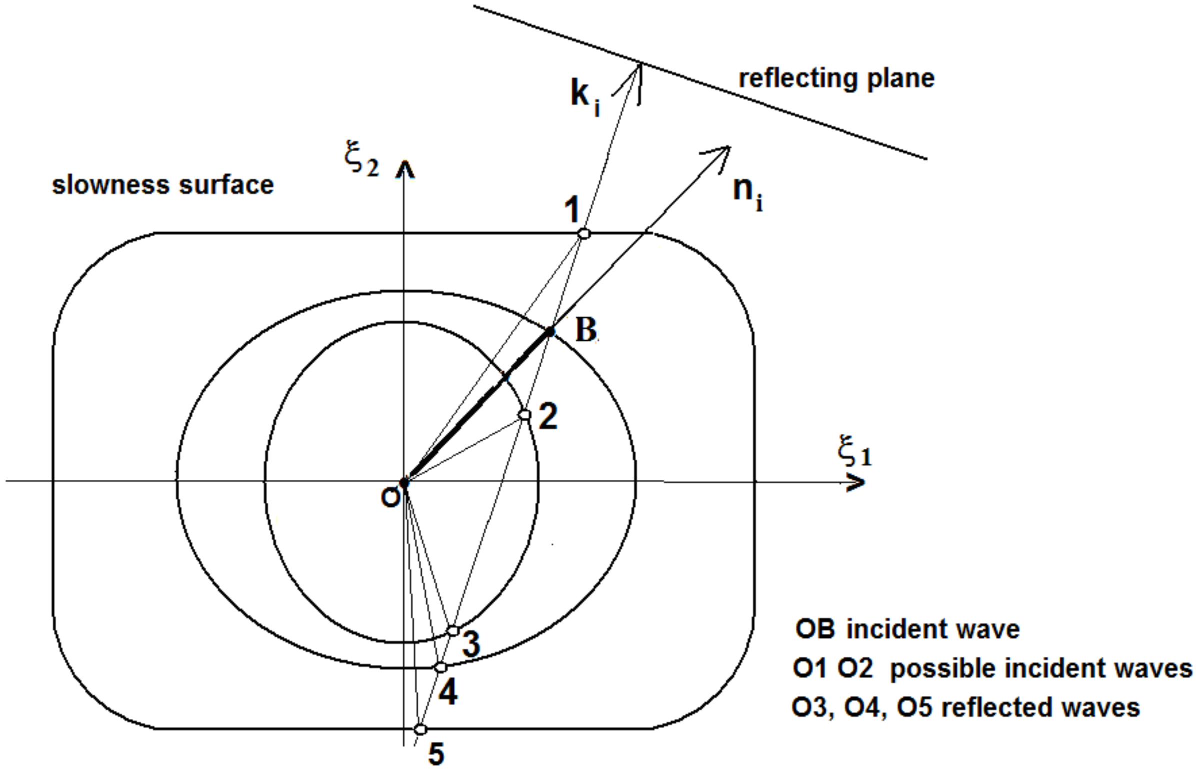




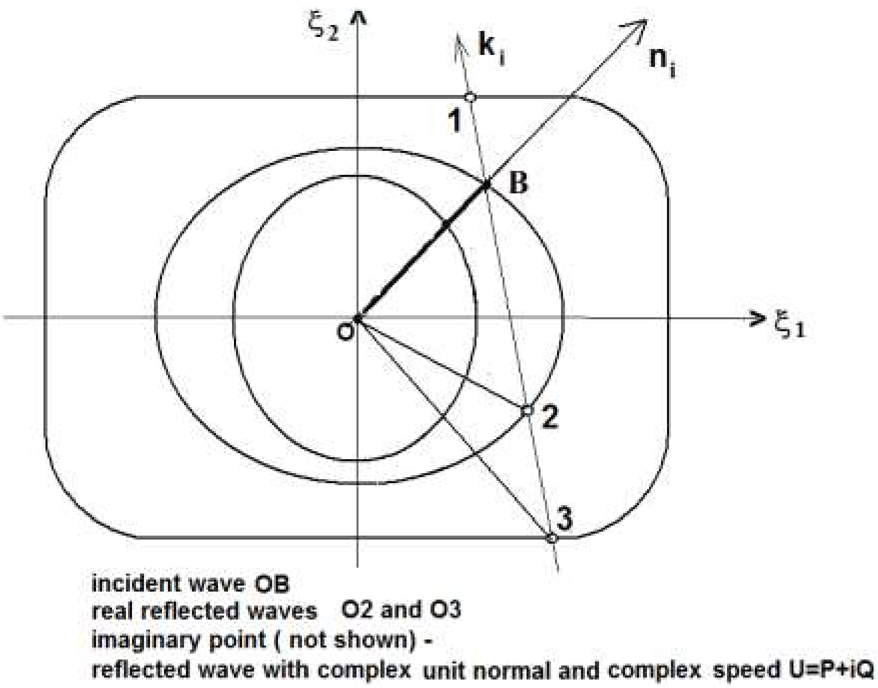
$$\frac{1}{U_1^2} = OB_1, \quad \frac{1}{U_2^2} = OB_2, \quad \frac{1}{U_3^2} = OB_3. \tag{18}$$

