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EXISTENCE RESULTS FOR A SUBLINEAR SECOND ORDER DIRICHLET BOUNDARY VALUE PROBLEM ON THE HALF-LINE

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Abstract. In this paper we study the existence of nontrivial solutions for a boundary value problem on the half-line, where the nonlinear term is sublinear, by using Ekeland's variational principle and critical point theory.

Keywords: Ekeland's variational principle, critical point.

Mathematics Subject Classification: 34B40, 35A15, 35B38, 45C05, 34B24, 46T20.

1. INTRODUCTION

We consider the problem

$$\begin{cases} -(p(x)u'(x))' = \lambda q(x)f(x,u(x)), & x \in [0,+\infty), \\ u(0) = u(+\infty) = 0, \end{cases} \tag{1.1}$$

where $f : [0,+\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and λ is a positive parameter.

Boundary value problems with Dirichlet conditions are an area of the fast developing differential equations theory. Problems of this type arise in various fields of physics, biology, biotechnology etc. Thus, existence and multiplicity of solutions to boundary value problems on the half-line were studied by many authors. These results were obtained using upper and lower solution techniques, fixed point theory and topological degree theory, see for example, [6–9, 14] and [13, 16, 18, 19]. However, as far as we know, the study of solutions for second order boundary value problems, possibly singular on the infinite intervals via variational methods has received considerably less attention. For example, see [5] and [1, 2, 4, 12, 15]. We are interested here in the existence of solutions for a second order boundary value problem with the Sturm–Liouville operator

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on the half-line. We take as a model a Dirichlet problem. Our results are obtained by applying Ekeland's variational principle technique, and under sublinear condition of the nonlinearity $f \in C([0,+\infty)\times\mathbb{R};\mathbb{R})$, with a positive parameter, the weighted q has a constant sign on the whole half-line, with $q>0$ for $x \geq 0$. We give some new criteria to guarantee that the problem (1.1) has at least one classical solution. Moreover, we assume that the following conditions are satisfied:

(H1) there exist constants $a, b \in \mathbb{R}^+ \setminus \{0\}$ and $r \in (0,1)$ such that

$$\forall x \in \mathbb{R}^+ \forall u \in \mathbb{R} \, : \, |f(x,u)| \leq a|u|^r + b,$$

(H2) $p : [0,+\infty) \longrightarrow (0,+\infty)$ is continuously differentiable, $q : [0,+\infty) \longrightarrow (0,+\infty)$ with $\frac{1}{p}$, $q \in L^1(0,+\infty)$, and

$$M_1 = \left(\int_0^{+\infty} \Big(\int_x^{+\infty} \frac{ds}{p(s)} \Big) dx \right)^{\frac{1}{r+1}} < +\infty,$$

$$M_2 = \left(\int_0^{+\infty} q(x) \Big(\int_x^{+\infty} \frac{ds}{p(s)} \Big)^{\frac{r+1}{2}} dx \right)^{\frac{1}{r+1}} < +\infty,$$

(H3) $f(x,0)=0$ and $\lim\limits_{u\to 0^+} \frac{f(x,u)}{u^r} = +\infty$, uniformly for $x \in [0,+\infty)$.

Let the space $H^1_{0,p}(0,+\infty)$ defined by

$$H^1_{0,p}(0,+\infty) = \Big\{ u \in AC([0,+\infty),\mathbb{R}) \, : \, \sqrt{p}u' \in \, L^2(0,+\infty), \; u(0)=u(+\infty)=0 \Big\}$$

be endowed with its natural norm

$$\|u\|_p = \left(\int_0^{+\infty} u^2(x)dx + \int_0^{+\infty} p(x)u'^2(x)dx \right)^{\frac{1}{2}}.$$

$H^1_{0,p}(0,+\infty)$ is a Hilbert space with the following inner product

$$\forall u,v \in H^1_{0,p}(0,+\infty) \, : \, (u,v) = \int_0^{+\infty} u(x)v(x)dx + \int_0^{+\infty} p(x)u'(x)v'(x)dx.$$

