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Optimality conditions for a set-valued optimization problem in terms of approximations*

by

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Abstract: In this paper, we are concerned with a constrained set-valued optimization problem (P). Using support functions, we give necessary optimality conditions in terms of Karush-Kuhn-Tucker (KKT) multipliers and approximations. Under generalized convexity, we investigate sufficient optimality conditions. An example illustrating our findings is also given.

Keywords: approximation; weak Pareto minimal point; A-pseudoconvexity; A-quasi-invexity

1. Introduction

Multiobjective optimization is known as a useful mathematical model, developed in order to investigate some real world problems with conflicting objectives, arising from economics, engineering and human decision making. We are rarely asked to make decisions based on only one criterion; most often, decisions are based on several conflicting criteria. A lot of research has been carried out and discussed by several authors at various levels of generality regarding multiobjective optimization problems (see, e.g., Allali and Amarhroq, 1997; Amahroq and Taa, 1997; Amahroq and Gadhi, 2001; Chuong and Kim, 2014; Gadhi, 2005a,b; Gadhi and Jawhar, 2013; Kwan and Kim, 2011; Luc, 1991; Luc and Jahn, 1992; Ta, 1996; Zhou and Xuan-wei, 2018). In construction of optimality conditions, convexity has been the most important concept during the last decades. Recently, there have been numerous attempts to generalize the concept of convexity in order to weaken the assumptions on the attained results. Different kinds of generalized convexities have proven to be the main tool in the construction of optimality conditions, particularly the sufficient conditions.

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In this paper, we are concerned with the constrained set-valued optimization problem

$$(P):\begin{cases}\min F(x)\\ \text{subject to } G(x)\cap(-Z^{+})\neq\emptyset\end{cases}$$

where $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ are set-valued mappings between Banach spaces X, Y, Z, and $Z^{+} \subset Z$ is a closed convex cone with non empty interior Z^{++}.

Let

$$\Omega=\left\{x\in X : G(x)\cap\left(-Z^{+}\right)\neq\emptyset\right\}$$

be the feasible set of (P) and let $Y^{+} \subset Y$ be a closed convex cone with non empty interior Y^{++}. A point $(\overline{x}, \overline{y}) \in X \times Y$, $\overline{x} \in \Omega$ and $\overline{y} \in F(\overline{x})$, is said to be a weak Pareto minimal point with respect to Y^{+} of the problem (P) if

$$F(x)-\overline{y}\notin -Y^{++},\ \text{for all } x\in\Omega.$$

Approximations are important tools of nonsmooth analysis. Notice that for a locally Lipschitz function, both the Clarke subdifferential and approximate Jacobians (Jeyakumar and Luc, 1998) are approximations. Since approximations are not necessarily closed, it may be advantageous to use them when investigating optimality conditions instead of approximate Jacobians (Jeyakumar and Luc, 1998), which are closed by definition. With the help of the concept of approximation (Allali and Amahroq, 1997; Bazine et al., 2011; Dempe and Gadhi, 2010; Jourahi and Thibault, 1993; Khanh and Dinh, 2008; Khanh and Tung, 2013; Khan and Tuan, 2006), using support functions of the set valued mappings F and G together with a scalarization technique, we give necessary optimality conditions for (P). In order to obtain sufficient optimality conditions, inspired by the work of Dutta and Chandra (2004), we introduce a generalized convexity of functions admitting prior approximations. Neither the closedness nor the compactness of the approximations is required to find these sufficient conditions. An example illustrating our findings is also given.

The rest of the paper is written as follows. Section 2 contains basic definitions and preliminary materials. Sections 3 and 4 are devoted to optimality conditions.

2. Preliminaries

We denote by $L(X,Y)$ the set of continuous linear mappings between X and Y, $\mathbb{B}_Y$ is the closed unit ball of Y centered at the origin, $\mathbb{S}_Y$ is the unit sphere of Y, and X^* is the continuous dual of X. We write $\langle .,. \rangle$ for the canonical bilinear form with respect to the duality $\langle X^*, X\rangle$. Let A be a nonempty subset of Y and let D be a nonempty convex subset of Y. A point $\overline{y} \in A$ is said to be a Pareto (respectively. a weak Pareto) minimal point of A with respect to D if

$$(A - \overline{y}) \cap (-D) = \{0\}, \qquad (\text{resp. } (A - \overline{y}) \cap (-IntD) = \emptyset\ ),$$

where Int denotes the topological interior.

