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THE OPERATIONAL CALCULUS MODEL FOR THE n^{TH}-ORDER BACKWARD DIFFERENCE

ABSTRACT

In this paper, there has been constructed such a model of the non-classical Bittner operational calculus, in which the derivative is understood as the backward difference $\nabla_n\{x(k)\} := \{x(k) - x(k-n)\}$. Next, the presented model has been generalized by considering the operation $\nabla_{n,b}\{x(k)\} := \{x(k) - b\,x(k-n)\}$, where $b \in \mathbb{C} \setminus \{0\}$.

Key words:
operational calculus, derivative, integrals, limit conditions, backward difference.

THE FOUNDATIONS OF THE NON-CLASSICAL BITTNER OPERATIONAL CALCULUS

The *Bittner operational calculus* [2–5] is a system

$$CO(L^0, L^1, S, T_q, s_q, Q), \tag{1}$$

in which L^0 and L^1 are linear spaces (over the same scalar field Γ) such that $L^1 \subset L^0$. The linear operation $S : L^1 \longrightarrow L^0$ (denoted as $S \in \mathscr{L}(L^1, L^0)$), called the (abstract) *derivative*, is a surjection. What is more, Q is a set of indices q for the operations $T_q \in \mathscr{L}(L^0, L^1)$ and $s_q \in \mathscr{L}(L^1, L^1)$ such that $S\,T_q f = f$, $f \in L^0$ and $s_q x = x - T_q S\,x$, $x \in L^1$. These operations are called *integrals* and *limit conditions*, respectively. The kernel of S, i.e. $\operatorname{Ker} S$ is a set of elements understood as *constants* for the derivative S. The limit conditions $s_q, q \in Q$ are projections of L^1 on the subspace $\operatorname{Ker} S$.

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If we define the objects (1), then we have in mind the *representation* or the *model* of the operational calculus.

An example of the operational calculus (1) is a *discrete* model, in which the derivative is the *backward difference* $\nabla\{x(k)\} := \{x(k) - x(k-1)\}$.

THE BACKWARD DIFFERENCE MODEL

Let $\mathbb{Z}$ and $\mathbb{C}$ mean the set of integers and the set of complexes, respectively. Moreover, let $C(\mathbb{Z}, \mathbb{C})$ be a linear space of two-sided complex sequences $x = \{x(k)\}_{k\in\mathbb{Z}}$ with usual sequences addition and sequences multiplication by complexes.

In [7], the author of the present paper has proven that the system (1), in which $L^0 = L^1 := C(\mathbb{Z}, \mathbb{C})$ and

$$S\,x \equiv \nabla x := \{x(k) - x(k-1)\}, \tag{2}$$

$$T_{k_0}x := \begin{cases} -\sum\limits_{i=k+1}^{k_0} x(i) & \text{for} \quad k < k_0 \\ \quad 0 & \text{for} \quad k = k_0 \\ \sum\limits_{i=k_0+1}^{k} x(i) & \text{for} \quad k > k_0 \end{cases}, \quad k \in \mathbb{Z}, \tag{3}$$

$$s_{k_0}x := \{x(k_0)\}, \tag{4}$$

where $x = \{x(k)\} \in L^0 = L^1$ and $x = \{x(k)\} \in L^0 = L^1$ and $k_0 \equiv q \in Q := \mathbb{Z}$[1], is an operational calculus model. In the same article, it has also been shown that to the so-called *backward difference on the basis* $b = \{b(k)\}$

$$S_b x := \{x(k) - b(k)x(k-1)\},$$

where $b(k) \neq 0$ for each $k \in \mathbb{Z}$ and $\{b(k)\}\{x(k)\} := \{b(k)\,x(k)\}$ means a usual multiplication of sequences b, x in the algebra L^0, there correspond the integrals

$$T_{b,k_0}x = \{e(k)\}T_{k_0}\left\{\frac{x(k)}{e(k)}\right\}$$

[1] Given the definition of integrals T_{k_0}, we assume that $\sum_{i=k_0+1}^{k_0} x(i) := 0$.