5. REFLECTION OF THE WEAK DISCONTINUITY WAVE

The wave OB is reflected from the plane of unit normal (k_1, k_2, k_3).



6. SURFACE WAVE



Four real points are shown. Since the 6-th order algebraic equation must have 6 roots, there exist complex points. A wave with complex propagation direction and complex propagation speed is called a surface wave, (cf. [7]). Note that the complex plane does not separate undisturbed and disturbed regions!

7. WAVES OF HIGHER ORDER

$$\begin{gathered}\text{On } \Psi,\ \text{the quantities } H=\frac{\partial^2 u_k}{\partial x_l \partial x_m}, \frac{\partial^2 u_k}{\partial x_l \partial t}, \frac{\partial^2 u_k}{\partial t^2} \quad \text{possess no jump.} \\ \text{On } \Psi,\ \text{the quantities } \frac{\partial^3 u_k}{\partial x_l \partial x_m \partial x_n}, \frac{\partial^3 u_k}{\partial x_l \partial x_m \partial t}, \frac{\partial^3 u_k}{\partial x_l \partial t^2}, \frac{\partial^3 u_k}{\partial t^3} \text{ possess a jump.}\end{gathered} \tag{19}$$

Differentiate the equation of motion. Its order increases by 1. Only the highest derivatives are discontinuous, therefore waves of higher order are the weak discontinuity waves. The propagation condition is exactly the same as the propagation condition of the acceleration wave.

8. EVOLUTION OF THE WEAK DISCONTINUITY WAVE

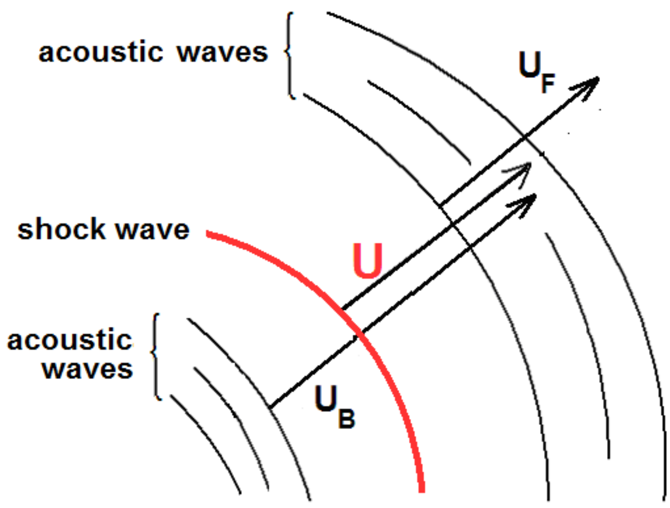
For a weak discontinuity wave there exists an acoustical ray, as in optics, but calculation is more complicated since the wave is not transverse. Analysis of evolution of the amplitude is time consuming. For engineers; a numerical approach is useful. Exact results are too long and too complex to be interesting.

