dr n. tech. Andrzej Antoni CZAJKOWSKI[a,b], dr inż. Wojciech Kazimierz OLESZAK[b], mgr inż. Jerzy DYRDAŁ[c]

[a] Higher School of Technology and Economics in Szczecin, Faculty of Automotive Systems
Wyższa Szkoła Techniczno-Ekonomiczna w Szczecinie, Wydział Systemów Automotive

[b] Higher School of Humanities of Common Knowledge Society in Szczecin
Wyższa Szkoła Humanistyczna Towarzystwa Wiedzy Powszechnej w Szczecinie

[c] Gryf Logistics Centre in Szczecin / Centrum Logistyczne Gryf Sp. z o.o. w Szczecinie

ANALYTICAL AND NUMERICAL SOLVING OF LINEAR NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE FIRST-ORDER WITH CHANGEABLE COEFFICIENTS BY USING CONSTANT VARIATION METHOD AND APPLICATION OF *MATHEMATICA* PROGRAM

Abstract

Introduction and aim: The paper presents the analytical and numerical algorithm of solving linear non-homogeneous equations of the first order with changeable coefficients. The aim of the work is to show the algorithms for solving equations both analytically and numerically. The additional aim is to show numerical algorithms and graphical interpretation of solutions.
Material and methods: Some selected equations have been chosen from the subject literature. In the solutions the constant variation method has been presented.
Results: The paper presents the selected linear non-homogeneous equations of the first order with changeable coefficients containing exponential, logarithmic, trigonometric and cyclometric functions.
Conclusion: Taking into account the constant variation method it is possible to solve the first order linear non-homogeneous differential equations with changeable coefficients. Using the *Mathematica* program it is possible quickly get a solution and create its graphical interpretation.

Keywords: Ordinary differential equations, linear equations, homogeneous equations, equations of the first order, changeable coefficients, variation constant method, analytical solution, numerical solution, *Mathematica*.

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ROZWIĄZYWANIE ANALITYCZNO-NUMERYCZNE LINIOWYCH NIEJEDNORODNYCH RÓWNAŃ RÓŻNICZKOWYCH PIERWSZEGO RZĘDU O ZMIENNYCH WSPÓŁCZYNNIKACH PRZY UŻYCIU METODY WARIACJI STAŁEJ I ZASTOSOWANIU PROGRAMU MATHEMATICA

Streszczenie

Wstęp i cel: *W pracy pokazano algorytmy analityczny i numeryczny rozwiązywania równań różniczkowych liniowych niejednorodnych pierwszego rzędu o zmiennych współczynnikach. Celem pracy jest pokazanie algorytmu rozwiązywania równań zarówno sposobem analitycznym jak i numerycznym. Ponadto również dodatkowym celem jest pokazanie algorytmów numerycznych oraz interpretacji graficznej rozwiązań.*
Materiał i metody: *Wybrane równania zaczerpnięto z literatury przedmiotu. W rozwiązaniach równań zastosowano metodę wariacji stałej.*
Wyniki: *W pracy opracowano wybrane równania różniczkowe liniowe niejednorodne pierwszego rzędu o zmiennych współczynnikach zawierających funkcje wykładnicze, logarytmiczne, trygonometryczne i arcus.*
Wniosek: *Stosując metodę uzmienniania stałej jest możliwe rozwiązywanie równań różniczkowych liniowych niejednorodnych pierwszego rzędu o zmiennych współczynnikach. Wykorzystując program Mathematica można szybko uzyskać rozwiązanie oraz sporządzić jego interpretację graficzną.*

Słowa kluczowe: *Równania różniczkowe zwyczajne, równania liniowe, równania niejednorodne, równania pierwszego rzędu, zmienne współczynniki, metoda wariacji stałej, rozwiązanie analityczne, rozwiązanie numeryczne, Mathematica.*

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1. Theoretical introduction

Definition 1.

The differential non-homogeneous linear equation of the first order with changeable coefficients has the following form:

$$\frac{dy}{dx} + p(x)y = q(x) \tag{1}$$

where p(x) and q(x) are some continuous functions in a certain interval (a, b) [2], [4].

Definition 2.

The differential homogeneous linear equation of the first order with changeable coefficient has the following form:

$$\frac{dy}{dx} + p(x)y = 0 \tag{2}$$

where p(x) is a changeable function in a certain interval (a, b) [2], [4].

Theorem 1.

The general solution of the non-homogeneous differential equation (1) is the sum of the general solution of the homogeneous differential equation (2) and the particular solution of the non-homogeneous differential equation (1) [8], [10].

The general solution of the homogeneous equation (2), after variables separation, is obtained from the following equation [9], [11], [12]:

$$\frac{dy}{y} = -p(x)dx. \tag{3}$$

The equation (3) we integrate on both sides respectively of the variables y and x:

$$\int \frac{dy}{y} = -\int p(x)\,dx. \tag{4}$$

Hence, after integration

$$\ln|y| = -\int p(x)dx + \ln|C|, \tag{5}$$

$$\ln\left|\frac{y}{C}\right| = -\int p(x)\,dx. \tag{6}$$

Using the definition of a logarithm, we get:

$$\frac{y}{C} = \exp(-\int p(x)\,dx). \tag{7}$$

Thus, the general solution of the homogeneous equation (2) has the form:

$$y(x) \equiv y_1(x) = C \cdot \exp(-\int p(x)\,dx) \tag{8}$$

where C is the real constant.

