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CLASSICAL AND NON-CLASSICAL PROCESSOR SHARING SYSTEMS WITH NON-HOMOGENEOUS CUSTOMERS

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Abstract

We discuss a processor sharing system with non-homogeneous customers. There are resources of two types for their service: 1) resource of the first type is discrete, there are N units (servers) of the resource; 2) resource of the second type (capacity) is not-necessary discrete. The type of a customer is defined by the amount of first type resource units which is used for the customer service. Each customer is also characterized by some random capacity or some amount of the second type resource which is also used for his service. The total capacity of customers present in the system is limited by some value $V > 0$, which is called the memory volume of the system. The customer capacity and length (the work necessary for service) are generally dependent. The joint distribution of these random variables also depends on the customer type. For such systems we determine the stationary distribution of the number of customers of each type present in the system and stationary loss probabilities for each type of customers.

1. Introduction

Egalitarian processor sharing systems have been used to model and solve the various practical problems occurring in the design of computer and communicating networks. They are used as models of situations when different customers are served simultaneously by limited amount of system resources [1], for example, in WEB-servers design [2].

The Egalitarian Processor Sharing (EPS) discipline was first introduced by Kleinrock [3] as a limit case for modelling time sharing systems.

The most important theoretical results in procesor sharing systems analysis (including a solution of the problem of sojourn time determination) were obtained by Yashkov (see [4, 5]). He is also an author of exhaustive surveys on mathematical methods of processor sharing systems analysis [2, 6, 7].

During some last years there appear a tremendous number of papers with new results in processor sharing (see surveys [2, 7]). But only a few of them consider systems with limited resources and customer length depending on capacity.

Later on, we shall call customer length the amount of work necessary for customer's service, i. e. the service time under condition that there are no other customers in the system during his presence in it. Analogously, we shall call residual length of the customer his residual service time after some time instant under the same condition (see [5]).

Assume that each customer in the system under consideration is characterized by some non-negative random capacity. This random variable can be interpreted as a part of system's memory space used by the customer during his presence in the system. A total sum of customers capacities $\sigma(t)$ in the system at arbitrary time t is referred as the total customers capacity. The random value $\sigma(t)$ can be limited by some constant value V $(0 < V \leq \infty)$, which is called the memory volume of the system.

The purpose of the paper is 1) to present a short survey of some previous results connected with determination of stationary total capacity characterisics for classical processor sharing systems ($V = \infty$); 2) to determine some estimations of loss characteristics for systems with limited memory space ($V < \infty$) based on the model with the unlimited one; 3) to investigate the processor sharing system with limited memory space and customers of different types.

For the last case, we obtain the stationary distribution of the number of customers of all types in the system and loss probabilities for each type of customers.

2. The case of unlimeted memory space and some estimations

A simplest processor sharing model with non-homogeneous customers is a classical system $M/G/1 - EPS$ in which each customer has additionally some random capacity ζ, and customer length ξ depends arbitrarily on his capacity, i.e. the following distribution function is defined:

$$F(x,t) = \mathbf{P}\{\zeta < x, \xi < t\}. \quad (1)$$

For this system, Sengupta [8] has obtained the Laplace-Stieltjes transform $\delta(s)$ of the total customers capacity in stationary mode:

$$\delta(s) = \frac{1-\rho}{1 + a\alpha'_q(s,q)|_{q=0}}, \tag{2}$$

where $\alpha(s,q)$ is the double Laplace-Stieltjes transform (with respect to x and t) of the distribution function (1):

$$\alpha(s,q) = \mathbf{E}e^{-s\sigma - q\xi} = \int_0^\infty \int_0^\infty e^{-sx-qt} dF(x,t), \;\; \rho = a\beta_1 < 1,$$

where a is an arrival rate of entrance flow of customers, β_1 is the first moment of customer length.

From formula (2), we can (in some cases) determine the relation for the distribution function $D(x)$ of the total customers capacity σ (this random variable is unlimited in the system, i.e. $V = \infty$ and customers are never lost).

We can use formula (2) for special cases analysis, therefore we can sometimes obtain the wiew of the function $D(x)$. For example, consider the case when the customer capacity ζ and his length ξ are connected by the relation $\xi = c\zeta + \xi_1$, $c \geq 0$, where the random variables ζ and ξ_1 are independent (such dependence between customer capacity and his length is true for many real information systems). Denote by $\kappa_1 = \mathbf{E}\xi_1$ the first moment of the random variable ξ_1. Let $\varphi(s) = \alpha(s,0)$ be the Laplace-Stieltjes transform of the customer capacity distribution function $L(x) = F(x,\infty)$. In this case we have [9]: $\alpha(s,q) = \varphi(s+cq)\kappa(s)$, where $\kappa(s)$ is the Laplace-Stieltjes transform of the distribution function of the random variable ξ_1. Then the relation (2) takes the following form:

$$\delta(s) = \frac{1-\rho}{1 + a[c\varphi'(s) - \kappa_1\varphi(s)]}, \tag{3}$$

Now, let us assume that customer capacity has an exponential distribution with the parameter f. In this case, from the relation (3) we obtain

$$\delta(s) = \frac{(1-\rho)(s+f)^2}{(s+f)^2 - \rho_1 f^2 - \rho_2 f(s+f)}, \tag{4}$$

where $\rho_1 = ac/f$, $\rho_2 = a\kappa_1$, so that $\rho = a\beta_1 = \rho_1 + \rho_2$.