9. STRONG DISCONTINUITY WAVE

For this wave on the discontinuity surface Ψ the displacement u_k is continuous and their derivatives with respect to x_i and time t are discontinuous. In this situation not only the second derivatives of u_k, but also the coefficients of the governing differential equations are discontinuous on $t=\Psi(x_1,x_2,x_3)$. Such a wave is usually called a shock wave, (cf. [8]).

$$\left\|\frac{\partial u_i}{\partial x_k}\right\| = A_i n_k, \left\|\frac{\partial u_i}{\partial x_k}\right\| = -A_i U \,. \tag{20}$$

Analysis and calculations lead to the following scheme for the eliminate weak and strong discontinuity waves



$U_F < U < U_B$

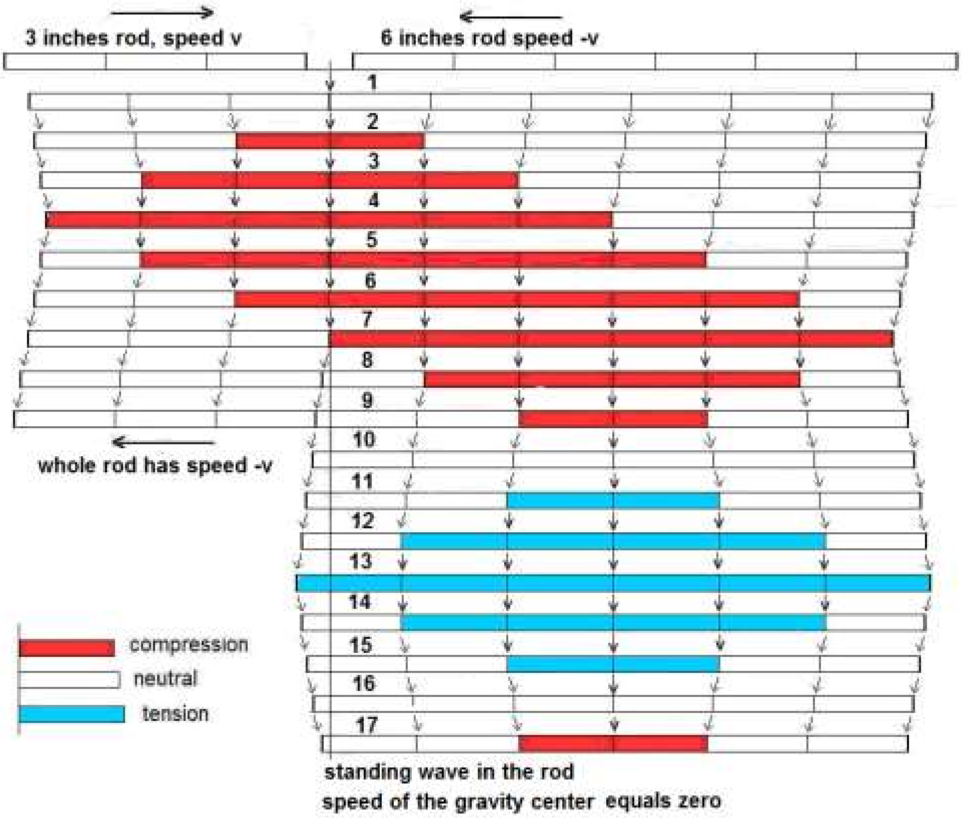
10. WAVES IN STRUCTURES

1.Torsional wave in a circulal cylinder	very easy
2. Bending wave in a rail	difficult
3. Wave in a pipeline	difficult
4. Thermal wave	very difficult
5. Wave in a bell	very difficult, since up to 7 frequencies must be matched and during matching, only removal of material is allowed. Adding material is impossible

example of a 470 kg bell:
main tone.................... a1 435 Hz.

jméno patrona	st. Hyacint	
nárazov tón	a1	(+0,25)
S8 (spodní oktáva)	a-3	(-0,75)
3 (tercie)..	c2	(+0,50)
5 (kvinta)...	e2	(-1,25)
8 (horní oktáva).....................................	a2	(+0,25)
10 (decima)	c3	(+2,00)
12 (duodecima)	e3	(+0,50)

6. Collision of billiards balls	An exact solution of this problem is unknown! The solution quoted in many textbooks is not correct. In general, after collision inside the balls there appear many reflected waves. Their energy must be accounted for the equations. This fact is evident in the example of the collision of two rods of different lengths shown below



After the collision both rods are compressed. The short rod, after the stage 7, is stress-free. Later, it moves with speed –v. Until stage 10, parts of the long rod until stage 10 are compressed. Later, the waves reflect from the ends and some parts are under tension. After the stage 7, the central point of the long rod does not move. The waves are reflected from the ends and a standing wave appears.