We define the weighted Lebesgue space

$$L^{r+1}_q(0,+\infty) = \left\{ u : (0,+\infty) \to \mathbb{R} \,\middle|\, u \text{ is measurable and } \int_0^{+\infty} q(x)|u(x)|^{r+1}dx < +\infty \right\}$$

equipped with the usual norm

$$\|u\|_{r+1,q} = \left(\int_0^{+\infty} q(x)|u(x)|^{r+1} dx \right)^{\frac{1}{r+1}}.$$

Consider also the space

$$C_0[0,+\infty) = \left\{ u \in C([0,+\infty),\mathbb{R}) \colon \lim_{x\to+\infty} u(x) = 0 \right\}$$

endowed with the norm

$$\|u\|_\infty = \sup_{x\in[0,+\infty)} |u(x)|.$$

In our considerations we shall need the following corollary and lemmas.

Lemma 1.1 ([5])**.** *On* $H^1_{0,p}(0,+\infty)$ *the quantity* $\|u\| = \left(\int_0^{+\infty} p(x)u'^2(x)dx \right)^{\frac{1}{2}}$ *is a norm which is equivalent to the norm* $\|u\|_p$*, i.e.*

$$\forall u \in H^1_{0,p}(0,+\infty) \, : \, \|u\| \le \|u\|_p \le \sqrt{1+M_1}\|u\|.$$

Lemma 1.2 ([5])**.** $H^1_{0,p}(0,+\infty)$ *is a reflexive Banach space.*

Lemma 1.3 ([5])**.** $H^1_{0,p}(0,+\infty)$ *embeds continuously in* $C_0[0,+\infty)$ *with*

$$\|u\|_\infty \le \sqrt{\left\|\frac{1}{p}\right\|_{L^1}}\|u\|.$$

Corollary 1.4 ([5])**.** $H^1_{0,p}(0,+\infty)$ *is compactly embedded in* $C_0[0,+\infty)$*.*

Further we will also use the following lemmas and corollaries.

Lemma 1.5. $C_0[0,+\infty)$ *is continuously embedded in* $L^{r+1}_q(0,+\infty)$*.*

Proof. For any $u \in C_0[0,+\infty)$, we have

$$\|u\|^{r+1}_{r+1,q} = \int_0^{+\infty} q(x)|u(x)|^{r+1} dx \le \sup_{x\in[0,+\infty)} |u(x)|^{r+1} \int_0^{+\infty} q(x)dx.$$

Consequently, we obtain $\|u\|_{r+1,q} \le C\|u\|_\infty$, where $C = \|q\|^{\frac{1}{r+1}}_{L^1}$. □

Lemma 1.6. *$H^1_{0,p}(0,+\infty)$ embeds continuously in $L^{r+1}_q[0,+\infty)$.*

Proof. For all $u \in H^1_{0,p}(0,+\infty)$, we get

$$|u(x)|^{r+1} = \Big|u(+\infty) - u(x)\Big|^{r+1} = \left|\int_x^{+\infty} u'(s)ds\right|^{r+1} = \left|\int_x^{+\infty} \sqrt{p(s)}u'(s)\frac{1}{\sqrt{p(s)}}ds\right|^{r+1}$$
$$\leq \left(\int_x^{+\infty} p(s)u'^2(s)ds\right)^{\frac{r+1}{2}} \left(\int_x^{+\infty} \frac{1}{p(s)}ds\right)^{\frac{r+1}{2}}.$$

Therefore, we obtain

$$\int_0^{+\infty} q(x)|u(x)|^{r+1}dx \leq \left(\int_0^{+\infty} p(s)u'^2(s)ds\right)^{\frac{r+1}{2}} \left(\int_0^{+\infty} q(x)\Big(\int_x^{+\infty} \frac{1}{p(s)}ds\Big)^{\frac{r+1}{2}} dx\right).$$

Thus, we have

$$\|u\|_{r+1,q} \leq M_2\|u\|$$

which finishes the proof. □

Corollary 1.7. *$H^1_{0,p}(0,+\infty)$ is compactly embedded in $L^{r+1}_q(0,+\infty)$, namely*

$$H^1_{0,p} \hookrightarrow\hookrightarrow C_0 \hookrightarrow L^{r+1}_q.$$

We are now concerned in the principal eigenvalue λ_1 of the nonlinear problem

$$\begin{cases} -(p(x)u'(x))' = \lambda q(x)|u(x)|^r, & x \geq 0, \\ u(0) = u(+\infty) = 0, \end{cases} \tag{1.2}$$

namely

$$\lambda_1 = \inf\left\{\int_0^{+\infty} p(x)u'(x)^2dx : u \in H^1_{0,p} \setminus \{0\},\ \int_0^{+\infty} q(x)|u(x)|^{r+1}dx = 1\right\}.$$

Arguing as in [17, Proposition 3.2], one can easily establish the following lemma.