as well as limit conditions

$$s_{b,k_0}x = \{e(k)\}s_{k_0}\left\{\frac{x(k)}{e(k)}\right\},$$

where

$$e(k) = \begin{cases} \dfrac{1}{\prod_{i=k+1}^{0} b(i)} & \text{for} \quad k < 0 \\ 1 & \text{for} \quad k = 0 \\ \prod_{i=1}^{k} b(i) & \text{for} \quad k > 0 \end{cases}, \quad k \in \mathbb{Z}.$$

THE HIGHER-ORDER BACKWARD DIFFERENCE MODEL

A generalization of the operation $\nabla \equiv \nabla_1$ is the below n^{th}*-order backward difference*

$$\nabla_n x(k) := x(k) - x(k-n), \tag{5}$$

where n is a given natural number, i.e. $n \in \mathbb{N}$.

Taking (5) as a derivative S, we shall determine the integrals T_{k_0} as well as their corresponding limit conditions s_{k_0}. Firstly, let us notice that any constant c for the derivative (5) is an n*-periodic* sequence, i.e. $c(k+n) = c(k)$ for each $k \in \mathbb{Z}$ (cf. [1]), because this condition is equivalent to $c(k) - c(k-n) = 0, k \in \mathbb{Z}$. What is more, for any sequence $c \in \operatorname{Ker} \nabla_n$ there exist numbers $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$ such that

$$c = \{a_0\varepsilon_0^k + a_1\varepsilon_1^k + \cdots + a_{n-1}\varepsilon_{n-1}^k\}, \tag{6}$$

where

$$\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1} \tag{7}$$

are n^{th} roots of unity, i.e.

$$\varepsilon_j = \cos\frac{2j\pi}{n} + \mathrm{i}\,\sin\frac{2j\pi}{n}, \quad j \in \overline{0, n-1}^{2},$$

whereas 'i' means the imaginary unit.

[2] $\overline{0, n-1} := \{0, 1, \ldots, n-1\}$.

In what follows, we shall use the below properties of the sequence (7):

$$\varepsilon_j^{k\pm n} = \varepsilon_j^k, \quad j \in \overline{0, n-1}, k \in \mathbb{Z},$$
$$\varepsilon_0^m + \varepsilon_1^m + \ldots + \varepsilon_{n-1}^m = 0, \quad m \neq \ell n, \ell, m \in \mathbb{Z}, n \in \mathbb{N} \setminus \{1\}.$$

We shall prove the following.

Theorem. The system (1), where $x = \{x(k)\} \in L^0 = L^1 := C(\mathbb{Z}, \mathbb{C})$, $k_0 \equiv q \in Q := \mathbb{Z}$ and

$$S\,x := \{x(k) - x(k-n)\}, \tag{8}$$

$$T_{k_0}x := \begin{cases} -\dfrac{1}{n}\displaystyle\sum_{j=0}^{n-1}\sum_{i=k+1}^{k_0} \varepsilon_j^{k-i}\, x(i) & for \quad k < k_0 \\ \qquad\qquad 0 & for \quad k = k_0 \\ \dfrac{1}{n}\displaystyle\sum_{j=0}^{n-1}\sum_{i=k_0+1}^{k} \varepsilon_j^{k-i}\, x(i) & for \quad k > k_0 \end{cases}, \quad k \in \mathbb{Z}, \tag{9}$$

$$s_{k_0}x := \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0-n+1}^{k_0} \varepsilon_j^{k-i}\, x(i)\right\} \tag{10}$$

constitutes the discrete Bittner operational calculus model.

Proof. It is obvious that the operations (8)–(10) are linear. Let $\{y(k)\} := T_{k_0}\{x(k)\}$. Therefore, for $k = k_0$ we get

$$S\{y(k)\}|_{k=k_0} = \{y(k_0) - y(k_0-n)\} = \left\{0 + \frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0-n+1}^{k_0} \varepsilon_j^{k_0-n-i}\, x(i)\right\}$$
$$= \left\{\frac{1}{n}\sum_{i=k_0-n+1}^{k_0-1} [\varepsilon_0^{k_0-i} + \varepsilon_1^{k_0-i} + \cdots + \varepsilon_{n-1}^{k_0-i}]\, x(i) + x(k_0)\right\} = \{x(k)\}|_{k=k_0}. \tag{11}$$