Now, we can determine the original of the Laplace transform $\delta(s)/s$, where $\delta(s)$ is defined by formula (4), and obtain the wiew of the stationary distribution function $D(x)$ of total customers capacity:

$$D(x) = 1 - \frac{(1-\rho)e^{-fx}}{2b}\left[\frac{(\rho_2+b)^2 e^{(\rho_2+b)fx/2}}{2-\rho_2-b} - \frac{(\rho_2-b)^2 e^{(\rho_2-b)fx/2}}{2-\rho_2+b}\right], \quad (5)$$

where $b = \sqrt{{\rho_2}^2 + 4\rho_1}$.

If customer length does not depend on the capacity ($c = 0$, $\rho = \rho_2$), we have from the relation (5) that

$$D(x) = 1 - \rho e^{-(1-\rho)fx}. \quad (6)$$

If customer length is proportional to the capacity ($c > 0$, $\kappa_1 = 0$), we have from (5) that

$$D(x) = 1 - \frac{\sqrt{\rho}}{2}\left[(1+\sqrt{\rho})e^{-(1-\sqrt{\rho})fx} - (1-\sqrt{\rho})e^{-(1+\sqrt{\rho})fx}\right], \quad (7)$$

where in this case $\rho = \rho_1$.

The generalization of the relation (2) to a non-stationary case was obtained in [10], where it was also shown how to estimate the memory volume V in order to guarantee inexceeding of given loss probability, if the characteristics of stationary total capacity are known for the system with unlimited memory space. For example, if we denote by $D_V(x)$ the stationary distribution function of the total customers capacity for the system that differ from the above classical one only with the fact that its total capacity is limited by the constant value V, then the loss probability P for such a system satisfies the following enequality:

$$P = 1 - \int_0^V D_V(V-x)dL(x) \leq 1 - \int_0^V D(V-x)dL(x) = P^*.$$

Thus, the value P^* is an upper estimation of loss probability for the system with memory space limited by V. If we choose V given P^* so that the equality

$$\int_0^V D(V-x)dL(x) = 1 - P^*$$

is satisfied, then the real loss probability does not exceed P^*. If in the system under consideration only very rare losses are permitted, then the difference between the values P and P^* is inessential.

Note that the loss probability is not exhaustive characteristic of losses, because its value shows the part of lossing customers, but not the part of lossing capacity (or, in other words, lossing information). Really, it is obvious that customers having large capacity will be lost more often. Therefore, more objective losses estimation is the value

$$Q = 1 - \frac{1}{\varphi_1}\int_0^V xD_V(V-x)dL(x),$$

where $\varphi_1 = \mathbf{E}\zeta$ is the first moment of the random variable ζ. The value Q is the probability of lossing of a unit of customer capacity. Obviously, the value Q satisfies the inequality

$$Q = 1 - \frac{1}{\varphi_1}\int_0^V xD_V(V-x)dL(x) \le 1 - \frac{1}{\varphi_1}\int_0^V xD(V-x)dL(x) = Q^*.$$

If in the system under consideration only very rare losses are permitted, then the difference between the values Q and Q^* is inessential.

For the case, described by the distribution function (6), we have

$$P^* = e^{-(1-\rho)fV},\ Q^* = e^{-fV}\left[1 + fV + \frac{1}{\rho}\left(e^{\rho fV} - 1 - \rho fV\right)\right].$$

For the distribution function (5), we obtain

$$P^* = \left\{1 - \frac{1-\rho}{b}\left[a_1\frac{1-e^{-(1-b_1)fV}}{b+\rho_2} + a_2\frac{1-e^{-(1-b_2)fV}}{b-\rho_2}\right]\right\}e^{-fV},$$

where $a_1 = \dfrac{(\rho_2+b)^2}{2-\rho_2-b}$, $a_2 = \dfrac{(\rho_2-b)^2}{2-\rho_2+b}$, $b_1 = -1+\dfrac{\rho_2+b}{2}$, $b_2 = -1+\dfrac{\rho_2-b}{2}$;

$$Q^* = \left\{1 + fV - \frac{2(1-\rho)}{b}\left[\frac{(a_1+a_2)fV}{8\rho_1} + \right.\right.$$

$$\left.\left. + a_1\frac{1-e^{-(1-b_1)fV}}{(b+\rho_2)^2} - a_2\frac{1-e^{-(1-b_2)fV}}{(b-\rho_2)^2}\right]\right\}e^{-fV}.$$

In particular, for the case, described by the distribution function (7), we find

$$P^* = e^{-fV}\left\{1 + \frac{1}{2}\left[(1+\sqrt{\rho})(e^{\sqrt{\rho}fV} - 1) - (1-\sqrt{\rho})(1 - e^{-\sqrt{\rho}fV})\right]\right\},$$

$$Q^* = e^{-fV}\left\{1 + fV + \frac{1}{2\sqrt{\rho}}\left[(1+\sqrt{\rho})(e^{\sqrt{\rho}fV} - 1 - \sqrt{\rho}fV) - \right.\right.$$

$$\left.\left. -(1-\sqrt{\rho})(1 - e^{-\sqrt{\rho}fV} - \sqrt{\rho}fV)\right]\right\}.$$

Note that in most cases the calculation and estimation of the probability Q is very complicated. Therefore, we must often restrict ourselves to the calculation and estimation of the loss probability P.