Lemma 1.8. *The value λ_1 is positive and is achieved for some positive function $\varphi_1 \in H^1_{0,p}(0,+\infty) \setminus \{0\}$.*

We need the following technique to prove our main result.

Theorem 1.9 (The weak Ekeland variational principle, [10])**.** *Let (E,d) be a complete metric space and let $J : E \to \mathbb{R}$ a functional that is lower semi-continuous, bounded from below. Then, for each $\varepsilon > 0$, there exists $u_\varepsilon \in E$ with*

$$J(u_\varepsilon) \leq \inf_E J + \varepsilon,$$

and whenever $w \in E$ with $w \neq u_\varepsilon$, then

$$J(u_\varepsilon) < J(w) + \varepsilon d(u_\varepsilon, w).$$

2. MAIN RESULT

To get the existence of solutions of problem (1.1), we consider the energy functional $J : H^1_{0,p}(0,+\infty) \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 - \lambda \int_0^{+\infty} q(x)F(x,u(x))dx, \quad u \in H^1_{0,p}(0,+\infty),$$

where F denotes the primitive of f with respect to its second variable, i.e. $F(x,u) = \int_0^u f(x,s)ds$.

Proposition 2.1. *Suppose that the conditions* (H1), (H2) *hold. Then the functional J is Fréchet differentiable on $H^1_{0,p}(0,+\infty)$. Moreover, one has*

$$\forall v \in H^1_{0,p}(0,+\infty) \,:\, \langle J'(u), v\rangle = \int_0^{+\infty} p(x)u'(x)v'(x)dx - \lambda \int_0^{+\infty} q(x)f(x,u(x))v(x)dx.$$

Proof. First we show that J is Gâteaux-differentiable. Indeed, for all $v \in H^1_{0,p}(0,+\infty)$ and for each $t > 0$, we have

$$\begin{aligned} J(u+tv) - J(u) &= \frac{1}{2}\int_0^{+\infty} p(x)((u'+tv')(x))^2dx - \lambda \int_0^{+\infty} q(x)F(x,(u+tv)(x))dx \\ &\quad - \frac{1}{2}\int_0^{+\infty} p(x)u'^2(x)dx + \lambda\int_0^{+\infty} q(x)F(x,u(x))dx \\ &= \frac{t^2}{2}\int_0^{+\infty} p(x)v'^2(x)dx + t\int_0^{+\infty} p(x)u'(x)v'(x)dx \\ &\quad - \lambda\int_0^{+\infty} q(x)\Big[F(x,(u+tv)(x)) - F(x,u(x))\Big]dx \\ &= \frac{t^2}{2}\int_0^{+\infty} p(x)v'^2(x)dx + t\int_0^{+\infty} p(x)u'(x)v'(x)dx \\ &\quad - t\lambda\int_0^{+\infty} q(x)f(x,(u+t\theta v)(x))v(x)dx, \end{aligned}$$

where $0 < \theta < 1$.

Then

$$\frac{1}{t}[J(u+tv)-J(u)] = \frac{t}{2}\int_0^{+\infty} p(x)v'^2(x)dx + \int_0^{+\infty} p(x)u'(x)v'(x)dx - \lambda\int_0^{+\infty} q(x)f(x,(u+t\theta v)(x))v(x)dx.$$

Let $t \to 0$, then, by using the Lebesgue dominated convergence theorem and (H1), we have

$$\forall v \in H^1_{0,p}(0,+\infty) \,:\, \langle J'(u), v\rangle = \int_0^{+\infty} p(x)u'(x)v'(x)dx - \lambda\int_0^{+\infty} q(x)f(x,u(x))v(x)dx.$$