For $k < k_0$, we have

$$S\{y(k)\} = \{y(k) - y(k-n)\}$$
$$= \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k-n+1}^{k_0} \varepsilon_j^{k-n-i}\, x(i) - \frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k+1}^{k_0} \varepsilon_j^{k-i}\, x(i)\right\}$$
$$= \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k-n+1}^{k} \varepsilon_j^{k-i}\, x(i)\right\}. \tag{12}$$

It is easy to notice that for $n = 1$ we get $S\{y(k)\} = \{x(k)\}$. This also holds for $n > 1$. Namely, by analogy with (11), we obtain

$$S\{y(k)\} = \frac{1}{n}\left\{ \sum_{i=k-n+1}^{k-1} [\varepsilon_0^{k-i} + \varepsilon_1^{k-i} + \cdots + \varepsilon_{n-1}^{k-i}]x(i) \right\} + \{x(k)\} = \{x(k)\}. \tag{13}$$

For $k > k_0$ and $k - n > k_0$, we have

$$S\{y(k)\} = \{y(k) - y(k-n)\}$$
$$= \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k_0+1}^{k} \varepsilon_j^{k-i} x(i) - \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k_0+1}^{k-n} \varepsilon_j^{k-n-i} x(i) \right\}$$
$$= \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k-n+1}^{k} \varepsilon_j^{k-i} x(i) \right\}.$$

Hence, similarly to (12) and (13), we infer that $S\{y(k)\} = \{x(k)\}$.

When, in turn, $k > k_0$ and $k - n = k_0$, then

$$S\{y(k)\} = \{y(k) - y(k-n)\} = \{y(k_0 + n) - y(k_0)\}$$
$$= \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k_0+1}^{k_0+n} \varepsilon_j^{k_0+n-i} x(i) - 0 \right\}.$$

By analogy with (11), we further get

$$S\{y(k)\} = \{x(k_0 + n)\}, \quad \text{that is} \quad S\{y(k)\} = \{x(k)\}.$$

For $k > k_0$ and $k - n < k_0$, we obtain

$$S\{y(k)\} = \{y(k) - y(k-n)\}$$
$$= \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k_0+1}^{k} \varepsilon_j^{k-i} x(i) + \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k-n+1}^{k_0} \varepsilon_j^{k-n-i} x(i) \right\}$$
$$= \left\{ \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=k-n+1}^{k} \varepsilon_j^{k-i} x(i) \right\}.$$

Hence, on the basis of (12) and (13), we deduce that $S\{y(k)\} = \{x(k)\}$.

Finally, we can state that the axiom $S\,T_{k_0}x = x$ is satisfied.

Let $\{f(k)\} := S\{x(k)\} = \{x(k) - x(k-n)\}$. Then, for $k < k_0$ we have

$$T_{k_0}S\{x(k)\} = T_{k_0}\{f(k)\} = \left\{-\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k+1}^{k_0}\varepsilon_j^{k-i}\,f(i)\right\}$$

$$= \left\{\frac{1}{n}\sum_{j=0}^{n-1}\Big[\sum_{i=k+1}^{k_0}\varepsilon_j^{k-i}\,x(i-n) - \sum_{i=k+1}^{k_0}\varepsilon_j^{k-i}\,x(i)\,\Big]\right\}$$

$$= \Bigg\{\frac{1}{n}\sum_{j=0}^{n-1}\Big[\sum_{i=k-n+1}^{k_0-n}\varepsilon_j^{k-i}\,x(i) + \sum_{i=k_0-n+1}^{k}\varepsilon_j^{k-i}\,x(i)$$

$$-\Big(\sum_{i=k_0-n+1}^{k}\varepsilon_j^{k-i}\,x(i) + \sum_{i=k+1}^{k_0}\varepsilon_j^{k-i}\,x(i)\,\Big)\Big]\Bigg\}$$

$$= \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k-n+1}^{k}\varepsilon_j^{k-i}\,x(i)\right\} - \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0-n+1}^{k_0}\varepsilon_j^{k-i}\,x(i)\right\}.$$

Proceeding by analogy with (12), we eventually get

$$T_{k_0}S\{x(k)\} = \{x(k)\} - \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0-n+1}^{k_0}\varepsilon_j^{k-i}\,x(i)\right\} = \{x(k)\} - s_{k_0}\{x(k)\}.$$