Definition 1 *(Allali and Amahroq, 1997) Let f be a mapping from X into Y, $\overline{x} \in X$ and $A_f(\overline{x}) \subset L(X,Y)$. $A_f(\overline{x})$ is said to be an approximation of f at $\overline{x}$ if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$f(x) - f(\overline{x}) \in A_f(\overline{x})(x - \overline{x}) + \varepsilon \|x - \overline{x}\| \mathbb{B}_Y \tag{1}$$

for all $x \in \overline{x} + \delta \mathbb{B}_X$.

It is easy to see that $f+g$ has $A_f(\overline{x}) + A_g(\overline{x})$ as an approximation at $\overline{x}$ whenever $A_f(\overline{x})$ and $A_g(\overline{x})$ are approximations of f and g at $\overline{x}$.

Remark 1 *The approximations are not necessarily closed sets. For example, we can easily show that the real function $f(x) = x - 1$ admits the open set $\left]\frac{1}{2}, \frac{3}{2}\right[$ as an approximation at $\overline{x} = 0$.*

Note that $A_f(\overline{x})$ is a singleton if and only if f is Fréchet differentiable at $\overline{x}$. In Allali and Amahroq (1997), it is shown that when f is a locally Lipschitz function, it admits as an approximation the Clarke subdifferential of f at $\overline{x}$; i.e.

$$A_f(\overline{x}) = \partial f(\overline{x}) := cl\ co \{lim\ \nabla f(x_n); x_n \in \mathrm{dom}\nabla f \text{ and } x_n \to \overline{x}\}.$$

Remark 2 *(Amahroq and Gadhi, 2001) Let $f : X \to \overline{\mathbb{R}} := [-\infty, +\infty]$ be a continuous function. Then, the symmetric subdifferential of f at $\overline{x}$ is an approximation of f at $\overline{x}$.*

Definition 2 *(Jeyakumar and Luc, 1998) Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a function. A closed subset $\partial^* f(\overline{u})$ of $L(\mathbb{R}^n, \mathbb{R}^p)$ is called an approximate Jacobian of f at $\overline{u}$ if for every $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^p_+$ one has*

$$\liminf_{t \searrow 0} \frac{\langle v,\ f(\overline{u} + tu)\rangle - \langle v,\ f(\overline{u})\rangle}{t} \leq \sup_{M \in \partial^* f(\overline{u})} \langle v,\ M(u)\rangle.$$

Remark 3 *(Dempe and Gadhi, 2010) For continuous functions, approximate Jacobians are examples of approximations.*

Lemma 1 *Let $f : X \to Y$ and $g : Y \to Y$ be two given mappings. Let $h : X \times Y \to Y$ be the mapping defined by*

$$h(x,y) = f(x) + g(y).$$

Suppose that f admits an approximation $A_f(\overline{x})$ at $\overline{x} \in X$ and that g admits an approximation $A_g(\overline{y})$ at $\overline{y} \in Y$. Then,

$$A_h(\overline{x}, \overline{y}) = A_f(\overline{x}) \times \{0\} + \{0\} \times A_g(\overline{y})$$

is an approximation of h at $(\overline{x}, \overline{y})$.

Proof. Let $\varepsilon > 0$. Since $A_f(\overline{x})$ and $A_g(\overline{y})$ are approximations of f and g at $\overline{x}$ and $\overline{y}$, respectively, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$f(x) - f(\overline{x}) \in A_f(\overline{x})(x - \overline{x}) + \varepsilon \|x - \overline{x}\| \mathbb{B}_Y$$

and

$$g(y) - g(\overline{y}) \in A_g(\overline{y})(y - \overline{y}) + \varepsilon \|y - \overline{y}\| \mathbb{B}_Y,$$
$$\text{for all } x \in \overline{x} + \delta_1 \mathbb{B}_X \text{ and } y \in \overline{y} + \delta_2 \mathbb{B}_Y.$$

Consequently,

$$h(x,y) - h(\overline{x},\overline{y}) \in A_f(\overline{x})(x - \overline{x}) + A_g(\overline{y})(y - \overline{y}) + \varepsilon (\|x - \overline{x}\| + \|y - \overline{y}\|) \mathbb{B}_Y$$

for all $x \in \overline{x} + \delta_1 \mathbb{B}_X$ and $y \in \overline{y} + \delta_2 \mathbb{B}_Y$. By setting $\delta = \min\{\delta_1, \delta_2\}$, one deduces that

$$h(x,y) - h(\overline{x},\overline{y}) \in A_h(\overline{x},\overline{y})(x - \overline{x}, y - \overline{y}) + \varepsilon \|(x - \overline{x}, y - \overline{y})\| \mathbb{B}_Y$$

for all $(x,y) \in (\overline{x},\overline{y}) + \delta \mathbb{B}_{X \times Y}$. The proof is thus complete. ■