If it is impossible to determine the wiew of the distribution function $D(x)$, we can estimate the value P^* with the help of approximation of the function

$$\Phi(x) = \int_0^x D(x-u)dL(u),$$

which is the distribution function of the sum of independent random variables σ and ζ, by the approximate function of gamma distribution $\Phi^*(x) = \gamma(q, rx)/\Gamma(q)$, where $\gamma(q, rx) = \int_0^{rx} t^{q-1}e^{-t}dt$ is the incomplete gamma function, $\Gamma(q) = \gamma(q, \infty)$ is the gamma function. The parameters q and r of the approximate distribution should be chosen so that its first and second moments $f_1^* = q/r$ and $f_2^* = q(q+1)/r^2$ should be equal to the first and second moments of the distribution with the distribution function $\Phi(x)$, repectively. It is obvious that these moments have the form

$$f_1 = \delta_1 + \varphi_1, \ \ f_2 = \delta_2 + \varphi_2 + 2\delta_1\varphi_1, \tag{8}$$

where φ_i is a moment of the ith order of the random variable ζ, δ_i is a moment of the ith order of the random value σ, $i = 1, 2$. As it is shown in [8], for the classical processor sharing system we have

$$\delta_1 = \frac{a\alpha_{11}}{1-\rho}, \quad \delta_2 = \frac{a\alpha_{21}}{1-\rho} + 2\delta_1^2, \tag{9}$$

where $\alpha_{ij} = \mathbf{E}(\zeta^i\xi^j) = (-1)^{i+j}\frac{\partial^{i+j}\alpha(s,q)}{\partial s^i \partial q^j}\Big|_{s=0,q=0}$ is the mixed moment of the $(i+j)$th order of random variables ζ and ξ, $i, j = 1, 2, \ldots$.

Thus, the parameters of the distribution function $\Phi^*(x)$ should be chosen as follows:

$$q = \frac{f_1^2}{f_2 - f_1^2}, \ \ r = \frac{f_1}{f_2 - f_1^2},$$

where f_1 and f_2 can be calculated from (8), (9). Hence, we have an approximate formula

$$P^* \cong 1 - \Phi^*(V).$$

Note that in the case of not very small permissible loss probabilities, using the estimation P^* instead of P, leads to unjustifiely surplus choise of the memory volume V. Therefore, the analysis of processor sharing systems with limited resources (including the memory space) is very important.

3. Processor sharing system with limited memory space

Consider a non-classical processor sharing system that differs from the classical system $M/G/1 - EPS$ in the following properties.

1. The system contains N units of some homogeneous discrete resource. There are K types of cutomers in the system. Each m-customer (or customer of type m), $m = \overline{1, K}$, independently of his arrival time and characteristics of other customers, requires b_m ($b_m \le N$) units of the resource for his service. Denote by a_m the arrival rate of (stationary Poisson) entrance flow of m-customers, $a = a_1 + \cdots + a_m$. Assume for simplisity that $b_1 < \cdots < b_K \le N$.

2. Independently of other customers and his arrival time, each m-customer is characterized by the random capacity ζ_m (the random variable ζ_m is not necessarily discrete) and random length ξ_m. The distribution functions

$$F_m(x, t) = \mathbf{P}\{\zeta_m < x, \xi_m < t\}, \ m = \overline{1, K},$$

are given. Denote by $\eta_m(t)$ the number of m-customers present in the system at time t. Then, the random vector $\boldsymbol{\eta}(t) = (\eta_1(t), \ldots, \eta_K(t))$ describes the state of the system, i.e. it shows how many customers of each type are in the system at time t. The total number of customers in the system at time t will be refered as $\eta(t) = \sum_{m=1}^{K} \eta_m(t)$. Denote by $\sigma(t)$ the total capacity, i.e. the total sum of capacities of customers that are in the system at this time.

3. The total capacuty $\sigma(t)$ in the system is limited by constant value $V > 0$ which is called the memory volume. The values N and V are not equal to infinity at the same time.

Denote by $L_m(x) = F_m(x, \infty)$ the distribution function of the capacity of m-customer and denote by $B_m(t) = F_m(\infty, t)$ the distribution function of his length.

If at the arrival time τ of m-customer there are less than b_m free units of the discrete resource in the system, then the customer will be lost having no effect on further system behaviour.

If the required amount of free resource units is available, the customer will be nevertheless lost, if his capacity x is such that $\sigma(\tau - 0) + x > V$. If there are sufficiently many free discrete resource units at the arrival time and the condition $\sigma(\tau - 0) + x \le V$ is satisfied, then immediately after the arrival of a customer his service starts; here $\eta_m(\tau) = \eta_m(\tau - 0) + 1$ and $\sigma(\tau) = \sigma(\tau - 0) + x$. If τ is the service termination epoch of m-customer having capacity x, then $\eta_m(\tau) = \eta_m(\tau - 0) - 1$ and $\sigma(\tau) = \sigma(\tau - 0) - x$.

Note that maximum number of customers that are served simultaneously is not more than $M = [N/b_1]$.

For the considered system, we find the distribution of the number of cusomers in the system at an arbitrary time instant in stationary mode, and stationary loss probabilities for customers of each type.

4. Process and characteristics

Assume that customers in the considered system at an arbitrary time t are enumerated as random; i.e. if the number of customers is k, then there are $k!$ ways to enumerate them, and each enumeration can be chosen with the same probability $1/k!$. Denote by $\nu_j(t)$ the number of resource units that are used by jth customer at time t ($\nu_j(t) = b_m$ if it is a customer of type m). Denote by $\sigma_j(t)$ the capacity of this customer. We denote by $\xi_j^*(t)$ the residual length of jth customer in the system from the time t.