Similarly, if $k > k_0$, then

$$T_{k_0}S\{x(k)\} = T_{k_0}\{f(k)\} = \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0+1}^{k}\varepsilon_j^{k-i}\,f(i)\right\}$$

$$= \left\{\frac{1}{n}\sum_{j=0}^{n-1}\Big[\sum_{i=k_0+1}^{k}\varepsilon_j^{k-i}\,x(i) - \sum_{i=k_0+1}^{k}\varepsilon_j^{k-i}\,x(i-n)\,\Big]\right\}$$

$$= \Bigg\{\frac{1}{n}\sum_{j=0}^{n-1}\Big[\sum_{i=k-n+1}^{k_0}\varepsilon_j^{k-i}\,x(i) + \sum_{i=k_0+1}^{k}\varepsilon_j^{k-i}\,x(i)$$

$$-\Big(\sum_{i=k_0-n+1}^{k-n}\varepsilon_j^{k-i}\,x(i) + \sum_{i=k-n+1}^{k_0}\varepsilon_j^{k-i}\,x(i)\,\Big)\Big]\Bigg\}$$

$$= \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k-n+1}^{k}\varepsilon_j^{k-i}\,x(i)\right\} - \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0-n+1}^{k_0}\varepsilon_j^{k-i}\,x(i)\right\} = \{x(k)\} - s_{k_0}\{x(k)\}.$$

Thus, the axiom $T_{k_0}S\,x = x - s_{k_0}x$ is also fulfilled.

Let us observe that (2)–(4) constitute the particular case of the above model for $n = 1$.

Example. The limit condition (10) allows to present any n-periodic two-sided sequence $c = \{c(k)\}$ with a recurring cycle $(c_{-n+1}, c_{-n+2}, \ldots, c_0)$, i.e.

$$\begin{aligned} c &= \{(c_{-n+1}, c_{-n+2}, \ldots, c_0)\} \\ &:= \{\ldots, c_{-n+1}, c_{-n+2}, \ldots, c_0, c_{-n+1}, c_{-n+2}, \ldots, c_0, \ldots\} \end{aligned}$$

in the form of (6). For $c \in \operatorname{Ker}\nabla_n$ we have $s_{k_0}c = c$, thus for $k_0 = 0$ we get

$$c = \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=-n+1}^{0}\varepsilon_j^{k-i}\,c_i\right\}. \tag{14}$$

The following table presents general terms of sequences $\{c(k)\}$ for which

$$c_{-i} = n - 1 - i, \quad i \in \overline{0, n-1}.$$

They have been obtained basing on (14) and using the *Mathematica*® program for $n = 2, 3, \ldots, 10$, respectively.

n	cycle	$c(k)$
2	$(0, 1)$	$\frac{1}{2}(1 + (-1)^k)$
3	$(0, 1, 2)$	$1 + \cos\frac{2k\pi}{3} - \frac{1}{\sqrt{3}}\sin\frac{2k\pi}{3}$
4	$(0, 1, 2, 3)$	$\frac{1}{2}(3 + (-1)^k) + \cos\frac{k\pi}{2} - \sin\frac{k\pi}{2}$
5	$(0, 1, 2, 3, 4)$	$\frac{2}{5}\Big(5 + 4\cos\frac{2k\pi}{5} + 4\cos\frac{4k\pi}{5} + 3\cos\frac{2(k+1)\pi}{5}$ $+3\cos\frac{4(k+1)\pi}{5} + 2\cos\frac{2(k+2)\pi}{5} + 2\cos\frac{4(k+2)\pi}{5}$ $+\cos\frac{2(k+3)\pi}{5} + \cos\frac{4(k+3)\pi}{5}\Big)$
6	$(0, 1, 2, 3, 4, 5)$	$\frac{1}{2}(5 + (-1)^k) + \cos\frac{k\pi}{3} + \cos\frac{2k\pi}{3}$ $-\sqrt{3}\sin\frac{k\pi}{3} - \frac{1}{\sqrt{3}}\sin\frac{2k\pi}{3}$