Let us recall the following notions. For every $\lambda^* \in Y^*$, the support function of F at x is defined by $C_F(\lambda^*, x) = \inf_{y \in F(x)} \langle \lambda^*, y \rangle$. The negative polar cone $(Y^+)^\circ$ of a cone Y^+ is as follows

$$(Y^+)^\circ = \{\lambda^* \in Y^* : \langle \lambda^*, y \rangle \leq 0 \text{ for all } y \in Y^+\}.$$

We suppose that the barrier cone of F, $Y_F^* = \{\lambda^* \in Y^* : C_F(\lambda^*, x) > -\infty\}$, is closed, convex and does not depend on x.

A set-valued mapping F from X into Y is said to be locally Lipschitz at $x \in X$ if there exists a neighborhood U of x such that for some constant α and for all $x_1, x_2 \in U$, we have

$$F(x_1) \subseteq F(x_2) + \alpha \|x_2 - x_1\| \mathbb{B}_Y.$$

The number α will be called a Lipschitz-constant for F at x. It is obvious that $C_F(\lambda^*, x)$ is locally Lipschitz in x, and $\alpha \|\lambda^*\|$ is a Lipschitz-constant for $C_F(\lambda^*, x)$ at x if α is a Lipschitz-constant for F at x.

Definition 3 *(Corley, 1988) Let $C \subseteq X$ be a convex set. The set valued mapping F from C into Y is said to be Y^+-convex on C, if $\forall x_1, x_2 \in C$, $\forall \alpha \in [0,1]$,*

$$\alpha F(x_1) + (1-\alpha) F(x_2) \subseteq F(\alpha x_1 + (1-\alpha) x_2) + Y^+.$$

Proposition 1 *(Gadhi, 2005b) Let $C \subseteq X$ be a convex set. Considering $y^* \in (-Y^+)^\circ$, if F is Y^+-convex on C, then $C_F(\lambda^*, .)$ is a convex function on C.*

For the rest of the paper, we suppose that X is separable and that the support functions $C_F(\lambda^*,.)$ and $C_G(\mu^*,.)$ admit approximations $A_{C_F(\lambda^*,.)}(\overline{x})$ and $A_{C_G(\mu^*,.)}(\overline{x})$ at $\overline{x}$. Moreover, in Section 3, $C_F(\lambda^*,.)$ and $C_G(\mu^*,.)$ are assumed to have the following properties:

- There exists $\delta > 0$ such that for every $x \in \overline{x}+\delta\mathbb{B}_X$, the functions $C_F(\lambda^*,.)$ and $C_G(\mu^*,.)$ admit bounded and w^*-closed approximations $A_{C_F(\lambda^*,.)}(x)$ and $A_{C_G(\mu^*,.)}(x)$ at x.
- If $a_n^* \in A_{C_G(\mu_n^*,.)}(x_n)$ and $b_n^* \in A_{C_F(\lambda_n^*,.)}(x_n)$, where $a_n^* \overset{w^*}{\to} a^*$, $b_n^* \overset{w^*}{\to} b^*$, in X^*, $\lambda_n^* \to \lambda^*$, $\mu_n^* \to \mu^*$ in $\mathbb{R}$ and $x_n \to \overline{x}$ in X, then
$$a^* \in A_{C_G(\mu^*,.)}(\overline{x}) \text{ and } b^* \in A_{C_F(\lambda^*,.)}(\overline{x}).$$
- For each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in \overline{x} + \delta\mathbb{B}_X$

$$A_{C_F(\lambda^*,.)}(x) \subset A_{C_F(\lambda^*,.)}(\overline{x})+\varepsilon\mathbb{B}_{X^*} \text{ and } A_{C_G(\mu^*,.)}(x) \subset A_{C_G(\mu^*,.)}(\overline{x})+\varepsilon\mathbb{B}_{X^*}.$$

For a nonempty subset S of Y, we consider the Hiriart-Urruty signed distance (Hiriart-Urruty, 1979; Hirart-Urruty and Lemarechal, 1993):

$$\Delta_S(y) = \begin{cases} d(y,\ S) & if \quad y \notin S \\ -d(y, Y\backslash S) & if \quad y \in S. \end{cases}$$

Let us recall the following results.

