PROJECTIVE COLLINEATIONS AS PRODUCTS OF TWO CYCLIC COLLINEATIONS.

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Abstract. The paper contains one theorem stating that, for arbitrary k, every projective collineation in the projective plane is a composition of two k-cyclic collineations.

Keywords: collineation, cyclic collineation, composition.

The problem of decomposing linear transformation into some special transformations has been investigated in many papers ([5], [6] for instance). This is also interesting what is the minimal number of such factors required. This task is known as the length problem.

The well known property of the projective geometry says that any projectivity on a projective line P_1 is a composition of two involutions. A generalisation of this fact was given in [2]. Namely, the following theorem was proved: Let $P_1(F)$ be one-dimensional projective space over an algebraically closed field F of characteristic 0, let k be an arbitrary integer not less than 2, and let f be a projective transformation of $P_1(F)$ onto itself. Then there exist exactly k-cyclic projective transformations g,h such that f=gh (a transformation $f:X\to X$ is called to be exactly k-cyclic if $f^k=id$, and $f^m\neq id$ for m<k). In the case of the real projective line $P_1(R)$ the following property [3] holds: Let f be a nonsingular projectivity in $P_1(R)$. If detF>0, then for every $2\leq k\neq 3$, there exist exactly k-cyclic projectivities g,h such that f-gh. If however detF<0, then for every $k\geq 2$ there exist an exactly k-cyclic projectivity g and an involution h such that f=gh (F denotes the matrix of f).

In this paper we shall deal with the real projective plane $P_2(R)$. E. W. Ellers [6] investigated a much more general situation namely, he considered a projective space of an arbitrary dimension (even infinite) over an arbitrary field (not necessarily commutative). However, in the particular case of $P_2(R)$, as it often occurs, the results from [6] can be improved. It should be also noticed that a projective plane (especially the real projective plane) plays a special role in the projective geometry. That is why we investigate the case of $P_2(R)$. We shall show that any nonsingular collineation in $P_2(R)$ is a composition of two exactly k-cyclic collineations for an arbitrary integer $k\geq 3$. Since the case k=3 was investigated in [4], we consider $k\geq 4$.

First of all, it should be noticed that we can distinguish, depending on the number of fixed points of a given collineation f, the following cases:

- three distinct points; (1)
- two distinct points; (2)
- one point lying on the invariant line; (3)
- one point not lying on the invariant line; (4)
- one line consisting of fixed points (elation); (5)
- one line consisting of fixed points and one point not on this line (homology); (6)
- each point of the plane (identity).

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Observe that if f is of type (1), then its matrix, in an allowable coordinate system, is of the form:

$$M_1=\begin{bmatrix} a^3 & 0 & 0 \\ 0 & b^3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 1 \neq a \neq b \neq 1, a,b \neq 0.$$

Similarly, the respective matrices for cases (2), (3), (4), (5), (6) are:

$$M_2=\begin{bmatrix} 1 & 0 & 1 \\ 0 & a^3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 0 \neq a \neq 1;;$$

$$M_3=\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}, a \neq 0; M_4=\begin{bmatrix} a^3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & b^3 & c \end{bmatrix}, \begin{matrix} a,b \neq 0, c^2+4b^2<0, \\ ca^3+b^3 \neq a^6; \end{matrix}$$

$$M_5=\begin{bmatrix} a^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 0 \neq a \neq 1; M_6=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}, a \neq 0.$$

Take into account matrices:

$$A_1=\begin{bmatrix} R\frac{ab-1}{a^3-1} & 1+R^2a^2\frac{(a-b^2)(ab-1)}{(a^3-1)(b^3-a^3)} & R^2a\frac{(ab-1)(a^2-b)}{(a^3-1)^2}-Ra^2\frac{a-b^2}{b^3-a^3}-R \\ 0 & 0 & 1 \\ 1 & Ra^2\frac{a-b^2}{b^3-a^3} & Ra\frac{a^2-b}{a^3-1} \end{bmatrix},$$

$$B_1=\begin{bmatrix} a^3R\frac{ab-1}{a^3-1} & b^3[1+R^2a^2\frac{(a-b^2)(ab-1)}{(a^3-1)(b^3-a^3)}] & R^2a\frac{(ab-1)(a^2-b)}{(a^3-1)^2}-Ra^2\frac{a-b^2}{b^3-a^3}-R \\ 0 & 0 & 1 \\ a^3 & Ra^2b^3\frac{a-b^2}{b^3-a^3} & Ra\frac{a^2-b}{a^3-1} \end{bmatrix},$$