n	cycle	$c(k)$
7	(0, 1, 2, 3, 4, 5, 6)	$\frac{1}{7}\Big(21 + 12\cos\frac{2k\pi}{7} + 12\cos\frac{4k\pi}{7} + 12\cos\frac{6k\pi}{7} + 10\cos\frac{2(k+1)\pi}{7} + 10\cos\frac{4(k+1)\pi}{7} + 10\cos\frac{6(k+1)\pi}{7} + 8\cos\frac{2(k+2)\pi}{7} + 8\cos\frac{4(k+2)\pi}{7} + 8\cos\frac{6(k+2)\pi}{7} + 6\cos\frac{2(k+3)\pi}{7} + 6\cos\frac{4(k+3)\pi}{7} + 6\cos\frac{6(k+3)\pi}{7} + 4\cos\frac{2(k+4)\pi}{7} + 4\cos\frac{4(k+4)\pi}{7} + 4\cos\frac{6(k+4)\pi}{7} + 2\cos\frac{2(k+5)\pi}{7} + 2\cos\frac{4(k+5)\pi}{7} + 2\cos\frac{6(k+5)\pi}{7}\Big)$
8	(0, 1, 2, 3, 4, 5, 6, 7)	$\frac{1}{2}(7 + (-1)^k) + \cos\frac{k\pi}{4} + \cos\frac{k\pi}{2} + \cos\frac{3k\pi}{4} -(1 + \sqrt{2})\sin\frac{k\pi}{4} - \sin\frac{k\pi}{2} - (-1 + \sqrt{2})\sin\frac{3k\pi}{4}$
9	(0, 1, 2, 3, 4, 5, 6, 7, 8)	$\frac{1}{9}\Big(36 + 12\cos\frac{2(4k-1)\pi}{9} + 8\cos\frac{4(2k-1)\pi}{9} + 2\cos\frac{2(k-2)\pi}{9} + 9\cos\frac{2k\pi}{9} + 9\cos\frac{4k\pi}{9} + 9\cos\frac{2k\pi}{3} + 9\cos\frac{8k\pi}{9} + 14\cos\frac{2(k+1)\pi}{9} + 14\cos\frac{4(k+1)\pi}{9} + 12\cos\frac{2(k+2)\pi}{9} - 14\cos\frac{(8k-1)\pi}{9} - 6\cos\frac{(2k+1)\pi}{9} + 6\cos\frac{2(2k+1)\pi}{9} + 6\cos\frac{4(2k+1)\pi}{9} - 2\cos\frac{(4k+1)\pi}{9} + 2\cos\frac{2(4k+1)\pi}{9} - 3\sqrt{3}\sin\frac{2k\pi}{9} + 3\sqrt{3}\sin\frac{4k\pi}{9} - 3\sqrt{3}\sin\frac{2k\pi}{3} - 3\sqrt{3}\sin\frac{8k\pi}{9} - 8\sin\frac{(4k+7)\pi}{18} + 8\sin\frac{(8k+5)\pi}{18} - 12\sin\frac{(8k+7)\pi}{18}\Big)$
10	(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)	$\frac{1}{2}(9 + (-1)^k) - \cos\frac{(k-2)\pi}{5} + \cos\frac{2(k-1)\pi}{5} + \cos\frac{k\pi}{5} + \frac{13}{5}\cos\frac{2k\pi}{5} + \cos\frac{3k\pi}{5} + \frac{13}{5}\cos\frac{4k\pi}{5} + \cos\frac{(k+1)\pi}{5} + \frac{11}{5}\cos\frac{2(k+1)\pi}{5} + \cos\frac{(k+2)\pi}{5} + \cos\frac{(3k+2)\pi}{5} - \frac{7}{5}\cos\frac{(2k+1)\pi}{5} + \frac{7}{5}\cos\frac{2(2k+1)\pi}{5} - \cos\frac{(3k+1)\pi}{5} - \cos\frac{(4k+1)\pi}{5} - \sin\frac{(2k+3)\pi}{10} - \frac{9}{5}\sin\frac{(4k+3)\pi}{10} - \frac{11}{5}\sin\frac{(8k+3)\pi}{10} + \sin\frac{3(2k+1)\pi}{10} - \sin\frac{(6k+1)\pi}{10} + \frac{9}{5}\sin\frac{(8k+1)\pi}{10}$

It is not difficult to see that

$$c(k) = k + n - 1 \pmod{n}, \quad k \in \mathbb{Z}.$$

Hence,

$$a(k) := c(k - n + 1) = k \pmod n, \quad k \in \mathbb{Z}. \tag{15}$$

The sequences $\{a(k)\}$ (for $n \in \overline{2, 10}$ and $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) are included in *The On-Line Encyclopedia of Integer Sequences* OEIS® [3].