Proposition 2 *(Hiriart-Urruty, 1979) Let $S \subset Y$ be a closed convex cone with nonempty interior and $S \neq Y$. The function Δ_S is convex, positively homogeneous, 1-Lipschitzian, decreasing on Y with respect to the order introduced by S. Moreover $(Y\backslash S) = \{y \in Y : \Delta_S(y) > 0\}$, $int(S) = \{y \in Y : \Delta_S(y) < 0\}$ and the boundary of $S : bd(S) = \{y \in Y : \Delta_S(y) = 0\}$.*

Proposition 3 *(Ciligot-Travain, 1994) Let $S \subset Y$ be a closed convex cone with nonempty interior. Then for all $y \in Y$,*

$$0 \notin \partial\Delta_S(y)$$

where "∂" is the Fenchel subdifferential since Δ_S is convex.

3. Necessary optimality conditions

Throughout this section, it is assumed that $\dim Y < +\infty$, $\dim Z < +\infty$ and that the unit dual ball $\mathbb{B}_{X^*}$ is w^*-sequentially compact. The following theorem gives, in terms of approximations and KKT multipliers, necessary optimality conditions of the unconstrained problem

$$(Q) : \begin{cases} Y^+ \text{ - Min } F(x) \\ \text{subject to : } \quad x \in X. \end{cases}$$

Theorem 1 *Suppose that* $(\overline{x},\overline{y}) \in X \times Y$ *and* $\overline{y} \in F(\overline{x})$ *is a weak local Pareto minimal point with respect to* Y^+ *of the problem* (Q) *. Then, there exists a vector* $\mu^* \in (-Y^+)^\circ,\ \mu^* \neq 0_Y$*, such that*

$$C_F(\mu^*,\bar{x}) = \langle \mu^*, \overline{y}\rangle \quad \textit{and} \quad 0 \in A_{C_F(\mu^*,.)}(\bar{x}).$$

Proof. Since $(\overline{x},\overline{y})$ is a weak local Pareto minimal point with respect to Y^+ of the problem (Q), there exists a neighborhood U of $\overline{x}$ such that

$$y - \overline{y} \notin -Y^{++},\ \text{for all } x \in U \text{ and } y \in F(x).$$

Then,

$$\Delta_{-Y^{++}}(y-\overline{y}) \geq 0,\ \text{for all } x \in U \text{ and } y \in F(x).$$

Consequently, $(\overline{x},\overline{y})$ minimizes locally the scalar function $\varphi(x,y) = \Delta_{-Y^{++}}(y-\overline{y})$ over the set

$$\overline{\Omega} = \{(x,y) \in X \times Y \mid 0 \in H(x,y)\},$$

where $H(x,y) = F(x) - y$. By applying Theorem 2 from Amahroq and Gadhi (2001), we obtain that there exist $\lambda^* \in \mathbb{R}^+,\ \mu^* \in Y_F^*,\ (\lambda^*,\mu^*) \neq (0,0)$, such that

$$0 \in \lambda^* A_\varphi(\overline{x},\overline{y}) + A_{C_H(\mu^*,.)}(\overline{x},\overline{y})$$

and

$$C_H(\mu^*,(\overline{x},\overline{y})) = 0. \tag{2}$$

- On the one hand, since

$$C_H(\mu^*,(x,y)) = \inf_{h\in F(x)-y}\langle \mu^*,h\rangle = \inf_{v\in F(x)}\langle \mu^*, v-y\rangle =$$
$$= \left(\inf_{v\in F(x)}\langle \mu^*,v\rangle\right) - \langle \mu^*,y\rangle,$$

one gets

$$C_H(\mu^*,(x,y)) = C_F(\mu^*,x) - \langle \mu^*,y\rangle. \tag{3}$$

On the other hand, since φ is a convex and Lipschitz function, by Lemma 1, one has

$$(0,0) \in \lambda^*\left(\ \{0\} \times \partial\Delta_{-Y^{++}}(0)\ \right) + A_{C_F(\mu^*,.)}(\overline{x}) \times \{0\} - \{(0,\mu^*)\}. \tag{4}$$

Then, there exists $a^* \in \partial\Delta_{-Y^{++}}(0)$ such that

$$0 \in \lambda^* a^* - \mu^* \text{ and } 0 \in A_{C_F(\mu^*,.)}(\overline{x}). \tag{5}$$