The particular solution of the non-homogeneous equation (1) is found by the constant variation method.

Therefore:

$$y(x) = C(x)\cdot \exp(-\int p(x)dx). \tag{9}$$

Both sides of the above equality we differentiate relative to x variable:

$$\frac{dy}{dx} = \frac{dC}{dx}\exp(-\int p(x)\,dx) - C(x)\cdot\exp(-\int p(x)\,dx)\cdot\frac{d}{dx}(-\int p(x)\,dx), \tag{10}$$

$$\frac{dy}{dx} = \frac{dC}{dx}\exp(-\int p(x)\,dx) - C(x)\cdot p(x)\cdot\exp(-\int p(x)\,dx). \tag{11}$$

After substituting functions (11) and (8) in equation (1) we get:

$$\frac{dC}{dx}\exp(-\int (x)\,dx) - C(x)\cdot p(x)\cdot\exp(-\int p(x)\,dx) + C(x)\cdot p(x)\cdot\exp(-\int p(x)\,dx) = g(x). \tag{12}$$

Hence

$$\frac{dC}{dx}\exp(-\int p(x)dx) = g(x), \tag{13}$$

$$\frac{dC}{dx} = g(x)\cdot\exp(\int p(x)dx). \tag{14}$$

After integration the equation (14) relative to x variable, we get:

$$C(x) = \int [g(x)\cdot\exp(\int p(x)dx)]\,dx. \tag{15}$$

Therefore, the particular solution of non-homogeneous equation (1) has the form:

$$y(x) \equiv y_2(x) = [\exp(-\int p(x)\,dx)]\cdot\int [g(x)\cdot\exp(\int p(x)dx)]\,dx. \tag{16}$$

Finally, the general solution of the non-homogeneous equation (1) has the following form:

$$y(x) \equiv y_1(x) + y_2(x) = C\cdot\exp(-\int p(x)\,dx) + [\exp(-\int p(x)\,dx)]\cdot\int [g(x)\cdot\exp(\int p(x)dx)]\,dx, \tag{17}$$

$$y(x) = [\exp(-\int p(x)\,dx)]\cdot\{C + \int [g(x)\cdot\exp(\int p(x)dx)]\,dx\} \tag{18}$$

where C is the real constant [13]-[16].

2. Analytical and numerical solving of the first order linear non-homogeneous differential equations with changeable coefficients by using the constant variation method

Example 1.

Let us consider the following equation [6]:

$$\frac{dy}{dx} - \frac{1}{x}y = x\cdot\cos(x). \tag{19}$$

The homogeneous equation has the form:

$$\frac{dy}{dx} - \frac{1}{x}y = 0. \tag{20}$$

• **Analytical solution**

The general solution of the equation (20) is obtained from the following equation:

$$\frac{dy}{dx} = \frac{y}{x}, \tag{21}$$

$$\frac{dy}{y} = \frac{dx}{x}. \tag{22}$$

The equation (22) we integrate on both sides respectively of the variables y and x:

$$\int \frac{dy}{y} = \int \frac{dx}{x}. \tag{23}$$

Hence, after integration we have:

$$\ln|y| = \ln|x| + \ln|C|, \tag{24}$$

$$\ln|y| = \ln|Cx|. \tag{25}$$

Thus, the general solution of the homogeneous equation (20) has the form:

$$y(x) \equiv y_1(x) = C \cdot x. \tag{26}$$

where C is the real constant. The particular solution of the non-homogeneous equation (19) is found by the constant variation method. Therefore:

$$y(x) = C(x) \cdot x. \tag{27}$$

Both sides of the above equality we differentiate relative to x variable:

$$\frac{dy}{dx} = \frac{dC}{dx} x + C(x). \tag{28}$$

After substituting functions (28) and (27) into equation (19) we get:

$$\frac{dC}{dx} x + C(x) - C(x) = x \cdot \cos(x), \tag{29}$$

$$\frac{dC}{dx} = \cos(x). \tag{30}$$

We integrate the equation (30) relative to x variable:

$$\int \frac{dC}{dx} dx = \int \cos(x) dx. \tag{31}$$

After integration relative to x variable we get:

$$C(x) = \sin(x). \tag{32}$$

Therefore, the particular solution of non-homogeneous equation (19) has the form:

$$y(x) \equiv y_2(x) = x \cdot \sin(x). \tag{33}$$

Finally, the general solution of the non-homogeneous equation(19) has the form:

$$y(x) \equiv y_1(x) + y_2(x) = Cx + x \cdot \sin(x), \tag{34}$$

$$y(x) = [C + \sin(x)] \cdot x \tag{35}$$

where C is the real constant.

• Numerical solution

For numerical analysis we take into account the solution (35) where the constant C∈{0.5, 1, 1.5, 2} [1], [3], [5], [7].