Next, we show that J' is continuous. Indeed, let $(u_n) \subset H^1_{0,p}(0,+\infty)$ with $u_n \to u$ when $n \to +\infty$. Then there exists $R > 0$ such that $\|u_n\| \le R$ for all $n \in \mathbb{N}$. Furthermore, from (H1), (H2) and Lemma 1.3 we derive that

$$\begin{aligned} q(x)|f(x,u_n(x))| &\le aq(x)|u_n(x)|^r + bq(x) \\ &\le a \sup_{x\in[0,+\infty)} |u_n(x)|^r q(x) + bq(x) \\ &= (a\|u_n\|^r_\infty + b)q(x) \\ &\le \left(a\left((R\sqrt{\|\frac{1}{p}\|_{L^1}}\right)^r + b\right)q(x) \in L^1(0,+\infty). \end{aligned}$$

Consequently, according to the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to+\infty}\int_0^{+\infty} q(x)f(x,u_n(x))dx = \int_0^{+\infty} q(x)f(x,u(x))dx.$$

So, we have

$$\begin{aligned} \langle J'(u_n)-J'(u), v\rangle &= \int_0^{+\infty} p(x)u_n'(x)v'(x)dx - \lambda\int_0^{+\infty} q(x)f(x,u_n(x))v(x)dx \\ &\quad - \int_0^{+\infty} p(x)u'(x)v'(x)dx + \lambda\int_0^{+\infty} q(x)f(x,u(x))v(x)dx \\ &= \int_0^{+\infty} p(x)(u_n'(x)-u'(x))v'(x)dx \\ &\quad - \lambda\int_0^{+\infty} q(x)(f(x,u_n(x))-f(x,u(x)))v(x)dx. \end{aligned}$$

Passing to the limit in $\langle J'(u_n) - J'(u), v\rangle$, when $n \to +\infty$, and taking into account that f is continuous, we obtain $J'(u_n) \to J'(u)$ as $n \to +\infty$. □

Definition 2.2. We say that $u \in H^1_{0,p}(0,+\infty)$ is a weak solution of problem (1.1) if for every $v \in H^1_{0,p}(0,+\infty)$ we have

$$\langle J'(u), v\rangle = \int_0^{+\infty} p(x)u'(x)v'(x)dx - \lambda \int_0^{+\infty} q(x)f(x,u(x))v(x)dx = 0.$$

Remark 2.3. Since the nonlinear term f is continuous, then a weak solution of problem (1.1) is a classical solution.

Our main result is as follows.

Theorem 2.4. *Assume* (H1)–(H3) *hold. Then there exists* $\overline{\lambda} > 0$ *such that problem* (1.1) *has at least one nontrivial solution* u_λ *for each* $\lambda \in (0, \overline{\lambda})$.

Proof. From (H1) there exists $\delta_1 > 0$ such that

$$|F(x,u)| \le a\frac{1}{r+1}|u|^{r+1} + b|u| \le K|u|^{r+1} \quad \text{for all } \ |u| > \delta_1, \text{ for some } K > 0.$$

Also, from $(H1)$, there exists $M_3 > 0$ such that

$$|F(x,u)| \le M_3 \quad \text{for all } \ u \in [-\delta_1, \delta_1] \quad \text{and all } x \in (0,+\infty).$$

Therefore,

$$|F(x,u)| \le M_3 + K|u|^{r+1} \quad \text{for all } \ u \in \mathbb{R} \ \text{ and all } \ x \in [0,+\infty). \tag{2.1}$$

From (2.1) and (H2) and by using the continuous embedding of $H^1_{0,p}(0,+\infty)$ in $L^{r+1}_q(0,+\infty)$ (i.e. note that $\|u\|_{r+1,q} \le M_2\|u\|$) we get

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_0^{+\infty} q(x)F(x,u(x))dx \\
&\ge \frac{1}{2}\|u\|^2 - \lambda \int_0^{+\infty} q(x)\Big(M_3 + K|u|^{r+1}(x)\Big)dx \\
&\ge \frac{1}{2}\|u\|^2 - \lambda M_3 \int_0^{+\infty} q(x)dx - \lambda K \int_0^{+\infty} q(x)|u|^{r+1}(x)dx \\
&= \frac{1}{2}\|u\|^2 - \lambda M_3 \int_0^{+\infty} q(x)dx - \lambda K\|u\|^{r+1}_{r+1,q} \\
&\ge \frac{1}{2}\|u\|^2 - \lambda K M_2^{r+1}\|u\|^{r+1} - \lambda M_3\|q\|_{L^1}.
\end{aligned}$$