7. Solitary waves

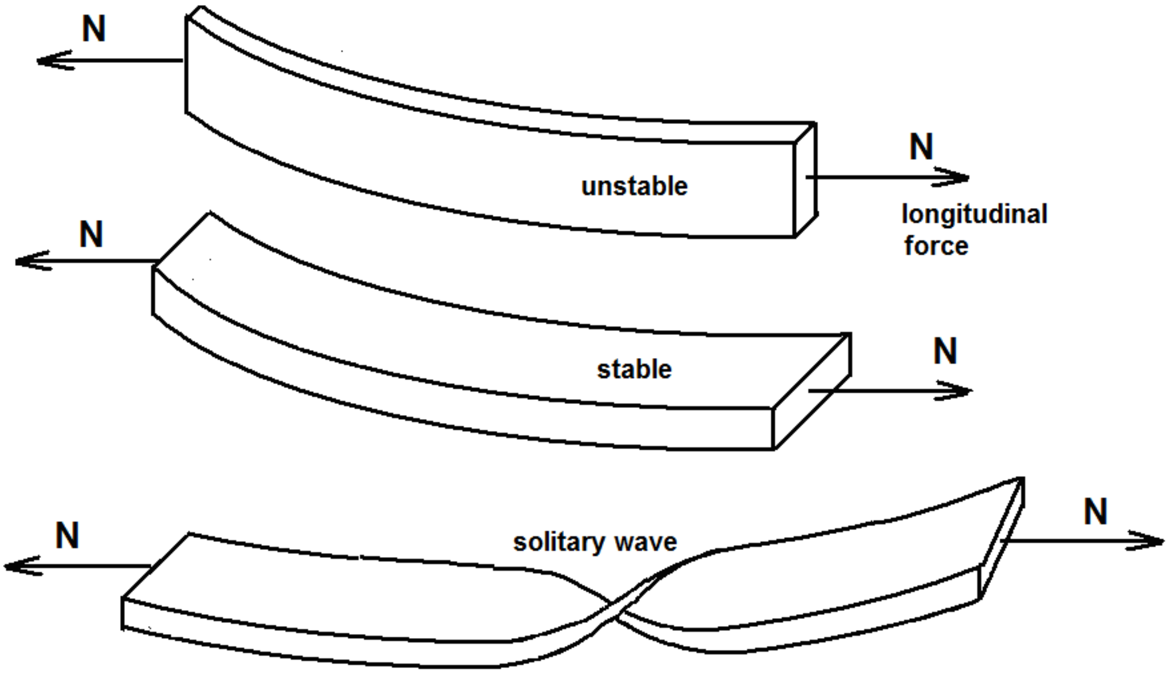
Consider a heavy long wire of rectangular cross-section in a gravitational field (such as, a wire hanging between two poles). Rotation of cross-section depends on the position and time and is denoted by φ(x,t), (cf. [7]). The motion is governed by the Sine - Gordon equation:

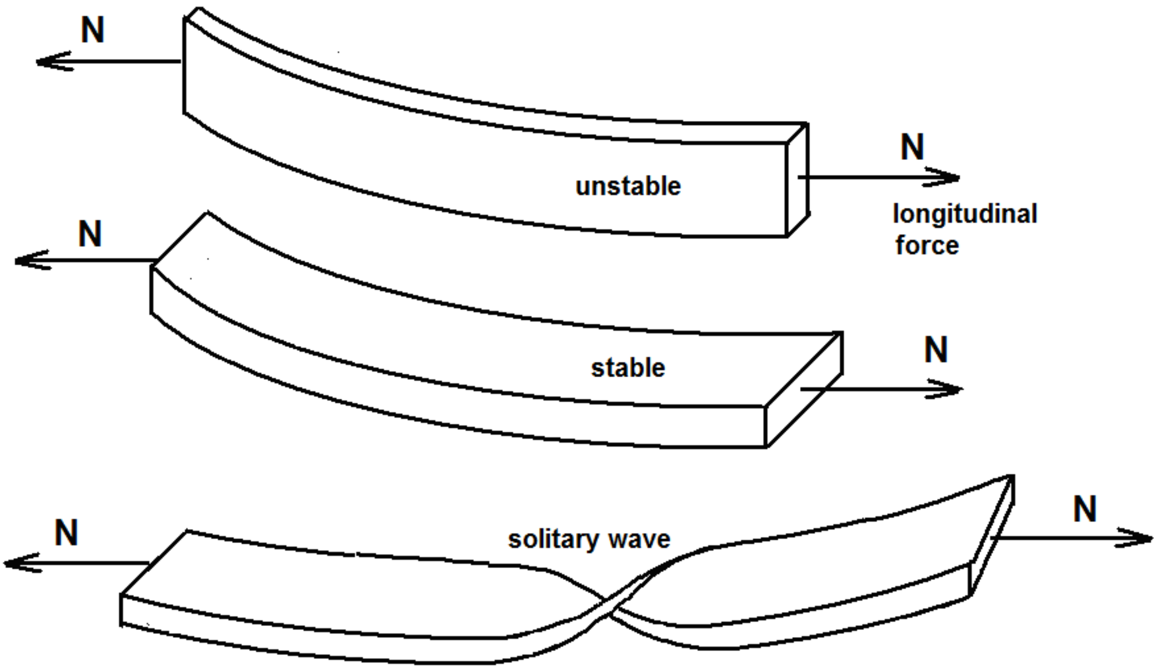
$$\frac{\partial^2\varphi(x,t)}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\varphi(x,t)}{\partial t^2}=\sin\varphi(x,t)\,. \tag{21}$$