One can easily show that the system under cosideration is described by the Markov process

$$\left(\eta(t), \nu_j(t), \sigma_j(t), \xi_j^*(t), j = \overline{1, \eta(t)}\right), \tag{10}$$

where components $\nu_j(t)$, $\xi_j^*(t)$ are absent if $\eta(t) = 0$.

Note that in our notations we have $\sigma(t) = \sum\limits_{j=1}^{\eta(t)} \sigma_j(t)$. In what follows, to simplify the notation, we denote

$$R_k = (r_1, \ldots, r_k),\ Y_k = (y_1, \ldots, y_k),\ R_k^j = (r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_k)$$

and, similarly,

$$Y_k^j = (y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_k).$$

We also assume that $r_{(k)} = r_1 + \cdots + r_k$.

For the components of the vector R_k, we assume that $r_i \in \{b_1, \ldots, b_K\}$, $i = \overline{1, k}$. Denote by $[r]$, where $r \in \{b_1, \ldots, b_K\}$, the number of type of customers, if r resource units are used for his service, i.e. $m = [r]$ if $r = b_m$.

Sometimes in the case $k = 1$, instead of R_1 and Y_1 we write, respectively, r_1 and y_1 ore the values that these components take, and in the case $k = 2$, instead of R_2 and Y_2 we write (r_1, r_2) and (y_1, y_2) or their values respectively. In other words, we sometimes specify vectors of small dimensions by indicating their components. We also use the notation $(R_k, b_m) = (r_1, \ldots, r_k, b_m)$ and $(Y_k, z) = (y_1, \ldots, y_k, z)$.

We characterize the process (10) by functions with the following probabilistic sense:

$$G_k(x, R_k, Y_k, t) = \mathbf{P}\{\eta(t) = k, \sigma(t) < x, \nu_i(t) = r_i, \xi_i^*(t) < y_i, i = \overline{1, k}\},$$

$$k = \overline{1, M},\ r_{(k)} \le N; \tag{11}$$

$$\Theta_k(R_k, Y_k, t) = \mathbf{P}\{\eta(t) = k, \nu_i(t) = r_i, \xi_i^*(t) < y_i, i = \overline{1, k}\} =$$

$$= G_k(V, R_k, Y_k, t),\ k = \overline{1, M},\ r_{(k)} \le N. \tag{12}$$

We also introduce the functions

$$\Pi_k(R_k, t) = \mathbf{P}\{\eta(t) = k, \nu_i(t) = r_i, i = \overline{1, k}\} = \lim_{y_1, \ldots, y_k \to \infty} \Theta_k(R_k, Y_k, t),$$

$$k = \overline{1, M},\ r_{(k)} \le N; \tag{13}$$

$$P_0(t) = \mathbf{P}\{\boldsymbol{\eta}(t) = \mathbf{0}\} = \mathbf{P}\{\eta(t) = 0\}, \tag{14}$$

where $\mathbf{0} = (\underbrace{0, \ldots, 0}_{K})$;

$$P_k(t) = \mathbf{P}\{\eta(t) = k\} = \sum_{r_{(k)} \le N} \Pi_k(R_k, t),\ k = \overline{1, M}. \tag{15}$$

Assume that $N < \infty$ or (and) $V < \infty$. Then, a stationary mode exists for the system if $\rho = a\beta_1 < \infty$, where $\beta_1 = (a_1\beta_{11} + \cdots + a_K\beta_{1K})/a$ is the first moment of the total service time in the system, i.e. if $t \to \infty$, then $\boldsymbol{\eta}(t) \Rightarrow \boldsymbol{\eta}$, $\sigma(t) \Rightarrow \sigma$, $\nu_j(t) \Rightarrow \nu_j$ and $\xi_j^*(t) \Rightarrow \xi_j^*$ in the weak convergence sense. So, the following limits exist:

$$g_k(x, R_k, Y_k) = \lim_{t \to \infty} G_k(x, R_k, Y_k, t),\ k = \overline{1, M},\ r_{(k)} \le N; \tag{16}$$

$$\theta_k(R_k, Y_k) = \lim_{t \to \infty} \Theta_k(R_k, Y_k, t) = g_k(V, R_k, Y_k),\ k = \overline{1, M},\ r_{(k)} \le N; \tag{17}$$

$$\pi_k(R_k) = \lim_{t \to \infty} \Pi_k(R_k, t) = \lim_{y_1, \ldots, y_k \to \infty} \theta_k(R_k, Y_k),\ k = \overline{1, M},\ r_{(k)} \le N; \tag{18}$$

$$p_0 = \lim_{t \to \infty} P_0(t); \tag{19}$$

$$p_k = \lim_{t \to \infty} P_k(t) = \sum_{r_{(k)} \le N} \pi_k(R_k),\ k = \overline{1, M}. \tag{20}$$

Note that the functions $G_k(x, R_k, Y_k, t)$, $g_k(x, R_k, Y_k)$ and $\Theta_k(R_k, Y_k, t)$, $\theta_k(R_k, Y_k)$ are symmetric with respect to simultaneous permutations of components with the same indices of the vectors R_k and Y_k due to our random enumeration of customers in the system.