- Since $\Delta_{(-Y^{++})}(.)$ is a convex function and $\Delta_{(-Y^{++})}(0) = 0$, we have

  $$\Delta_{(-Y^{++})}(\upsilon) \geq \langle a^*,\ \upsilon\rangle,\ \forall \upsilon \in Y,$$

  and thus, for all $\upsilon \in -Y^{++}$, one gets

  $$0 \geq \Delta_{(-Y^{++})}(\upsilon) = -d_{(Y\setminus -Y^{++})}(\upsilon) \geq \langle a^*,\ \upsilon\rangle.$$

  Since $\overline{-Y^{++}} = -Y^+$, one gets $a^* \in (-Y^+)^\circ$. Since $\mu^* = \lambda^* a^*$, one deduces that $\mu^* \in (-Y^+)^\circ$.
- From Proposition 3, we have that $a^* \neq 0$. Now, since $(\lambda^*, \mu^*) \neq (0, 0)$, one deduces from (4) that $\mu^* \neq 0_Y$ and $\lambda^* \neq 0_{\mathbb{R}}$.

■

Remark 4 *In Amahroq and Gadhi (2001), Theorem 2, it was demonstrated that $\mu^* \in Y_F^*$, but it was not specified that $\mu^* \neq 0_Y$. Looking closely at the proof of Theorem 2 in Amahroq and Gadhi (2001), one remarks that the authors had initially shown the existence of a sequence $\mu_n^* \in Y_F^* \cap \mathbb{S}_{Y^*}$; we refer interested readers to page 440, line 18, of this reference for more details. Since we restricted ourselves to the case of* $\dim Y < +\infty$, *taking a subsequence if necessary, we can assume that $\mu_n^* \to \mu^* \in Y_F^* \cap \mathbb{S}_{Y^*}$ when n tends to $+\infty$. This result, in finite dimension, could have been incorporated as a corollary of Theorem 2, in Amahroq and Gadhi (2001).*

The following regularity condition has been introduced by Amahroq and Gadhi (2003); see Definition 2.3 there.

Definition 4 *(Amahroq and Gadhi, 2003) The problem (P) is said to be regular at $\bar{x} \in X$ if*

$$\left.\begin{array}{c} 0 \in A_{C_G(\mu^*,.)}(\overline{x}) \\ C_G(\mu^*, \overline{x}) = \langle \mu^*, \overline{z}\rangle \end{array}\right\} \Longrightarrow \mu^* = 0.$$

In the following theorem, we give necessary optimality conditions for (P).

Theorem 2 *Suppose that $(\overline{x}, \overline{y}) \in X \times Y$, $\overline{x} \in \Omega$ and $\overline{y} \in F(\overline{x})$, is a weak local Pareto minimal point with respect to Y^+ of the problem (P). Then, for all $\bar{z} \in G(\bar{x}) \cap (-Z^+)$, there exist vectors $\lambda^* \in (-Y^+)^\circ$ and $\mu^* \in (-Z^+)^\circ$, $(\lambda^*, \mu^*) \neq (0_Y, 0_Z)$ such that $C_F(\lambda^*, \overline{x}) = \langle \lambda^*, \overline{y}\rangle$, $C_G(\mu^*, \overline{x}) = \langle \mu^*, \overline{z}\rangle$ and*

$$0 \in A_{C_F(\lambda^*,.)}(\overline{x}) + A_{C_G(\mu^*,.)}(\overline{x}). \tag{6}$$

If, in addition, (P) is regular at $\overline{x}$, one gets $\lambda^* \neq 0_Y$.

Proof. Let $(\bar{x}, \bar{y})$ be a weak local Pareto minimal point of (P) with respect to Y^+ and let $\bar{z} \in G(\bar{x}) \cap (-Z^+)$ be an arbitrary element. By Amahroq and Taa (1997, Proposition 3.1), $(\bar{x}, \bar{y}, \bar{z})$ is a weak local Pareto minimal point of

$$(\Gamma) : \left\{ \begin{array}{l} \min\ (F(x), G(x)) \\ \text{subject to } x \in X, \end{array}\right.$$

with respect to $Y^+ \times (Z^+ + \bar{z})$.