<u>Program 1</u>. (Mathematica 7.0)

```
In[1]:= DSolve[y'[x] == (1/x)*y[x] + x*Cos[x], y, x] /.C[1] → {0.5, 1, 1.5, 2}]
        Plot[Evaluate[y[x] /. %], {x, –10, 10},
        Background → RGBColor[0.95,0,1],
        PlotStyle → {{RGBColor[1,0,0], Thickness[0.009]},
        {RGBColor[0,0,1], Thickness[0.009]}, {RGBColor[0,1,0], Thickness[0.009]},
        {RGBColor[1,0,1], Thickness[0.009]}},
        PlotRange → {–25, 25}, AxesOrigin → {0, 0},
        AxesStyle → Thickness[0.004], Axeslabel → {"x", "y"},
        GridLines → Automatic, TextStyle → {FontFamily → "Arial", FontSize → 12}]

Out[1] = {{y → Function[{x}, x{0.5, 1, 1.5, 2} + xSin[x]] }}
Out[2] =  = Graphics =
```

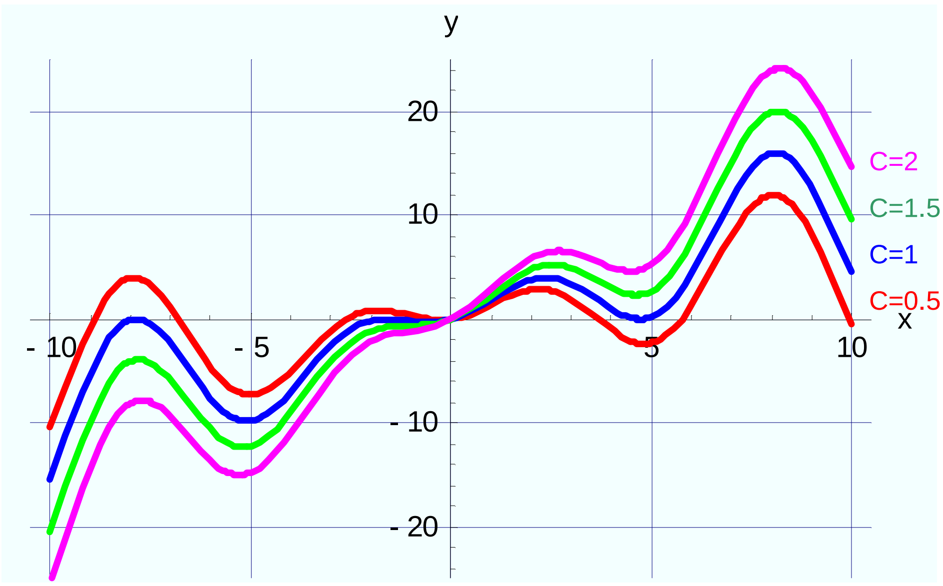

Fig. 1. The graphs of the function (35) as a solution of the first order linear non-homogeneous differential equation (19) for constant: C = 0.5, C = 1, C = 1.5, C = 2

Source: Program and graphs in Mathematica elaborated by the Authors

<u>Example 2</u>. Let us consider the following equation [6]:

$$\frac{dy}{dx}+\frac{1}{1+x^2}y=\frac{\mathrm{arctg}(x)}{1+x^2}. \quad (36)$$

The homogeneous equation has form:

$$\frac{dy}{dx}+\frac{1}{1+x^2}y=0. \quad (37)$$

• **Analytical solution**

The general solution of the equation (37) is obtained from the following equation:

$$\frac{dy}{dx} = -\frac{y}{1+x^2}, \tag{38}$$

$$\frac{dy}{y} = -\frac{dx}{1+x^2}. \tag{39}$$

The equation (39) we integrate on both sides respectively of the variables y and x:

$$\int \frac{dy}{y} = -\int \frac{dx}{1+x^2}. \tag{40}$$

Hence, after integration we have:

$$\ln|y| = -\mathrm{arctg}(x) + \ln|C|, \tag{41}$$

$$\ln\left|\frac{y}{C}\right| = -\mathrm{arctg}(x). \tag{42}$$

Using the definition of a logarithm, we get:

$$\frac{y}{C} = \exp[-\mathrm{arctg}(x)]. \tag{43}$$

Thus, the general solution of the homogeneous equation (37) has the form:

$$y \equiv y_1 = C \cdot \exp[-\mathrm{arctg}(x)] \tag{44}$$

where C is the real constant.

The particular solution of the non-homogeneous equation (31) is found by the constant variation method.

Therefore:

$$y(x) = C(x) \cdot \exp[-\mathrm{arctg}(x)]. \tag{45}$$

Both sides of the above equality we differentiate relative to x variable:

$$\frac{dy}{dx} = \frac{dC}{dx}\exp[-\mathrm{arctg}(x)] - C(x) \cdot \exp[-\mathrm{arctg}(x)] \cdot \frac{1}{1+x^2}. \tag{46}$$

After substituting functions (46) and (45) in equation (36) we get:

$$\frac{dC}{dx}\exp[-\mathrm{arctg}(x)] - C(x) \cdot \frac{\exp[-\mathrm{arctg}(x)]}{1+x^2} + C(x) \cdot \frac{\exp[-\mathrm{arctg}(x)]}{1+x^2} = \frac{\mathrm{arctg}(x)}{1+x^2}. \tag{47}$$

Hence

$$\frac{dC}{dx} = \frac{\mathrm{arctg}(x) \cdot \exp[\mathrm{arctg}(x)]}{1+x^2}. \tag{48}$$

We integrate the equation (48) and we get:

$$\int \frac{dC}{dx}\,dx = \int \frac{\mathrm{arctg}(x) \cdot \exp[\mathrm{arctg}(x)]}{1+x^2}\,dx. \tag{49}$$