Denote by $H_m(x)$ the probability that an arbitrary m-customer has a capacity less than x and a length greater than or equal to y, i.e.

$$H_m(x,y) = \mathbf{P}\{\zeta_m < x, \xi_m \ge y\} = \int_{v=0}^{x}\int_{u=y}^{\infty} dF_m(v,u) =$$
$$= \mathbf{P}\{\zeta_m < x\} - \mathbf{P}\{\zeta_m < x, \xi_m < y\} = L_m(x) - F_m(x,y). \tag{21}$$

5. Stationary distribution of the number of customers

Using the method of auxiliary variables [11] and taking into account the symmetric properties, we can write out partial differential equations for the functions (11), (12) and (14):

$$\frac{\partial P_0(t)}{\partial t} = -P_0(t)\sum_{m=1}^{K} a_m L_m(V) + \sum_{m=1}^{K} \left.\frac{\partial \Theta_1(b_m,y,t)}{\partial y}\right|_{y=0}; \tag{22}$$

$$\frac{\partial \Theta_1(b_m,y,t)}{\partial t} - \frac{\partial \Theta_1(b_m,y,t)}{\partial y} + \left.\frac{\partial \Theta_1(b_m,y,t)}{\partial y}\right|_{y=0} =$$
$$= a_m P_0(t) F_m(V,y) - \sum_{j:\, b_j \le N-b_m} a_j \int_0^V G_1(V-x,b_m,y,t)\,dL_j(x) +$$
$$+ \sum_{j:\, b_j \le N-b_m} \left.\frac{\partial \Theta_2((b_m,b_j),(y,z),t)}{\partial z}\right|_{z=0},\ m=\overline{1,K}; \tag{23}$$

$$\frac{\partial \Theta_k(R_k,Y_k,t)}{\partial t} - \frac{1}{k}\sum_{j=1}^{k}\left[\frac{\partial \Theta_k(R_k,Y_k,t)}{\partial y_j} - \left.\frac{\partial \Theta_k(R_k,Y_k,t)}{\partial y_j}\right|_{y_j=0}\right] =$$
$$= \frac{1}{k}\sum_{j=1}^{k} a_{[r_j]} \int_0^V G_{k-1}(V-x,R_k^j,Y_k^j,t)\,d_x F_{[r_j]}(x,y_j) -$$
$$- \sum_{m:\, b_m \le N-r_{(k)}} a_m \int_0^V G_k(V-x,R_k,Y_k,t)\,dL_m(x) +$$
$$+ \sum_{m:\, b_m \le N-r_{(k)}} \left.\frac{\partial \Theta_{k+1}((R_k,b_m),(Y_k,z),t)}{\partial z}\right|_{z=0},\ k=\overline{2,M},\ r_{(k)} \le N. \tag{24}$$

Passing to the limit as $t \to \infty$ in the equations (22)–(24), we obtain the following equations for stationary functions (16), (17), (19):

$$0 = -p_0\sum_{m=1}^{K} a_m L_m(V) + \sum_{m=1}^{K}\left.\frac{\partial \theta_1(b_m,y)}{\partial y}\right|_{y=0}; \tag{25}$$

$$-\frac{\partial\theta_1(b_m,y)}{\partial y}+\frac{\partial\theta_1(b_m,y)}{\partial y}\bigg|_{y=0}=a_m p_0 F_m(V,y)-$$

$$-\sum_{j:\,b_j\le N-b_m} a_j\int_0^V g_1(V-x,b_m,y)dL_j(x)+\sum_{j:\,b_j\le N-b_m}\frac{\partial\theta_2((b_m,b_j),(y,z))}{\partial z}\bigg|_{z=0},$$

$$m=\overline{1,K}; \tag{26}$$

$$-\frac{1}{k}\sum_{j=1}^{k}\left[\frac{\partial\theta_k(R_k,Y_k)}{\partial y_j}-\frac{\partial\theta_k(R_k,Y_k)}{\partial y_j}\bigg|_{y_j=0}\right]=$$

$$=\frac{1}{k}\sum_{j=1}^{k}a_{[r_j]}\int_0^V g_{k-1}(V-x,R_k^j,Y_k^j)d_xF_{[r_j]}(x,y_j)-$$

$$-\sum_{m:\,b_m\le N-r_{(k)}} a_m\int_0^V g_k(V-x,R_k,Y_k)dL_m(x)+$$

$$+\sum_{m:\,b_m\le N-r_{(k)}}\frac{\partial\theta_{k+1}((R_k,b_m),(Y_k,z))}{\partial z}\bigg|_{z=0},\ k=\overline{2,M},\ r_{(k)}\le N. \tag{27}$$

In stationary mode, we have boundary conditions for equations (25)–(27) described by the following equilibrium equations:

$$\frac{\partial\theta_1(b_m,y)}{\partial y}\bigg|_{y=0}=a_m p_0 L_m(V),\ m=\overline{1,K}; \tag{28}$$

$$\frac{\partial\theta_k((R_{k-1},b_m),(Y_{k-1},z))}{\partial z}\bigg|_{z=0}=a_m\int_0^V g_{k-1}(V-x,R_{k-1},Y_{k-1})dL_m(x),$$

$$k=\overline{2,M},\ b_m\le N-r_{(k-1)}. \tag{29}$$

Let us explain the meaning of equations (28) and (29) for the case $k=\overline{2,M}$, $N\ge 2$. If, at some time t in stationary mode, the considered system was in the state $\{\eta=k-1,\nu_j=r_j,\xi_j^*<y_j,j=\overline{1,k-1}\}$, then the probability that m-customer will enter the system during small time interval Δt equals

$$a_m\Delta t\int_0^V g_{k-1}(V-x,R_{k-1},Y_{k-1})dL_m(x)+o(\Delta t).$$

In stationary mode this probability, obviously, coincides with the probability of the inverse transition, i.e. with the probability that within time interval Δt the system will pass to this state upon termination of service of an m-customer. Taking into account the above-mentioned symmetry of the functions $\theta_k(R_k, Y_k)$ with respect to permutations, the latter probability equals $\left.\frac{\partial\theta_k((R_{k-1}, b_m),(Y_{k-1}, z))}{\partial z}\right|_{z=0} \Delta t + o(\Delta t)$, whence (29) follows. Equality (28) is deduced similarly.