Fix large enough $R > 0$. Then for $u \in H^1_{0,p}(0,+\infty)$ such that $\|u\| \geq R$ we have

$$J(u) \geq \frac{1}{2}\|u\|^2 - \lambda\Big(KM_2^{r+1}\|u\|^{r+1} + M_3\|q\|_{L^1}\Big).$$

Note that if $\lambda < \overline{\lambda} := \frac{R^2}{2(KM_2^{r+1}R^{r+1}+M_3\|q\|_{L^1})}$, this gives the existence of a real number $\rho > R$ satisfying

$$J(u) > 0 \quad \text{if } \|u\| = \rho, \quad \text{and} \quad \inf_{u\in\partial\overline{B}_\rho(0)} J(u) > 0, \tag{2.2}$$

and

$$J(u) \geq -C \ \text{ if } \ \|u\| \leq \rho, \ \text{for some } \ C > 0.$$

Since J is a Fréchet differentiable functional, hence it is lower semicontinuous and bounded from below on $\overline{B}_\rho(0)$. We claim that

$$\inf_{u\in\overline{B}_\rho(0)} J(u) < 0. \tag{2.3}$$

Indeed, let $\varphi_1 \in H^1_{0,p}(0,+\infty)$ be the eigenfunction corresponding to the principal eigenvalue of problem (1.2) as described in Lemma 1.8. Then for any fixed λ in $\big(0,\overline{\lambda}\big)$, by (H3), for any D,

$$D > \frac{t^{1-r}\|\varphi_1\|^2}{2\lambda\|\varphi_1\|^{r+1}_{r+1,q}}, \tag{2.4}$$

there exists $0 < \epsilon_D < 1$ such that

$$f(x,u) \geq Du^r \quad \text{for} \quad 0 < u < \epsilon_D.$$

Note that since $0 < u < 1$, therefore

$$f(x,u) \geq Du^r \quad \text{implies} \quad F(x,u) \geq Du^{r+1}. \tag{2.5}$$

Since the function φ_1 is continuous on $[0,+\infty)$ and $\varphi_1(0) = \varphi_1(+\infty) = 0$, there exists $\widehat{c} > 0$ such that $\sup_{x\in[0,+\infty)} \varphi_1(x) \leq \widehat{c}$. Thus, for every $0 < t < 1/\widehat{c}$ (t near 0), owing to (2.4), (2.5) and Lemma 1.8 we have

$$\begin{aligned} J(t\varphi_1) &= \frac{t^2}{2}\|\varphi_1\|^2 - \lambda\int_0^{+\infty} q(x)F(x,t\varphi_1(x))dx \\ &\leq \frac{t^2}{2}\|\varphi_1\|^2 - \lambda Dt^{r+1}\int_0^{+\infty} q(x)\varphi_1^{r+1}(x)dx \\ &= \frac{t^2}{2}\|\varphi_1\|^2 - \lambda Dt^{r+1}\|\varphi_1\|^{r+1}_{r+1,q} < 0. \end{aligned}$$

So (2.3) is proved.

From (2.2) together with (2.3) we obtain

$$\inf_{u\in\overline{B}_\rho(0)} J(u) < 0 < \inf_{u\in\partial\overline{B}_\rho(0)} J(u).$$

We define a distance on $\overline{B}_\rho(0)$ as follows:

$$d(u,v) = \|u - v\| \quad \text{for } u, v \in \overline{B}_\rho(0).$$

Clearly, $\overline{B}_\rho(0)$ is a complete metric space. It is known that $J \in C^1(\overline{B}_\rho(0), \mathbb{R})$, therefore J is lower semicontinuous and bounded from below on $\overline{B}_\rho(0)$. We put

$$c_\lambda = \inf\{J(u) : u \in \overline{B}_\rho(0)\}.$$

By the Ekeland variational principle (see Theorem 1.9) in $\overline{B}_\rho(0)$, there is a minimizing sequence $(u_n) \subset \overline{B}_\rho(0)$ for all $n \in \mathbb{N} \setminus \{0\}$ such that

$$c_\lambda < J(u_n) \le \inf_{u\in\overline{B}_\rho(0)} J(u) + \frac{1}{n} \le c_\lambda + \frac{1}{n}, \tag{2.6}$$

$$\forall w \in \overline{B}_\rho(0) \,:\, J(u_n) \le J(w) + \frac{1}{n}\|w - u_n\|. \tag{2.7}$$