The below table contains the general terms of $a(k), k \in \mathbb{Z}$ obtained on the basis of (15), again by using *Mathematica*®.

n	OEIS® ID No.	$a(k) = k \pmod n$
2	A000035	$\frac{1}{2}(1 - (-1)^k)$
3	A010872	$1 - \cos\frac{2k\pi}{3} - \frac{1}{\sqrt{3}}\sin\frac{2k\pi}{3}$
4	A010873	$\frac{1}{2}(3 - (-1)^k) - \cos\frac{k\pi}{2} - \sin\frac{k\pi}{2}$
5	A010874	$\frac{2}{5}\Big(5 + 4\cos\frac{4(k-4)\pi}{5} + 3\cos\frac{2(k-3)\pi}{5} + 3\cos\frac{4(k-3)\pi}{5} + 2\cos\frac{2(k-2)\pi}{5} + 2\cos\frac{4(k-2)\pi}{5} + \cos\frac{2(k-1)\pi}{5} + \cos\frac{4(k-1)\pi}{5} + 4\cos\frac{2(k+1)\pi}{5}\Big)$
6	A010875	$\frac{1}{2}(5 - (-1)^k) - \cos\frac{k\pi}{3} - \cos\frac{2k\pi}{3} - \sqrt{3}\sin\frac{k\pi}{3} - \frac{1}{\sqrt{3}}\sin\frac{2k\pi}{3}$
7	A010876	$\frac{1}{7}\Big(21 + 12\cos\frac{4(k-6)\pi}{7} + 12\cos\frac{6(k-6)\pi}{7} + 12\cos\frac{2(k+1)\pi}{7} + 10\cos\frac{2(k-5)\pi}{7} + 10\cos\frac{4(k-5)\pi}{7} + 10\cos\frac{6(k-5)\pi}{7} + 8\cos\frac{2(k-4)\pi}{7} + 8\cos\frac{4(k-4)\pi}{7} + 8\cos\frac{6(k-4)\pi}{7} + 6\cos\frac{2(k-3)\pi}{7} + 6\cos\frac{4(k-3)\pi}{7} + 6\cos\frac{6(k-3)\pi}{7} + 4\cos\frac{2(k-2)\pi}{7} + 4\cos\frac{4(k-2)\pi}{7} + 4\cos\frac{6(k-2)\pi}{7} + 2\cos\frac{2(k-1)\pi}{7} + 2\cos\frac{4(k-1)\pi}{7} + 2\cos\frac{6(k-1)\pi}{7}\Big)$

[3] https://oeis.org.