$$A_2=\begin{bmatrix} Ra\frac{a+1}{a^2+a+1} & R\frac{a^3+a^2}{(a^2+a+1)^2(a-1)}+1-\frac{a}{a-1} & -R\,a\frac{a+1}{a^2+a+1}+\frac{a}{R(a\ -1)} \\ R\frac{a-1}{a} & \frac{R}{a^2+a+1} & -R\frac{a-1}{a} \\ 0 & 1 & 0 \end{bmatrix},$$

$$B_2=\begin{bmatrix} Ra\dfrac{a+1}{a^2+a+1} & R\dfrac{a^6+a^5}{(a^2+a+1)^2(a-1)}+a^3-\dfrac{a^4}{a-1} & \dfrac{a}{R(a-1)} \\ R\dfrac{a-1}{a} & \dfrac{Ra^3}{a^2+a+1} & 0 \\ 0 & a^3 & 0 \end{bmatrix},$$

$$A_3=\begin{bmatrix} 2\cos\dfrac{2\pi}{k} & 0 & -1 \\ 0 & 1 & -a \\ 1 & 0 & 0 \end{bmatrix},\ B_3=\begin{bmatrix} 2\cos\dfrac{2\pi}{k} & 2a\cos\dfrac{2\pi}{k} & -1 \\ 0 & 1 & 0 \\ 1 & a & 0 \end{bmatrix},$$

$$A_4=\begin{bmatrix} R-1 & 0 & 1 \\ \dfrac{-R(R-1)a^2b-(R-1)^2a^4+ac-Rb^2}{ab^3} & 1 & \dfrac{-Rab-(R-1)a^3}{b^3} \\ -1 & 0 & 0 \end{bmatrix},$$

$$B_4=\begin{bmatrix} (R-1)a^3 & b^3 & c \\ \dfrac{-R(R-1)a^4b-(R-1)^2a^6+a^3c-Ra^2b^2}{b^3} & -Rab-(R-1)a^3 & \dfrac{-Rabc-(R-1)a^3c}{b^3}+1 \\ -a^3 & 0 & 0 \end{bmatrix},$$

$$A_5=\begin{bmatrix} \dfrac{R}{a^2+a+1} & 0 & 1 \\ R\dfrac{a^2+3a+1}{a^3+a^2+a}-\dfrac{R^2(a+1)^2}{(a^2+a+1)^2}-\dfrac{a^2+a+1}{Ra^2} & \dfrac{Ra(a+1)}{a^2+a+1} & 1 \\ R^2\dfrac{a(a+1)}{(a^2+a+1)^2}-R\dfrac{a+1}{a^2+a+1} & \dfrac{-Ra^2}{a^2+a+1} & 0 \end{bmatrix},$$

$$B_5=\begin{bmatrix} \dfrac{Ra^3}{a^2+a+1} & 0 & 1 \\ R\dfrac{a^4+3a^3+a^2}{a^2+a+1}-\dfrac{R^2(a+1)^2a^3}{(a^2+a+1)^2}-\dfrac{a^3+a^2+a}{R} & \dfrac{Ra(a+1)}{a^2+a+1} & 1 \\ R^2\dfrac{a^4(a+1)}{(a^2+a+1)^2}-Ra^3\dfrac{a+1}{a^2+a+1} & \dfrac{-Ra^2}{a^2+a+1} & 0 \end{bmatrix},$$

$$A_6=\begin{bmatrix} 2\cos\frac{2\pi}{k} & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ B_6=\begin{bmatrix} 2\cos\frac{2\pi}{k} & 0 & 1 \\ 0 & 1 & a \\ -1 & 0 & 0 \end{bmatrix},\ \text{where } R=1+2\cos\frac{2\pi}{k}.$$

Apparently $A_iM_i = B_i$ for i=1,2,...,6. According to Theorem II [1] p. 353, all the matrices A_i, B_i represent exactly k-cyclic collineations. In fact, their invariant factors have no repeated divisors and are divisors of a polynomial of type $x^k-\beta$. Hence we can formulate

Theorem

If f is a nonsingular projective collineation in $P_2(R)$ and k is an arbitrary integer not less than 3, then f is a composition of two exactly k-cyclic collineations.

As it can be seen the length problem has been solved in an optimal way. Naturally, the above holds in the case of complex projective plane.

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KOLINEACJE RZUTOWE JAKO ZŁOŻENIA DWÓCH KOLINEACJI CYKLICZNYCH.

W pracy pokazano, że każda kolineacja rzutowa rzeczywistej płaszczyzny rzutowej jest złożeniem dwóch kolineacji k-cyklicznych. Przy tym ma to miejsce dla dowolnej liczby naturalnej nie mniejszej niż 3.