- Since

  $$A_{C_{(F,G)}((\lambda^*,\mu^*),.)}(\overline{x}) = A_{C_F(\lambda^*,.)}(\overline{x}) + A_{C_G(\mu^*,.)}(\overline{x}),$$

  by applying Theorem 1, we get that there exists $(\lambda^*, \mu^*) \in (-Y^+)^\circ \times (-Z^+)^\circ$, $(\lambda^*, \mu^*) \neq (0_Y, 0_Z)$ such that

  $$0 \in A_{C_F(\lambda^*,.)}(\overline{x}) + A_{C_G(\mu^*,.)}(\overline{x}) \tag{7}$$

  and

  $$C_F(\lambda^*, \overline{x}) + C_G(\mu^*, \overline{x}) = \langle \lambda^*, \overline{y} \rangle + \langle \mu^*, \overline{z} \rangle. \tag{8}$$

- Let us prove that

  $$\langle \lambda^*, \overline{y} \rangle = C_F(\lambda^*, \overline{x}) \text{ and } C_G(\mu^*, \overline{x}) = \langle \mu^*, \overline{z} \rangle.$$

  - Since $\overline{z} \in G(\overline{x})$, one gets

    $$C_G(\mu^*, \overline{x}) \leq \langle \mu^*, \overline{z} \rangle.$$

    Then,

    $$\langle \lambda^*, \overline{y} \rangle \leq C_F(\lambda^*, \overline{x}).$$

  - Since $\overline{y} \in F(\overline{x})$, one deduces that

    $$\langle \lambda^*, \overline{y} \rangle = C_F(\lambda^*, \overline{x});$$

    and consequently,

    $$C_G(\mu^*, \overline{x}) = \langle \mu^*, \overline{z} \rangle. \tag{9}$$

- Let us prove that $\lambda^* \neq 0_Y$ when (P) is regular at $\overline{x}$. By contrary, suppose that $\lambda^* = 0_Y$. Then, relations (7) and (9) reduce to

  $$0 \in A_{C_G(\mu^*,.)}(\overline{x}) \text{ and } C_G(\mu^*, \overline{x}) = \langle \mu^*, \overline{z} \rangle.$$

  Since (P) is regular at $\overline{x}$, we deduce that $\mu^* = 0_Z$, which is a contradiction with $(\lambda^*, \mu^*) \neq (0_Y, 0_Z)$.

■

Remark 5 *In the case where F and G are locally Lipschitz set-valued mappings, the support functions $C_F(\lambda^*, .)$ and $C_G(\mu^*, .)$ will be locally Lipschitz. By adopting the Clarke subdifferentials $\partial C_F(\lambda^*, .)(\overline{x})$ and $\partial C_G(\mu^*, .)(\overline{x})$ as approximations of the support functions $C_F(\lambda^*, .)$ and $C_G(\mu^*, .)$ at $\overline{x}$, we would get the result of Gadhi (2005a, Corollary 1). If in addition, F is Y^+-convex and G is Z^+-convex, the support functions $C_F(\lambda^*, .)$ and $C_G(\mu^*, .)$ will be convex and we could use the convex analysis subdifferential instead of the Clarke subdifferential in* (6).

4. Sufficient optimality conditions

In Dutta and Chandra (2004), the authors defined generalized convexities using convexificators and used them to show sufficient optimality conditions. Taking inspiration from Dutta and Chandra (2004), we introduce the following generalized convexity of a function admitting an approximation.

Definition 5 *Let $g : X \to \mathbb{R}$ and $\overline{x} \in X$. We assume that g admits an approximation $A_g(\overline{x}) \subset L(X, \mathbb{R})$ at $\overline{x}$.*

- *g is said to be A_g-convex at $\overline{x}$ iff for all $x \in X$, one has*

$$\langle \xi, x - \overline{x} \rangle \leq g(x) - g(\overline{x}), \qquad \text{for all } \xi \in A_g(\overline{x}). \tag{10}$$

- *g is said to be A_g-quasiconvex at $\overline{x}$ iff for all $x \in X$, one has*

$$g(x) - g(\overline{x}) \leq 0 \Longrightarrow \langle \xi, x - \overline{x} \rangle \leq 0, \qquad \text{for all } \xi \in A_g(\overline{x}).$$

- *g is said to be A_g-pseudoconvex at $\overline{x}$ iff for all $x \in X$, one has*

$$g(x) - g(\overline{x}) < 0 \Longrightarrow \langle \xi, x - \overline{x} \rangle < 0, \qquad \text{for all } \xi \in A_g(\overline{x}).$$

Dutta and Chandra (2004) introduced ∂^*-pseudoconvex functions by using this concept of convexifactor. Later on, with the help of ∂^*-convexity, ∂^*-pseudo-convexity and ∂^*-quasiconvexity of bifunctions in terms of convexifactors, Suneja and Kohli (2011) gave optimality and duality results for a bilevel programming problem. Then, in Suneja and Kohli (2013), they used ∂^*-pseudoconvexity and ∂^*-quasiconvexity in terms of convexifactors for studying duality results of multiobjective fractional programming problem.