There exist infinitely many static stable solutions, infinitely many static unstable solutions and infinitely many other solutions, e. g., the solution

$$\varphi = 4\operatorname{arctg}\left[\exp\frac{x-ct}{\sqrt{1-c^2}}\right], \tag{22}$$

where c is constant.

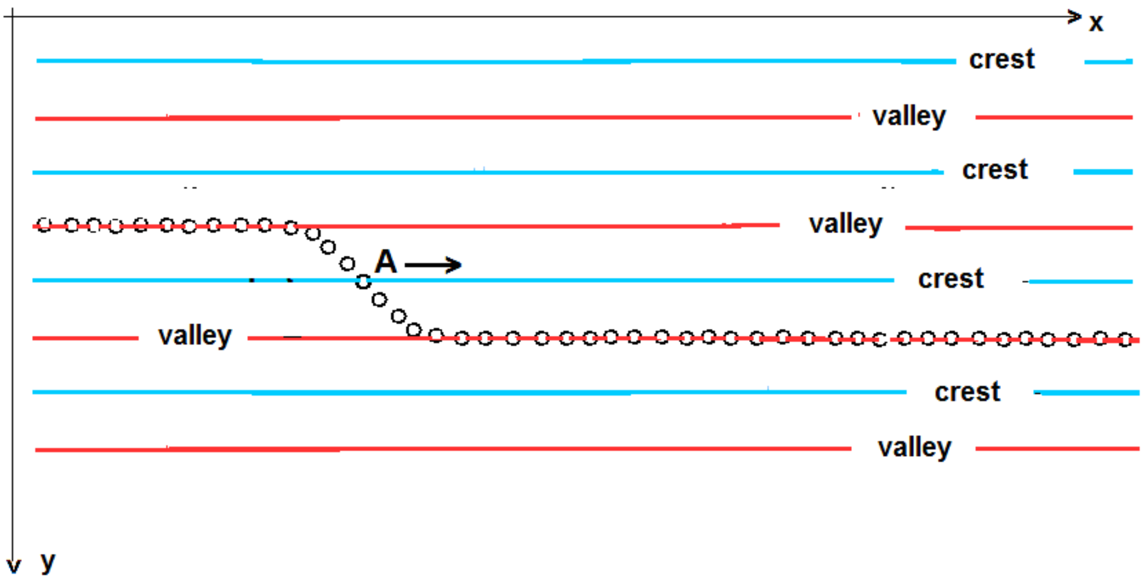




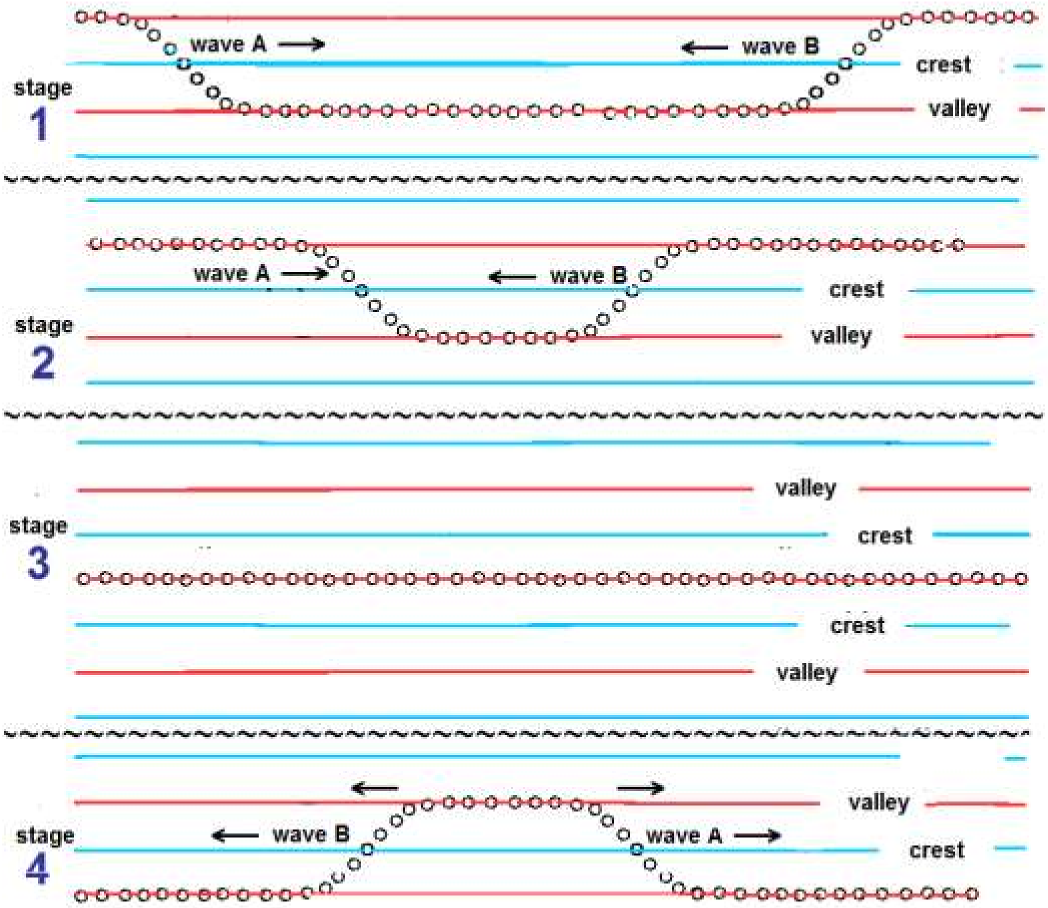
The same Sine - Gordon equation governs the motion of a chain situated on sinusoidal hill z=sin(y). The chain is almost parallel to the x-direction and moves in the y-direction. Motion in the x-direction is negligible.

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = \sin u(x,t) \, . \tag{23}$$

We face the solitary wave, when the left end of the chain is situated in one valley and the right part is situated in the next valley, as on the following picture. Most of the energy is localized near the point A.



Two solitary waves are shown on the following picture. They are running in opposite directions. The waves collide and annihilate each other (see stage 3). Later both waves recover (see stage 4). Solitary waves which recover after annihilation are called solitons.



Shallow water waves display similar behavior The water speed of such a wave is governed by the Korteweg – de Vries equation

$$u = u(x, t), \quad \frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{24}$$