We integrate (49) the right-hand side of the equation using integration by substitution:

$$\int \frac{\operatorname{arctg}(x)\cdot \exp[\operatorname{arctg}(x)]}{1+x^2}dx = \left\langle \begin{matrix} \operatorname{arctg}(x)=t \\ \dfrac{1}{1+x^2}=dt \end{matrix} \right\rangle = \int t\cdot \exp(t)\,dt. \tag{50}$$

We integrate (50) the right-hand side of the equation using integration by parts method:

$$\int t\cdot \exp(t)\,dt = \left\langle \begin{matrix} u=t \\ v'=\exp(t) \end{matrix} \middle| \begin{matrix} u'=1 \\ v=\exp(t) \end{matrix} \right\rangle = t\cdot \exp(t) - \int \exp(t)\,dt = t\cdot \exp(t) - \exp(t). \tag{51}$$

Hence, we get:

$$\int \frac{\operatorname{arctg}(x)\cdot \exp[\operatorname{arctg}(x)]}{1+x^2}dx = (t-1)\cdot \exp(t) = [\operatorname{arctg}(x)-1]\cdot \exp[\operatorname{arctg}(x)]. \tag{52}$$

After integration equation (49) relative to x variable and taking into account (52) we get:

$$C(x) = [\operatorname{arctg}(x)-1]\cdot \exp[\operatorname{arctg}(x)]. \tag{53}$$

Therefore, the particular solution of non-homogeneous equation (36) has the form:

$$y(x) \equiv y_2(x) = [\operatorname{arctg}(x)-1]\cdot \exp[\operatorname{arctg}(x)]\cdot \exp[-\operatorname{arctg}(x)], \tag{54}$$

$$y(x) \equiv y_2(x) = \operatorname{arctg}(x)-1. \tag{55}$$

The general solution of the non-homogeneous equation (36) has the following form:

$$y(x) \equiv y_1(x) + y_2(x) = C\cdot \exp[-\operatorname{arctg}(x)] + \operatorname{arctg}(x) - 1, \tag{56}$$

$$y(x) = C\cdot \exp[-\operatorname{arctg}(x)] + \operatorname{arctg}(x) - 1 \tag{57}$$

where C is the real constant.

• **Numerical solution**

For numerical analysis we take into account the solution (57) where the constant $C \in \{1, 2, 3, 4\}$ [1], [3], [5], [7].

<u>Program 2</u>. (Mathematica 7.0)

```
In[1]:= DSolve[y'[x] == (–1/(1 + x^2))*y[x] + (1/(1 + x^2))*ArcTan[x], y, x] /.C[1] → {1, 2, 3, 4}]
        Plot[Evaluate[y[x] /. %], {x, –4, 4},
        Background → RGBColor[0.95,0,1],
        PlotStyle → {{RGBColor[1,0,0], Thickness[0.009]},
        {RGBColor[0,0,1], Thickness[0.009]}, {RGBColor[0,1,0], Thickness[0.009]},
        {RGBColor[1,0,1], Thickness[0.009]}},
        PlotRange → {0, 14}, AxesOrigin → {0, 0},
        AxesStyle → Thickness[0.004], Axeslabel → {"x", "y"},
        GridLines → Automatic, TextStyle → {FontFamily → "Arial", FontSize → 12}]

Out[1] = {{y → Function[{x}, –1 + ArcTan[x] + e^(-ArcTan[x]){1, 2, 3, 4}] }}
Out[2] =  = Graphics =
```

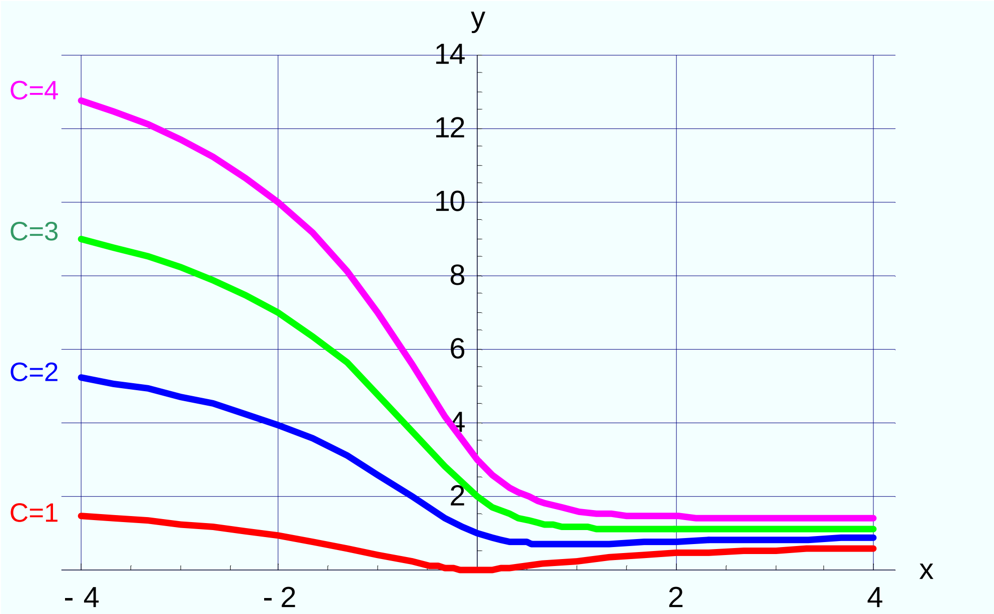

Fig. 2. The graphs of the function (57) as a solution of the first order linear non-homogeneous differential equation (36) for constant: C = 1, C = 2, C = 3, C = 4