n	OEIS® ID No.	$a(k) = k \pmod n$
8	A010877	$\frac{1}{2}(7-(-1)^k) - \cos\frac{k\pi}{4} - \cos\frac{k\pi}{2} - \cos\frac{3k\pi}{4}$ $-(1+\sqrt{2})\sin\frac{k\pi}{4} - \sin\frac{k\pi}{2} - (-1+\sqrt{2})\sin\frac{3k\pi}{4}$
9	A010878	$\frac{1}{9}\Big(36 - 3\sqrt{3}\sin\frac{2(k-8)\pi}{3} - 3\sqrt{3}\sin\frac{8(k-8)\pi}{9}$ $-3\sqrt{3}\sin\frac{2(k+1)\pi}{9} + 3\sqrt{3}\sin\frac{4(k+1)\pi}{9} + 9\cos\frac{2(k-8)\pi}{3}$ $+9\cos\frac{8(k-8)\pi}{9} + 14\cos\frac{4(k-7)\pi}{9} + 12\cos\frac{2(k-6)\pi}{9}$ $+2\cos\frac{2(k-1)\pi}{9} + 2\cos\frac{4(k-1)\pi}{9} + 9\cos\frac{2(k+1)\pi}{9}$ $+9\cos\frac{4(k+1)\pi}{9} + 14\cos\frac{2(k+2)\pi}{9} + 6\cos\frac{2(2k-15)\pi}{9}$ $-8\cos\frac{(2k+1)\pi}{9} + 8\cos\frac{2(2k+1)\pi}{9} + 8\cos\frac{4(2k+1)\pi}{9}$ $-6\cos\frac{(2k+3)\pi}{9} - 12\cos\frac{(4k+3)\pi}{9} - 2\cos\frac{(8k+1)\pi}{9}$ $-6\cos\frac{(8k+3)\pi}{9} + 14\cos\frac{2(4k-1)\pi}{9} - 12\cos\frac{(8k-3)\pi}{9}\Big)$
10	A010879	$\frac{1}{2}(9-(-1)^k) + \sin\frac{3(2k+1)\pi}{10} + \frac{11}{5}\sin\frac{(8k+1)\pi}{10}$ $-\cos\frac{k\pi}{5} + \cos\frac{2k\pi}{5} - \cos\frac{3k\pi}{5} + \cos\frac{4k\pi}{5} - \cos\frac{(k-11)\pi}{5}$ $+\cos\frac{3(k-9)\pi}{5} + \frac{13}{5}\cos\frac{4(k-9)\pi}{5} + \frac{11}{5}\cos\frac{2(k-8)\pi}{5}$ $+\cos\frac{(k-7)\pi}{5} + \frac{7}{5}\cos\frac{2(k-1)\pi}{5} + \cos\frac{(k+1)\pi}{5}$ $+\frac{13}{5}\cos\frac{2(k+1)\pi}{5} + \cos\frac{(k+2)\pi}{5} - \frac{9}{5}\cos\frac{(2k+1)\pi}{5}$ $+\frac{9}{5}\cos\frac{2(2k+1)\pi}{5} - \cos\frac{(3k+1)\pi}{5} + \cos\frac{(3k+2)\pi}{5}$ $-\frac{7}{5}\cos\frac{(4k+1)\pi}{5}$

A CERTAIN GENERALIZATION

The operation

$$S_b\{x(k)\} := \{x(k) - b\,x(k-n)\}, \tag{16}$$

where $\{x(k)\} \in L^0 = L^1 := C(\mathbb{Z}, \mathbb{C})$, $b \in \mathbb{C} \setminus \{0\}$, is a generalization of the derivative (8).

While constructing an operational calculus model corresponding to the derivative (16), we shall use the method of solving the equation $x(k+1) - b(k)\,x(k) = f(k)$, described in [6], as well as the following auxiliary theorems:

Lemma 1 (Th. 3 [5]). *An abstract differential equation*

$$S\,x = f, \quad f \in L^0, x \in L^1$$

with the limit condition

$$s_q x = x_{0,q}, \quad x_{0,q} \in \operatorname{Ker} S$$

has exactly one solution

$$x = x_{0,q} + T_q f. \tag{17}$$

Lemma 2 (Th. 4 [5]). *With a given derivative $S \in \mathscr{L}(L^1, L^0)$, the projection $s_q \in \mathscr{L}(L^1, \operatorname{Ker} S)$ determines the integral $T_q \in \mathscr{L}(L^0, L^1)$ from the condition*

$$x = T_q f \quad \textit{if and only if} \quad S\,x = f, s_q x = 0.$$

Moreover, s_q is a limit condition corresponding to the integral T_q.

One of the elements of the space $\operatorname{Ker} S_b$ is the sequence

$$e(k) := b^{\frac{k}{n}}, \quad k \in \mathbb{Z}.$$

Then

$$e(k) = b\,e(k-n), \quad k \in \mathbb{Z}.$$

Let us consider the below difference equation

$$S_b\{x(k)\} = \{f(k)\},$$

i.e.

$$x(k) - b\,x(k-n) = f(k), \quad k \in \mathbb{Z}. \tag{18}$$

Hence we obtain

$$\frac{x(k)}{e(k)} - \frac{x(k-n)}{e(k-n)} = \frac{f(k)}{e(k)}, \quad k \in \mathbb{Z},$$

so

$$y(k) - y(k-n) = g(k), \quad k \in \mathbb{Z}, \tag{19}$$

where

$$y(k) := \frac{x(k)}{e(k)}, \quad g(k) := \frac{f(k)}{e(k)}, \quad k \in \mathbb{Z}. \tag{20}$$

The equation (19) can be presented as

$$S\{y(k)\} = \{g(k)\}, \tag{21}$$

where $S \equiv \nabla_n$ is the operation (8).