If we put $w = u_n + th$ in (2.7) for $t > 0$, $h \in H^1_{0,p}(0, +\infty)$ and $n \in \mathbb{N} \setminus \{0\}$, then we get $J(u_n) \le J(u_n + th) + \frac{1}{n}t\|h\|$. Thus, we have

$$\frac{1}{t}[J(u_n) - J(u_n + th)] < \frac{1}{n}\|h\|,$$

and taking into account that J is a Fréchet differentiable functional we see that

$$-\langle J'(u_n), h\rangle \le \frac{1}{n}\|h\| \ \text{ for all } n \in \mathbb{N} \setminus \{0\},$$

Similarly, if we put $w = u_n - th$, then we get

$$\langle J'(u_n), h\rangle \le \frac{1}{n}\|h\| \ \text{ for all } n \in \mathbb{N} \setminus \{0\}.$$

So

$$\sup_{\|h\|\le 1} |\langle J'(u_n), h\rangle| \le \frac{1}{n} \ \text{ for all } n \in \mathbb{N} \setminus \{0\}.$$

Therefore, we have

$$\|J'(u_n)\| \to 0 \ \text{ and } \ J(u_n) \to c_\lambda \ \text{ as } \ n \to +\infty,$$

where c_λ stands for the infimum of $J(u)$ on $\overline{B}_\rho(0)$. From the above discussion, we know that (u_n) is a bounded $(P.S)_{c_\lambda}$ sequence (see [11, Definition 2.3]), and $\overline{B}_\rho(0)$ is a closed convex set, and from Corollary 1.7, then there exist $u_\lambda \in \overline{B}_\rho(0) \subset H^1_{0,p}(0,+\infty)$ and a convergent subsequence still denoted by (u_n), such that

$$\begin{cases} u_n \to u_\lambda, & \text{weakly in } H^1_{0,p}(0,+\infty), \\ u_n(x) \to u_\lambda(x), & \text{for a.e. in } (0,+\infty), \\ u_n \to u_\lambda, & \text{strongly in } L^{r+1}_q(0,+\infty). \end{cases}$$

Consequently, passing to the limit in $\langle J'(u_n), v\rangle$ as $n \to +\infty$, we deduce that

$$\int_0^{+\infty} p(x)u'_\lambda(x)v'(x)dx - \lambda \int_0^{+\infty} q(x)f(x,u_\lambda(x))v(x)dx = 0,$$

for all $v \in H^1_{0,p}(0,+\infty)$, namely $\langle J'(u_\lambda), v\rangle = 0$ for all $v \in H^1_{0,p}(0,+\infty)$.

Now it remains to show that $J(u_\lambda) = c_\lambda$. Indeed, by (2.1) and Lemma 1.3, for all $x \in [0,+\infty)$ and $n \in \mathbb{N} \setminus \{0\}$, we obtain

$$\begin{aligned} q(x)|F(x,u_n(x))| &\le M_3 q(x) + Kq(x)|u_n(x)|^{r+1} \\ &\le M_3 q(x) + Kq(x) \sup_{x\in[0,+\infty)} |u_n(x)|^{r+1} \\ &\le M_3 q(x) + Kq(x)\|u_n\|_\infty^{r+1} \\ &\le \left(M_3 + K\left(\rho\sqrt{\|\frac{1}{p}\|_{L^1}}\right)^{r+1}\right) q(x) \in L^1(0,+\infty). \end{aligned}$$

Thus, by the Lebesgue dominated convergence theorem and (2.6), we conclude that

$$c_\lambda \le \lim_{n\to+\infty} J(u_n) = J(u_\lambda) \le c_\lambda.$$

Hence, c_λ is a critical value of the functional J at the point u_λ in $H^1_{0,p}(0,+\infty)$. □

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