Source: Program and graphs in Mathematica elaborated by the Authors

Example 3. Let us consider the following equation [6]:

$$\frac{dy}{dx} - \frac{1}{x\ln(x)} y = x\ln(x). \tag{58}$$

The homogeneous equation has form:

$$\frac{dy}{dx} - \frac{1}{x\ln(x)} y = 0. \tag{59}$$

- **Analytical solution**

The general solution of the equation (59) is obtained from the following equation:

$$\frac{dy}{dx} = \frac{y}{x\ln(x)}, \tag{60}$$

$$\frac{dy}{y} = \frac{dx}{x\ln(x)}. \tag{61}$$

The equation (61) we integrate on both sides respectively of the variables y and x:

$$\int \frac{dy}{y} = \int \frac{dx}{x\ln(x)}. \tag{62}$$

We integrate the right-hand side of the equation (62) using integration by substitution:

$$\int \frac{dx}{x\ln(x)} = \left\langle \begin{matrix} \ln(x) = t \\ \frac{dx}{x} = dt \end{matrix} \right\rangle = \int \frac{dt}{t} = \ln|t| = \ln|\ln(x)|. \tag{63}$$

Hence, after integration the equation (62), we have:

$$\ln|y| = \ln|\ln(x)| + \ln|C|, \tag{64}$$

$$\ln\left|\frac{y}{C}\right| = \ln|\ln(x)|. \tag{65}$$

Using the definition of a logarithm, we get:

$$\frac{y}{C} = \ln(x). \tag{66}$$

Thus, the general solution of the homogeneous equation (59) has the form:

$$y(x) \equiv y_1(x) = C \cdot \ln(x) \tag{67}$$

where C is the real constant.

The particular solution of the non-homogeneous equation (58) is found by the constant variation method. Therefore:

$$y(x) = C(x) \cdot \ln(x). \tag{68}$$

Both sides of the above equality we differentiate relative to x variable:

$$\frac{dy}{dx} = \frac{dC}{dx}\ln(x) - C(x)\frac{1}{x}. \tag{69}$$

After substituting functions (69) and (68) into equation (58) we get:

$$\frac{dC}{dx}\ln(x) - C(x)\frac{1}{x} + C(x)\frac{1}{x} = x \cdot \ln(x). \tag{70}$$

Hence

$$\frac{dC}{dx} = x. \tag{71}$$

We integrate the equation (71) and we get:

$$\int \frac{dC}{dx} dx = \int x\, dx. \tag{72}$$

After integration (72) we have:

$$C(x) = \frac{x^2}{2}. \tag{73}$$

Therefore, the particular solution of non-homogeneous equation (58) has the form:

$$y(x) \equiv y_2(x) = \frac{x^2}{2} \cdot \ln(x). \tag{74}$$

The general solution of the non-homogeneous equation (58) has the following form:

$$y(x) \equiv y_1(x) + y_2(x) = C \cdot \ln(x) + \frac{x^2}{2} \cdot \ln(x), \tag{75}$$

$$y(x) \equiv y_1(x) + y_2(x) = \left(C + \frac{x^2}{2}\right) \cdot \ln(x) \tag{76}$$

where C is the real constant.

• Numerical solution

For numerical analysis we take into account the solution (76) where the constant $C\in\{0.5, 1, 1.5, 2\}$ [1], [3], [5], [7].

Program 3. (Mathematica 7.0)

```
In[1]:= DSolve[y'[x] == (1/(x*Log[x]))*y[x] + x*Log[x], y, x] /.C[1] → {0.5, 1, 1.5, 2}]
        Plot[Evaluate[y[x] /. %], {x, 0, 3},
        Background → RGBColor[0.95,0,1],
        PlotStyle → {{RGBColor[1,0,0], Thickness[0.009]},
        {RGBColor[0,0,1], Thickness[0.009]}, {RGBColor[0,1,0], Thickness[0.009]},
        {RGBColor[1,0,1], Thickness[0.009]}},
        PlotRange → {–6, 10}, AxesOrigin → {0, 0},
        AxesStyle → Thickness[0.004], Axeslabel → {"x", "y"},
        GridLines → Automatic, TextStyle → {FontFamily → "Arial", FontSize → 12}]
Out[1] = {{y → Function[{x}, ½ x²Log[x] + {0.5, 1, 1.5, 2}Log[x]] }}
Out[2] =  = Graphics =
```