where u(x,t) is the water speed. One of the solutions is

$$u(x, t) = \frac{1}{2} c \operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct)\right), \tag{25}$$

where c is the speed of the solitary wave. Far from the point x=ct, water speed is small. Near the point x=ct water speed is large; most of the energy is localized near the point x=ct.

11. SUMMARY

In this paper, waves in elastic material and elastic structures have been considered. The propagation speeds and amplitudes of sinusoidal waves, acceleration waves, waves of higher order, strong discontinuity waves, surface waves and solitary waves have been calculated. Additionally, the mathematical relations between such waves have been discussed.

REFERENCES

[1] C. Truesdell, R. Toupin, The Classical Field Theories, Handbuch der Physik, III/1, Berlin 1960.
[2] J. M. P. Musgrave, Crystal Acoustics, Holden-Day, San Francisco 1970.
[3] J. D. Achenbach, Wave Propagation in Elastic Solids, North-Holland, New York 1973.
[4] B. A. Auld, Acoustic Fields and Waves in Solids, John Willey&Sons, New York 1973.
[5] G. B. Whitham, Linear and Nonlinear Waves, John Willey&Sons, New York 1974.
[6] M. Suffczyński, Elektrodynamika, PWN Warszawa 1980.
[7] Z. Wesołowski, Dynamics of a bar of asymmetric cross-section, J. Eng. Math., 17, 1983.
[8] Z. Wesołowski, Akustyka Ciała Sprężystego, PWN Warszawa 1989.