To the relations (25)–(29), we should add the normalization condition which can be represented as follows:

$$p_0 + \sum_{k=1}^{M} \sum_{r_{(k)} \leq N} \pi_k(R_k) = 1. \tag{30}$$

Now introduce the function $\Phi_j^y(x) = \int_0^y H_j(x,u)du$. Its meaning becomes obvious if we use the representation $F_j(x,u) = L_j(x)B_j(u \,|\, \zeta_j < x)$, where $B_j(u \,|\, \zeta_j < x) = \mathbf{P}\{\xi_j < u \,|\, \zeta_j < x\}$ is the conditional distribution function of the length of a j-customer given that his capacity is less than x. Then (21) implies

$$\Phi_j^y(x) = L_j(x) \int_0^y [1 - B_j(u \,|\, \zeta_j < x)]du.$$

Let us also introduce the following notation for the Stieltjes convolution:

$$F_1 * \cdots * F_n(x) = \mathop{*}_{j=1}^{n} F_j(x).$$

In the particular case $F_1 = \cdots = F_n = F$, the value of n-order convolution of the function F at the point x will be denoted by $F_*^{(n)}(x)$.

Then, using the above-mentioned symmetry property of functions (16) and (17) and taking into account the boundary conditions (28) and (29), one can show by direct substitution that the solution of equations (25)–(27) can be represented as

$$g_k(x, R_k, Y_k) = C \mathop{*}_{j=1}^{k} \Phi_{[r_j]}^{y_j}(x) \prod_{j=1}^{k} a_{[r_j]}, \; k = \overline{1, M}, \; r_{(k)} \leq N, \tag{31}$$

where C is a constant to be specified from the normalization condition (30). From (17) it follows that

$$\theta_k(R_k, Y_k) = C \mathop{*}_{j=1}^{k} \Phi_{[r_j]}^{y_j}(V) \prod_{j=1}^{k} a_{[r_j]}, \; k = \overline{1, M}, \; r_{(k)} \leq N. \tag{32}$$

Introduce the notation

$$A_j(x) = \int_{u=0}^{x} \int_{y=0}^{\infty} u\, dF_j(u,y), \;\; j = \overline{1,K}.$$

The function $A_j(x)$ has meaning of "partial" mathematical expectation [12] of the random variable ξ_j with respect to the event $\{\zeta_j < x\}$:

$$A_j(x) = \mathbf{E}(\xi_j, \zeta_j < x) = \mathbf{E}(\xi_j \,|\, \zeta_j < x) L_j(x),$$

where $\mathbf{E}(\xi_j \,|\, \zeta_j < x)$ is the conditional mathematical expectation of the length of a customer of type j given that his capacity is less than x. It is easily seen that

$$A_j(x) = \lim_{y\to\infty} \Phi_j^y(x) = L_j(x) \int_0^{\infty} [1 - B_j(u \,|\, \zeta_j < x)] du.$$

Using relation (18), we obtain

$$\pi_k(R_k) = C \mathop{*}_{j=1}^{k} A_{[r_j]}(V) \prod_{j=1}^{k} a_{[r_j]}, \;\; k = \overline{1,M}, \;\; r_{(k)} \le N. \tag{33}$$

It follows from (20) that

$$p_k = C \sum_{r_{(k)} \le N} \mathop{*}_{j=1}^{k} A_{[r_j]}(V) \prod_{j=1}^{k} a_{[r_j]}, \;\; k = \overline{1,M}.$$

The latter relation and normalization condition (30) finally yield

$$p_k = p_0 \sum_{r_{(k)} \le N} \mathop{*}_{j=1}^{k} A_{[r_j]}(V) \prod_{j=1}^{k} a_{[r_j]}, \;\; k = \overline{1,M}, \tag{34}$$

where

$$p_0 = C = \left[1 + \sum_{k=1}^{L} \sum_{r_{(k)} \le N} \mathop{*}_{j=1}^{k} A_{[r_j]}(V) \prod_{j=1}^{k} a_{[r_j]} \right]^{-1}. \tag{35}$$