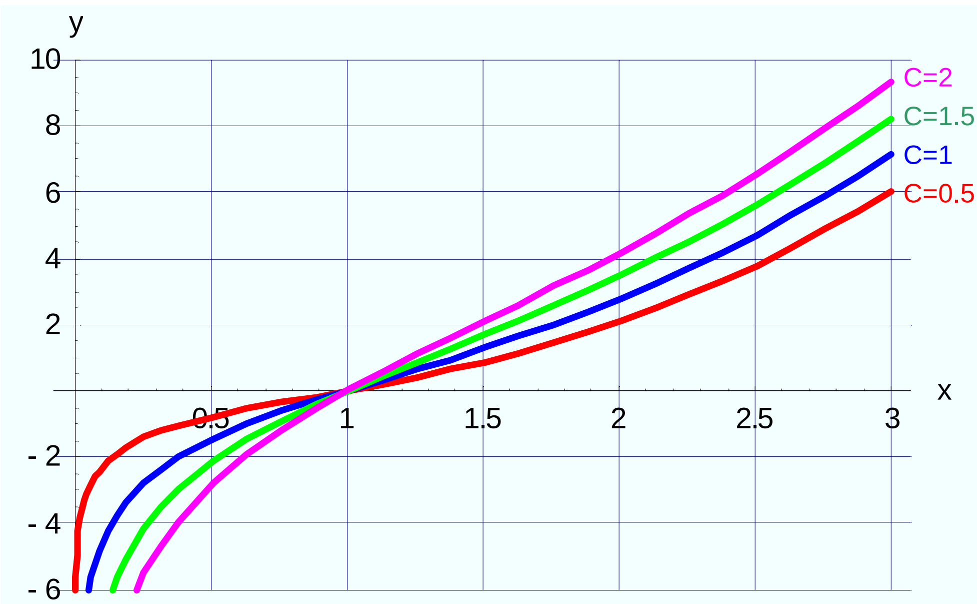

Fig. 3. The graphs of the function (76) as a solution of the first order linear non-homogeneous differential equation (58) for constant: C = 0.5, C = 1, C = 1.5, C = 2

Source: Program and graphs in Mathematica elaborated by the Authors

Example 4. Let us consider the following equation [6]:

$$\frac{dy}{dx}+2xy=x\cdot\exp(-x^2). \qquad (77)$$

The homogeneous equation has form:

$$\frac{dy}{dx}+2xy=0. \qquad (78)$$

• **Analytical solution**

The general solution of the equation (78) is obtained from the following equation:

$$\frac{dy}{dx} = -2xy, \tag{79}$$

$$\frac{dy}{y} = -2x\,dx. \tag{80}$$

The equation (80) we integrate on both sides respectively of the variables y and x:

$$\int \frac{dy}{y} = -2\int x\,dx. \tag{81}$$

Hence, after integration the equation (81), we have:

$$\ln|y| = -2\cdot\frac{x^2}{2} + \ln|C|, \tag{82}$$

$$\ln\left|\frac{y}{C}\right| = -x^2. \tag{83}$$

Using the definition of a logarithm, we get:

$$\frac{y}{C} = \exp(-x^2). \tag{84}$$

Thus, the general solution of the homogeneous equation (78) has the form:

$$y(x) \equiv y_1(x) = C\cdot\exp(-x^2) \tag{85}$$

where C is the real constant. The particular solution of the non-homogeneous equation (77) is found by the constant variation method. Therefore:

$$y(x) = C(x)\cdot\exp(-x^2). \tag{86}$$

Both sides of the above equality we differentiate relative to variable x:

$$\frac{dy}{dx} = \frac{dC}{dx}\exp(-x^2) - 2xC(x)\cdot\exp(-x^2), \tag{87}$$

After substituting functions (87) and (86) into equation (77) we get:

$$\frac{dC}{dx}\exp(-x^2) - 2xC(x)\cdot\exp(-x^2) + 2xC(x)\cdot\exp(-x^2) = x\cdot\exp(-x^2). \tag{88}$$

Hence

$$\frac{dC}{dx} = x. \tag{89}$$

We integrate the equation (89) and we get:

$$\int \frac{dC}{dx}dx = \int x\,dx. \tag{90}$$

After integration (90) we have:

$$C(x) = \frac{x^2}{2}. \tag{91}$$

Therefore, the particular solution of non-homogeneous equation (78) has the form:

$$y(x) \equiv y_2(x) = \frac{x^2}{2} \cdot \exp(-x^2). \tag{92}$$

The general solution of the non-homogeneous equation (77) has the following form:

$$y(x) \equiv y_1(x) + y_2(x) = C \cdot \exp(-x^2) + \frac{x^2}{2} \cdot \exp(-x^2), \tag{93}$$

$$y(x) = \left(\frac{x^2}{2} + C\right) \cdot \exp(-x^2) \tag{94}$$

where C is the real constant.

- **Numerical solution**

For numerical analysis we take into account the solution (94) where the constant $C \in \{1, 2, 3, 4\}$ [1], [3], [5], [7].

<u>Program 4</u>. (Mathematica 7.0)

```
In[1]:= DSolve[y'[x] == (–2)*x*y[x] + x*Exp[–x^2], y, x] /.C[1] → {1, 2, 3, 4}]
        Plot[Evaluate[y[x] /. %], {x, –3, 3},
        Background → RGBColor[0.95,0,1],
        PlotStyle → {{RGBColor[1,0,0], Thickness[0.009]},
        {RGBColor[0,0,1], Thickness[0.009]}, {RGBColor[0,1,0], Thickness[0.009]},
        {RGBColor[1,0,1], Thickness[0.009]}},
        PlotRange → {0, 4}, AxesOrigin → {0, 0},
        AxesStyle → Thickness[0.004], Axeslabel → {"x", "y"},
        GridLines → Automatic, TextStyle → {FontFamily → "Arial", FontSize → 12}]
Out[1] = {{y → Function[{x}, 1/2 Exp[–x^2] x^2 + Exp[–x^2] {1,2,3,4} }}
Out[2] =  = Graphics =
```