Remark 6 *If $g : X \to \mathbb{R}$ is locally Lipschitz and if $A_g(\overline{x}) = \partial g(\overline{x})$, then the A_g-pseudoconvex functions are termed ∂-pseudoconvex or non-smooth pseudoconvex functions.*

The following theorem represents sufficient optimality conditions for (P).

Theorem 3 *Let $\overline{x} \in \Omega$ be a feasible solution of (P) and let $\overline{y} \in F(\overline{x})$. Suppose that there exist vectors $\lambda^* \in (-Y^+)^\circ$, $\lambda^* \neq 0_{Y^*}$, and $\mu^* \in (-Z^+)^\circ$ such that*

$$0 \in A_{C_F(\lambda^*,.)}(\overline{x}) + A_{C_G(\mu^*,.)}(\overline{x}) \tag{11}$$

and

$$C_F(\lambda^*, \overline{x}) = \langle \lambda^*, \overline{y} \rangle, \; C_G(\mu^*, \overline{x}) = 0. \tag{12}$$

Suppose also that :

- *$C_F(\lambda^*, .)$ is $A_{C_F(\lambda^*,.)}$-pseudoconvex at $\overline{x}$.*
- *$C_G(\mu^*, .)$ is $A_{C_G(\mu^*,.)}$-quasiconvex at $\overline{x}$.*

Then $\overline{x}$ is a weak Pareto minimal point of (P) with respect to Y^+.

Proof. To the contrary, suppose that $(\overline{x},\overline{y})$ is not a weak Pareto minimal point with respect to Y^+ of (P). Then, there exists $x\in\Omega$ and $y\in F(x)$ such that $y-\overline{y}\in -Y^{++}$. Since $\lambda^*\in\left(-Y^+\right)^\circ$ with $\lambda^*\neq 0_{Y^*}$, one has $\langle\lambda^*,y\rangle < \langle\lambda^*,\overline{y}\rangle$. Then,

$$C_F(\lambda^*,x)\leq\langle\lambda^*,y\rangle<\langle\lambda^*,\overline{y}\rangle=C_F(\lambda^*,\overline{x}).$$

Consequently,

$$C_F(\lambda^*,x)<C_F(\lambda^*,\overline{x}). \tag{13}$$

Since

$$0\in A_{C_F(\lambda^*,.)}(\overline{x})+A_{C_G(\mu^*,.)}(\bar{x}),$$

there exist $a\in A_{C_F(\lambda^*,.)}(\overline{x})$ and $b\in A_{C_G(\mu^*,.)}(\bar{x})$ such that

$$a+b=0. \tag{14}$$

On the one hand, since $C_F(\lambda^*,.)$ is $A_{C_F(\lambda^*,.)}$-pseudoconvex at $\overline{x}$, by (13), one has

$$\langle a,x-\bar{x}\rangle \ < \ 0. \tag{15}$$

On the other hand, since $G(x)\cap(-Z^+)\neq\emptyset$, there exists $z\in G(x)\cap(-Z^+)$ such that

$$C_G(\mu^*,x)\leq\langle\mu^*,z\rangle.$$

As $C_G(\mu^*,\overline{x})=0$ and since $\mu^*\in(-Z^+)^\circ$, one gets

$$C_G(\mu^*,x)-C_G(\mu^*,\bar{x})\leq\langle\mu^*,z\rangle\leq 0.$$

As $C_G(\mu^*,.)$ is $A_{C_G(\mu^*,.)}$-quasiconvex at $\overline{x}$, we have

$$\langle b,x-\bar{x}\rangle \ \leq \ 0. \tag{16}$$

Combining (15) and (16), one deduces

$$\langle a+b,x-\bar{x}\rangle \ < \ 0.$$

Using (14) we obtain $\langle 0,x-\bar{x}\rangle<0$ a contradiction. The proof is thus complete. ∎