Now, let us pass to the traditional [13] representation of states of the system under consideration, using a stationary analogue $\boldsymbol{\eta}$ of the vector $\boldsymbol{\eta}(t)$ introduced in section 3. Denote by k_m, $m = \overline{1,K}$, the number of m-customers present in the system at an arbitrary time in stationary mode ($k_m \ge 0$). Then, we have $k = \sum\limits_{m=1}^{K} k_i$, where $k = \overline{0,M}$. We shall characterize a state of the system by the vector $\mathbf{k} = (k_1, \ldots, k_K)$. We introduce also the vector $\mathbf{b} = (b_1, \ldots, b_K)$ and scalar product $(\mathbf{b}, \mathbf{k}) = \sum_{m=1}^{K} k_m b_m$. Denote by $\mathbf{N}_0$ the set of integer non-negative numbers. Obviously, the space of values of the vector $\mathbf{k}$ is the set $\mathbf{S} := \{\mathbf{k} \in \mathbf{N}_0^K : (\mathbf{b}, \mathbf{k}) \le N\}$, where $\mathbf{N}_0^K$ is K-order Cartesian product $\mathbf{N}_0^K = \underbrace{\mathbf{N}_0 \times \cdots \times \mathbf{N}_0}_{K}$. Then, from (33), taking into account the symmetry of the functions $\pi_k(R_k)$ with respect to permutations, we obtain that stationary probability $g(\mathbf{k}) = \mathbf{P}\{\boldsymbol{\eta} = \mathbf{k}\}$ that there are k_m customers of type m, $m = \overline{1, K}$, in the system is defined by the following relation:

$$g(\mathbf{k}) = p_0 k! \mathop{*}_{m=1}^{K} A_{m*}^{(k_m)}(V) \prod_{m=1}^{K} \frac{a_m^{k_m}}{k_m!}, \ \mathbf{k} \in \mathbf{S}. \tag{36}$$

For such a representation, the probability p_0 takes the form

$$p_0 = \left[\sum_{\mathbf{k} \in \mathbf{S}} k! \mathop{*}_{m=1}^{K} A_{m*}^{(k_m)}(V) \prod_{m=1}^{K} \frac{a_m^{k_m}}{k_m!} \right]^{-1} \tag{37}$$

that is equivalent to the relation (35).

6. Loss probability

Finding the stationary loss probability P_m, $m = \overline{1, K}$, for a customer of type m is based on the fact that in stationary mode the average number of customers accepted for service within a time unit (i.e. customers that entered the system during this time period and were not lost) is equal to the average number of customers whose service was terminated within this time period. Thus, taking into account the symmetry of $\theta_k(R_k, Y_k)$ with respect to the above-mentioned permutations of components of vectors (R_k, Y_k), we obtain the following equilibrium equation:

$$a_m(1 - P_m) = \sum_{j=1}^{M} \sum_{r_{(j-1)} \le N - b_m} \left. \frac{\partial \theta_j((R_{j-1}, b_m), (\infty_{j-1}, z))}{\partial z} \right|_{z=0}, \ m = \overline{1, K},$$

where $\infty_i = (\underbrace{\infty, \ldots, \infty}_{i})$. Taking into account (32) and (35), the latter relation yields

$$P_m = 1 - p_0 \left[L_m(V) + \sum_{j=1}^{M} \sum_{r_{(j)} \le N - b_m} L_m * \left(\mathop{*}_{l=1}^{j} A_{[r_l]} \right)(V) \prod_{l=1}^{j} a_{[r_l]} \right]. \tag{38}$$

Then, the total loss probability is defined by the relation

$$P = \frac{1}{a}\sum_{m=1}^{K} a_m P_m =$$

$$= 1 - \frac{p_0}{a}\sum_{m=1}^{K} a_m \left[L_m(V) + \sum_{j=1}^{M} \sum_{r_{(j)} \le N-b_m} L_m * \left(\mathop{*}_{l=1}^{j} A_{[r_l]} \right)(V) \prod_{l=1}^{j} a_{[r_l]} \right].$$

If customers lengths are independent of their capacities for customers of all types, then we obviously have $A_m(x) = \beta_{m\,1} L_j(x)$, $m = \overline{1,K}$.

For presenting the loss probability of m-customer in another (traditional) form, we introduce subsets of states $\mathbf{S}_m :\ = \{\mathbf{k} \in \mathbf{S} : (\mathbf{b},\mathbf{k}) \le N - b_m\}$, $m = \overline{1,K}$. Then, from relation (38), we have

$$P_m = 1 - p_0 \sum_{\mathbf{k}\in\mathbf{S}_m} k!\, L_m * \left(\mathop{*}_{j=1}^{K} A_{j*}^{(k_j)} \right)(V) \prod_{j=1}^{K} \frac{{a_j}^{k_j}}{k_j!}. \tag{39}$$

Note that direct application of obtained relations happens to be inconvenient for calculation in the general case. However, their direct application is possible in certain particular cases.

7. Analysis of particular cases

We analyze here only a few of simple particular cases.

1. There is only one type of customers. Let us assume that each customer needs only one unit of the discrete resource ($K = 1$, $b_1 = 1$, $a_1 = a$), $N \le \infty$, $V < \infty$. For example, if $N < \infty$, then we have a system with limited number of customers.

We omit indexes in the notation for such systems. For example, we write $A(x)$ instead of $A_1(x)$, $F(x,t)$ instead of $F_1(x,t)$, etc.