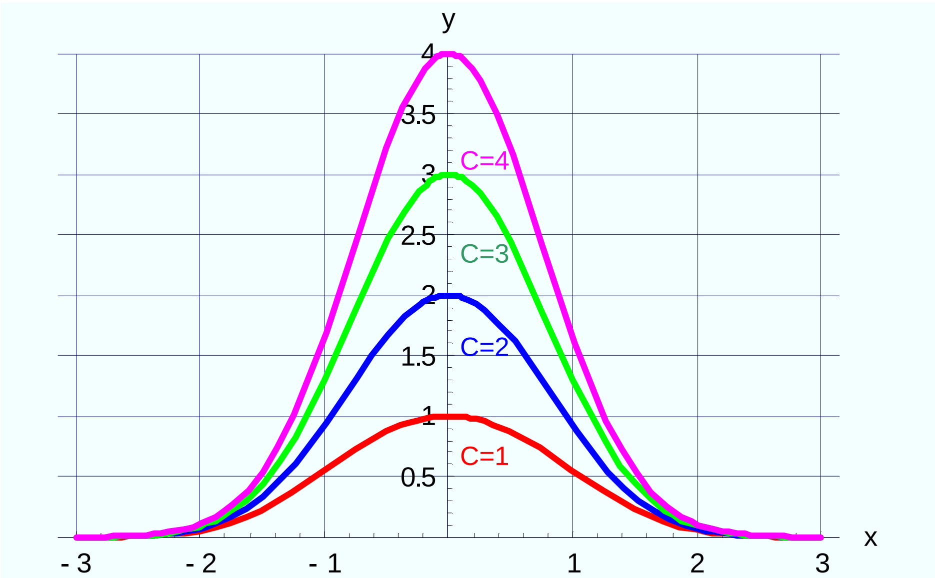

Fig. 4. The graphs of the function (94) as a solution of the first order linear non-homogeneous differential equation (77) for constant: C = 1, C = 2, C = 3, C = 4

Source: Program and graphs in Mathematica elaborated by the Authors

Example 5. Let us consider the following equation [6]:

$$\frac{dy}{dx}+\frac{1}{\sqrt{1-x^2}}y=\frac{\arcsin(x)}{\sqrt{1-x^2}}. \qquad (95)$$

The homogeneous equation has form:

$$\frac{dy}{dx}+\frac{1}{\sqrt{1-x^2}}y=0. \qquad (96)$$

• **Analytical solution**

The general solution of the equation (96) is obtained from the following equation:

$$\frac{dy}{dx}=-\frac{y}{\sqrt{1-x^2}}, \qquad (97)$$

$$\frac{dy}{y}=-\frac{dx}{\sqrt{1-x^2}}. \qquad (98)$$

The equation (98) we integrate on both sides respectively of the variables y and x:

$$\int\frac{dy}{y}=-\int\frac{dx}{\sqrt{1-x^2}}. \qquad (99)$$

Hence, after integration the equation (99), we have:

$$\ln|y|=-\arcsin(x)+\ln|C|, \qquad (100)$$

$$\ln\left|\frac{y}{C}\right|=-\arcsin(x). \qquad (101)$$

Using the definition of a logarithm, we get:

$$\frac{y}{C}=\exp[-\arcsin(x)]. \qquad (102)$$

Thus, the general solution of the homogeneous equation (96) has the form:

$$y(x)\equiv y_1(x)=C\cdot\exp[-\arcsin(x)] \qquad (103)$$

where C is the real constant.

The particular solution of the non-homogeneous equation (95) is found by the constant variation method. Therefore:

$$y(x)=C(x)\cdot\exp[-\arcsin(x)]. \qquad (104)$$

Both sides of the above equality we differentiate relative to variable x:

$$\frac{dy}{dx}=\frac{dC}{dx}\cdot\exp[-\arcsin(x)]-C(x)\cdot\exp[-\arcsin(x)]\cdot\frac{1}{\sqrt{1-x^2}}. \qquad (105)$$

After substituting functions (105) and (104) into equation (95) we get:

$$\frac{dC}{dx}\cdot\exp[-\arcsin(x)]-C(x)\cdot\frac{\exp[-\arcsin(x)]}{\sqrt{1-x^2}}+C(x)\cdot\frac{\exp[-\arcsin(x)]}{\sqrt{1-x^2}}=\frac{\arcsin(x)}{\sqrt{1-x^2}}. \qquad (106)$$

Hence

$$\frac{dC}{dx}\cdot\exp[-\arcsin(x)]=\frac{\arcsin(x)}{\sqrt{1-x^2}},\qquad(107)$$

$$\frac{dC}{dx}=\frac{\arcsin(x)\cdot\exp[\arcsin(x)]}{\sqrt{1-x^2}}.\qquad(108)$$

We integrate the equation (108) and we get:

$$\int\frac{dC}{dx}dx=\int\frac{\arcsin(x)\cdot\exp[\arcsin(x)]}{\sqrt{1-x^2}}dx.\qquad(109)$$