Example 1 *We consider the following constrained set-valued optimization problem*

$$(P):\begin{cases}\ \min\ F(x_1,x_2)\\ \textit{subject to } G(x_1,x_2)\cap\left(-\mathbb{R}^2_+\right)\neq\emptyset,\end{cases}$$

where

$$F(x_1,x_2)=(F_1(x_1,x_2),F_2(x_1,x_2)),\ G(x_1,x_2)=(G_1(x_1,x_2),G_2(x_1,x_2)),$$

$$F_1(x_1,x_2)=\left[\frac{1}{2}|x_1|+x_2^2,\ \sqrt{x_2}+x_2^2+|x_1|+\frac{1}{2}\right],\ F_2(x_1,x_2)$$
$$=\left[x_2^3+x_2\left(x_1^2+1\right)-1,\ \left(x_1^2+1\right)e^{x_2}+x_2^3\right]$$

$$G_1(x_1,x_2)=\left[|x_1|+x_1x_2-2,\ |x_1|+\frac{1}{4}\left(x_1^2-1\right)+x_2^2\right]\ \text{and}\ G_2(x_1,x_2)$$
$$=\left[x_1^2-\frac{1}{2}x_2,\ x_1^2+|x_2|+x_2^2e^{x_1}\right].$$

- *On the one hand,* $(\overline{x},\overline{y})=((0,0),(0,-1))$ *is a weak local Pareto minimal point of* (P^*) *with respect to* $\mathbb{R}^2_+$*. Indeed,*

$$F(0,0)=\left[0,\frac{1}{2}\right]\times[-1,1].$$

Then, for all $x\in\Omega,\ y=(y_1,y_2)\in F(x)$*, we have*

$$0\le\frac{1}{2}|x_1|+x_2^2\le y_1-\overline{y}_1\le\sqrt{x_2}+x_2^2+|x_1|+\frac{1}{2}.$$

Hence

$$y-\overline{y}\notin -int\ \left(\mathbb{R}^2_+\right).$$

- *On the other hand, for* $\lambda^*=(\lambda_1^*,\lambda_2^*)\in\left(-\mathbb{R}^2_+\right)^\circ$ *and* $\mu^*=(\mu_1^*,\mu_2^*)\in\left(-\mathbb{R}^2_+\right)^\circ$*, the support functions of* F *and* G *at* $x=(x_1,x_2)$ *are given by*

$$C_F(\lambda^*,x)=\lambda_1^*\left(\frac{1}{2}|x_1|+x_2^2\right)+\lambda_2^*\left(x_2^3+x_2\left(x_1^2+1\right)-1\right)$$

and

$$C_G(\mu^*,x)=\mu_1^*(|x_1|+x_1x_2-2)+\mu_2^*\left(x_1^2-\frac{1}{2}x_2\right).$$

Moreover,

$$A_{C_F(\lambda^*,.)}(\overline{x})=\left\{\left(\frac{1}{2}\lambda_1^*,\lambda_2^*\right),\left(-\frac{1}{2}\lambda_1^*,\lambda_2^*\right)\right\}$$

and

$$A_{C_G(\mu^*,.)}(\overline{x})=\left\{\left(\mu_1^*,-\frac{1}{2}\mu_2^*\right),\ \left(-\mu_1^*,-\frac{1}{2}\mu_2^*\right)\right\}.$$

- *Let* $\lambda^* = \left(0, \frac{1}{2}\right)$ *and* $\mu^* = (0, 1)$.
  - $C_F(\lambda^*, .)$ *is* $A_{C_F(\lambda^*,.)}$*-pseudoconvex at* $\overline{x}$ *and* $C_G(\mu^*, .)$ *is* $A_{C_G(\mu^*,.)}$*-quasiconvex at* $\overline{x}$.
  - (P) *is regular at* $\overline{x}$.
  - *Since* $G(\overline{x}) \cap \left(-\mathbb{R}^2_+\right) = \left[-2, -\frac{1}{4}\right] \times \{0\}$, *for any* $\overline{z} = (\alpha, 0)$, *with* $\alpha \in \left[-2, -\frac{1}{4}\right]$, *one has*

$$\begin{cases} 0 \in A_{C_F(\lambda^*,.)}(\overline{x}) + A_{C_G(\mu^*,.)}(\overline{x}) \\ C_F(\lambda^*, \overline{x}) = -\frac{1}{2} = \langle \lambda^*, \overline{y} \rangle, \\ C_G(\mu^*, \overline{x}) = 0 = \langle \mu^*, \overline{z} \rangle. \end{cases} \tag{17}$$

Consequently, necessary and sufficient optimality conditions are satisfied.

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