In this case, relations (34), (35) and (38) take the form

$$p_k = p_0 a^k A_*^{(k)}(V),\ k = \overline{1,N},\ p_0 = \left[1 + \sum_{k=1}^{N} a^k A_*^{(k)}(V) \right]^{-1},$$

$$P = 1 - p_0 \left[L(V) - \sum_{j=1}^{N-1} L * A_*^{(j)}(V) \right].$$

Assume additionally that customer capacity has an exponential distribution with parameter f, and let the customer length be proportional to his capacity ($\xi = c\zeta$, $c > 0$). Introduce the notation $\rho = ac/f$. Then, after some calculations, we obtain:

$$p_k = p_0 \rho^k \left[1 - e^{-fV} \sum_{i=0}^{2k-1} \frac{(fV)^i}{i!}\right], \ k = \overline{1, N};$$

$$p_0 = \left\{\sum_{k=0}^{N} \rho^k \left[1 - e^{-fV} \sum_{i=0}^{2k-1} \frac{(fV)^i}{i!}\right]\right\}^{-1};$$

$$P = 1 - p_0 \sum_{k=0}^{N-1} \rho^k \left[1 - e^{-fV} \sum_{i=0}^{2k} \frac{(fV)^i}{i!}\right].$$

In particular, if $N = \infty$, we have

$$p_0 = \begin{cases} \dfrac{1-\rho}{1 - \sqrt{\rho} e^{-fV} \left[\sinh(\sqrt{\rho} fV) + \sqrt{\rho} \cosh(\sqrt{\rho} fV)\right]}, & \text{if } \rho \neq 1, \\ \dfrac{1 + e^{-2fV}}{1 + fV}, & \text{if } \rho = 1; \end{cases}$$

$$P = p_0 e^{-fV} \cosh(\sqrt{\rho} fV).$$

2. Discrete case ($V = \infty$). Now, from the definition of the function $A_j(x)$ we have that $A_j(\infty) = \lim_{x\to\infty} A_j(x) = \beta_{j\,1}$. Therefore, when $V = \infty$, relations (34), (35), (38) take the form

$$p_k = p_0 \sum_{r_{(k)} \leq N} \prod_{j=1}^{k} a_{[r_j]} \beta_{[r_j]\,1}, \ k = \overline{1, M}; \tag{40}$$

$$p_0 = \left[1 + \sum_{k=1}^{M} \sum_{r_{(k)} \leq N} \prod_{j=1}^{k} a_{[r_j]} \beta_{[r_j]\,1}\right]^{-1}; \tag{41}$$

$$P_m = 1 - p_0 \left[1 + \sum_{j=1}^{M} \sum_{r_{(j)} \leq N-m} \prod_{l=1}^{j} a_{[r_l]} \beta_{[r_l]\,1}\right]. \tag{48}$$

Denoting by $\rho_m = a_m \beta_{m1}$, $m = \overline{1, K}$, we obtain that in this case relations (36), (37), (39) take the form:

$$g(\mathbf{k}) = p_0 k! \prod_{m=1}^{K} \frac{{\rho_m}^{k_m}}{k_m!}, \ \mathbf{k} \in \mathbf{S}; \ p_0 = \left[\sum_{\mathbf{k} \in \mathbf{S}} k! \prod_{m=1}^{K} \frac{{\rho_m}^{k_m}}{k_m!}\right]^{-1};$$

$$P_m = 1 - p_0 \sum_{\mathbf{k} \in \mathbf{S}_m} k! \prod_{j=1}^{K} \frac{{\rho_j}^{k_j}}{k_j!}.$$

In particular, if all the customers are the customers of the same type ($K = 1$, $b_1 = 1$), then we have the usual processor sharing system with limeted number of customers ($\rho = a\beta_1$):

$$p_k = \frac{(1-\rho)\rho^k}{1-\rho^{N+1}},\ k = \overline{0, N};\ P = \frac{(1-\rho)\rho^N}{1-\rho^{N+1}}.$$

3. There are two types of customers, a customer of the second type needs all the units of discrete resource for his service. Let $K = 2$. Assume that a customer of the first type needs one unit of discrete resource for his service, but a customer of the second type needs all the units of discrete resource, i.e. $b_1 = 1$, $b_2 = N$.

In this case, from relations (34), (35) we find that the distribution of number of customers in the system takes the form

$$p_0 = \left[1 + \sum_{k=1}^{N} {a_1}^k A_{1*}^{(k)}(V) + a_2 A_2(V)\right]^{-1};$$

$$p_1 = p_0 \left[a_1 A_1(V) + a_2 A_2(V)\right];\ p_k = p_0 {a_1}^k A_{1*}^{(k)}(V),\ k = \overline{2, N}.$$

From relation (38) we have the following formulas for loss probabilities:

$$P_1 = 1 - p_0 \left[L_1(V) + \sum_{j=1}^{N-1} {a_1}^j L_1 * A_{1*}^{(j)}(V)\right];\ P_2 = 1 - p_0 L_2(V).$$

Let, for example, $L_i(x) = 1 - e^{-f_i x}$ and customers lengths are proportional to their capacities: $\xi_i = c_i \zeta_i$, $i = 1, 2$. In this case, we have

$$A_i(x) = c_i \int_0^x u dL_i(u) = \frac{c_i}{f_i}\left[1 - (1 + f_i x)e^{-f_i x}\right],\ i = 1, 2.$$

The formulas for p_0 and loss probabilities take the form

$$p_0 = \left\{1 + \sum_{k=1}^{N} {\rho_1}^k \left[1 - e^{-f_1 V} \sum_{i=0}^{2k-1} \frac{(f_1 V)^i}{i!}\right] + \rho_2 \left[1 - (1 + f_2 V)e^{-f_2 V}\right]\right\}^{-1};$$

$$P_1 = 1 - p_0 \left\{1 - e^{-f_1 V} + \sum_{j=1}^{N-1} {\rho_1}^j \left[1 - e^{-f_1 V} \sum_{i=0}^{2j} \frac{(f_1 V)^i}{i!}\right]\right\};$$

$$P_2 = 1 - p_0 \left(1 - e^{-f_2 V}\right).$$

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