We integrate a right-hand side of equation (109) using integration by substitution method:

$$\int\frac{\arcsin(x)\cdot\exp[\arcsin(x)]}{\sqrt{1-x^2}}dx=\left\langle\begin{array}{c}\arcsin(x)=t\\ \frac{dx}{\sqrt{1-x^2}}=dt\end{array}\right\rangle=\int t\cdot\exp(t)dt=\left\langle\begin{array}{c|c}u=t & u'=1\\ v'=\exp(t) & v=\exp(t)\end{array}\right\rangle=\qquad(110)$$

$$=t\cdot\exp(t)-\int\exp(t)dt=t\cdot\exp(t)-\exp(t)=(t-1)\cdot\exp(t)=[\arcsin(x)-1]\cdot\exp[\arcsin(x)]$$

Hence, we get:

$$C(x)=[\arcsin(x)-1]\cdot\exp[\arcsin(x)].\qquad(111)$$

Therefore, the particular solution of non-homogeneous equation (95) has the form:

$$y(x)\equiv y_2(x)=[\arcsin(x)-1]\cdot\exp[\arcsin(x)]\cdot\exp[-\arcsin(x)],\qquad(112)$$

$$y(x)\equiv y_2(x)=\arcsin(x)-1.\qquad(113)$$

The general solution of the non-homogeneous equation (95) has the following form:

$$y(x)\equiv y_1(x)+y_2(x)=C\cdot\exp[-\arcsin(x)]+\arcsin(x)-1,\qquad(114)$$

$$y(x)=C\cdot\exp[-\arcsin(x)]+\arcsin(x)-1\qquad(115)$$

where C is the real constant.

• **Numerical solution**

For numerical analysis we take into account the solution (115) where the constant $C\in\{1, 2, 3, 4\}$ [1], [3], [5], [7].

Program 5. (Mathematica 7.0)

```
In[1]:= DSolve[y'[x] == (–1)*y[x] /Sqrt[1-x^2]+ ArcSin[x]/Sqrt[1-x^2], y, x] /.C[1] → {1, 2, 3, 4}]
        Plot[Evaluate[y[x] /. %], {x, –1, 1},
        Background → RGBColor[0.95,0,1],
        PlotStyle → {{RGBColor[1,0,0], Thickness[0.009]},
        {RGBColor[0,0,1], Thickness[0.009]}, {RGBColor[0,1,0], Thickness[0.009]},
        {RGBColor[1,0,1], Thickness[0.009]}},
        PlotRange → {0, 6}, AxesOrigin → {0, 0},
        AxesStyle → Thickness[0.004], Axeslabel → {"x", "y"},
        GridLines → Automatic, TextStyle → {FontFamily → "Arial", FontSize → 12}]
Out[1] = {{y → Function[{x},-1 + ArcSin[x] + e^-ArcSin[x] {1,2,3,4} }}
Out[2] =  = Graphics =
```

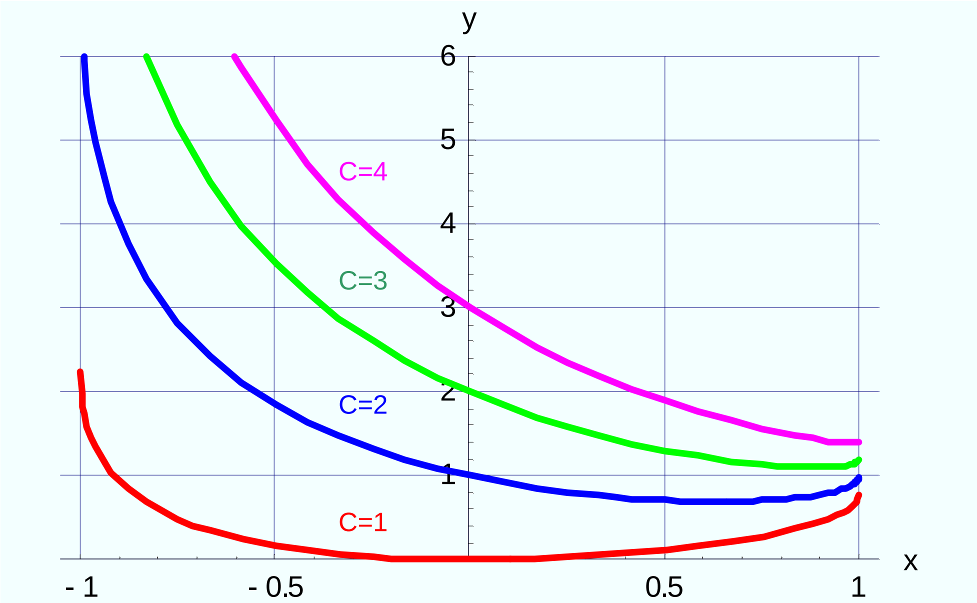

Fig. 5. The graphs of the function (115) as a solution of the first order linear non-homogeneous differential equation (95) for constant: C = 1, C = 2, C = 3, C = 4

Source: Program and graphs in Mathematica elaborated by the Authors

4. Conclusions

- Taking into account the constant variation method it is possible to solve the first order linear non-homogeneous differential equations with changeable coefficients. Using the *Mathematica* program you can quickly get a solution and create a graphical interpretation of solutions.
- In particular, solving of the first order non-homogeneous differential equations requires knowledge of the integration of exponential, logarithmic, polynomial, trigonometric and irrational functions.

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