From Lemma 1 it follows that the sequence

$$\{y(k)\} = s_{k_0}\{y(k)\} + T_{k_0}\{g(k)\},$$

where T_{k_0} and s_{k_0} are operations (9) and (10), constitutes the solution of the equation (21).

From (20) we get $x(k) = e(k)\,y(k), k \in \mathbb{Z}$. Eventually,

$$\{x(k)\} = \{e(k)\}s_{k_0}\left\{\frac{x(k)}{e(k)}\right\} + \{e(k)\}T_{k_0}\left\{\frac{f(k)}{e(k)}\right\} \tag{22}$$

is the solution of (18).

If we take

$$\{\widetilde{c}(k)\} := s_{k_0}\left\{\frac{x(k)}{e(k)}\right\},$$

then it follows that the sequence $\{\widetilde{c}(k)\} \in \operatorname{Ker} S$ is n-periodic, that is

$$\widetilde{c}(k) = \widetilde{c}(k-n), \quad k \in \mathbb{Z}.$$

Let

$$s_{b,k_0}\{x(k)\} := \{e(k)\}s_{k_0}\left\{\frac{x(k)}{e(k)}\right\}, \quad k_0 \in Q := \mathbb{Z}, \{x(k)\} \in L^1. \tag{23}$$

Thus, for each $k \in \mathbb{Z}$ we get

$$\begin{aligned} S_b s_{b,k_0} x(k) &= e(k)\,\widetilde{c}(k) - b\,e(k-n)\,\widetilde{c}(k-n) \\ &= e(k)(\widetilde{c}(k) - \widetilde{c}(k-n)) = e(k)\cdot 0 = 0, \end{aligned}$$

i.e. $s_{b,k_0} \in \mathscr{L}(L^1, \operatorname{Ker} S_b)$.

What is more, for each $k \in \mathbb{Z}$ holds the following

$$s_{b,k_0}^2 x(k) = s_{b,k_0}[e(k)\widetilde{c}(k)] = e(k)s_{k_0}\left[\frac{e(k)\widetilde{c}(k)}{e(k)}\right]$$
$$= e(k)s_{k_0}\widetilde{c}(k) = e(k)\widetilde{c}(k) = s_{b,k_0}x(k),$$

since $s_{k_0}\{\widetilde{c}(k)\} = \{\widetilde{c}(k)\}$. Finally, s_{b,k_0} is a projection of L^1 onto $\operatorname{Ker} S_b$ for each $k_0 \in \mathbb{Z}$. From Lemma 2 it follows that the projection s_{b,k_0} determines the *integral* T_{b,k_0} from the formula (22).
Namely,

$$T_{b,k_0}\{f(k)\} := \{e(k)\}T_{k_0}\left\{\frac{f(k)}{e(k)}\right\}, \quad k_0 \in Q, \{f(k)\} \in L^0. \tag{24}$$

Moreover, s_{b,k_0} is the *limit condition* corresponding to the integral (24). Thus, we arrive at the

Corollary. *The system* (16), (23), (24) *forms the discrete model of the Bittner operational calculus*

$$CO(C(\mathbb{Z},\mathbb{C}), C(\mathbb{Z},\mathbb{C}), S_b, T_{b,k_0}, s_{b,k_0}, \mathbb{Z}).$$

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MODEL RACHUNKU OPERATORÓW DLA RÓŻNICY WSTECZNEJ RZĘDU *n*

STRESZCZENIE

W artykule skonstruowano model nieklasycznego rachunku operatorów Bittnera, w którym pochodna rozumiana jest jako różnica wsteczna $\nabla_n\{x(k)\} := \{x(k) - x(k-n)\}$. Następnie dokonano uogólnienia opracowanego modelu, rozważając operację $\nabla_{n,b}\{x(k)\}:= \{x(k) - b\,x(k-n)\}$, gdzie $b \in \mathbb{C} \setminus \{0\}$.

Słowa kluczowe:

rachunek operatorów, pochodna, pierwotne, warunki graniczne, różnica wsteczna.