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THE INFLUENCE FUNCTION IN ANALYSIS OF BENDING CURVE AND REACTIONS OF ELASTIC SUPPORTS OF BEAM WITH VARIABLE PARAMETERS

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In the paper, theoretical knowledge about base solution of common differential equations with variable parameters is presented. The solutions are applied to mechanical problems of discrete, distributed and discrete-distributed homogeneous elastic models of structures and structural components. In this paper, the influence function and its properties are presented. The influence function is applied to analysis of the bending curve of a beam with constants and variable parameters. The presented method of the influence function is based on the mathematical similarity of differential equations describing free vibrations and deflection of beams, which are fourth-order equations with variable coefficients. In this paper, examples of calculation of support reactions as function of stiffness of the beam and elastic supports are presented.

Key words: influence function, bending curve, reactions of elastic supports

1. Introduction

In innovation technology, solution methods of static and dynamic problems have application to design of drive assemblies of machines which can be modeled as elastic systems with variable parameters. Solutions to problems such as an increase of rotor velocity and acceleration or loading of bearings is very important (Lazopoulos, 2010; Jaroszewicza nd Żur, 2012a,b,c).

In a previous paper (Jaroszewiczet *et al.*, 2008), the authors analyzed the simplest lower estimator to calculate the basic frequency of axi-symmetrical vibration of plates with variable thickness of a circular diaphragm type. The existence of the simplest estimator of the actual value of the parameter depending on the frequency rate of change characterized by a thick plate was analyzed. The accuracy of the method differed from the FEM, and in order to improve the accuracy of the estimators, it was decided to use a higher order, in that case double. Using the bilateral estimator, the similar problem arose in calculation of the exact solution in a paper by Conway (1958).

The aim of this paper is to present theoretical knowledge about the base solution (influence function) to common differential equations with variable parameters. The advantage of the influence function is the possibility of omission of the boundary of conjugation in the solution to the boundary value problem. The solutions are applied to mechanical problems of discrete, distributed and discrete-distributed homogeneous elastic models of structures and structural components like beams, shafts and plates (Timoshenko, 1940; Solecki and Szymkiewicz, 1964).

2. The solution by means influence function

In many works (Zoryj, 1987c) an important property of the influence function whoose base was constructed by solutions to the boundary value problem of the bending curve, vibrations and stability was presented. It is known that the base solution to the equation

$$L[y(x)] = \delta(x-\alpha) \tag{2.1}$$

is the influence function $\phi(x,\alpha)$ in which the linear operator have analytical parameters

$$L[y(x)] = p_0(x)y(x)^n + p_1(x)y(x)^{n-1} + \ldots + p_n(x)y(x) \tag{2.2}$$

which are finite in intervals (a,b), where $a < x < b$, $a < \alpha < b$, x is the independent coordinate, α – parameter, $\delta(x)$ – Dirac's delta, $p_i(x) > 0$ – variable coefficients.

The fundamental solution (influence function) to Eq. (2.1) is defined in the following form

$$\phi(x,\alpha) = K(x,\alpha)\Theta(x-\alpha) \tag{2.3}$$

where $\Theta(x)$ is the Heaviside step function, $K(x,\alpha)$ – Cauchy function which is the solution to homogeneous Euler's equation $L[y(x)] = 0$ which satisfies following conditions

$$K(x,\alpha) = K'(x,\alpha) = \ldots = K^{n-2}(x,\alpha) = 0 \qquad K^{n-1}(x,\alpha) = \frac{1}{p_0(x)} \tag{2.4}$$

In the case of equations

$$L[y(x)] = \delta^j(x-\alpha) \qquad j = 1,2,3\ldots \tag{2.5}$$

the solutions to them are partial derivativeswith respect to the parameter α of the influence function

$$y^j(x) = (-1)^j \frac{\partial^j \phi(x,\alpha)}{\partial \alpha^j} \tag{2.6}$$

In a general equation with the variable right part

$$L[y(x)] = g(x) \tag{2.7}$$

we have the following solution

$$y = \sum_{k=0}^{n-1} A_k \frac{\partial^k K}{\partial \alpha^k} + y_*(x,\alpha) \tag{2.8}$$

where A_k are arbitrary constants, $y_*(x,\alpha)$ – particular solution to Eq. (2.7) well known by the Cauchy formula

$$y_*(x,\alpha) = \int\limits_{\alpha}^{x} K(x,s)g(s)\;ds \tag{2.9}$$

We can concludde that the partial derivative of the Cauchy function relative α parameter (arbitrary steps) satisfies the equation $L[y(x)]$.

Below, in Table 1, we present six formulas for the Cauchy function corresponding to chosen differential operators which have practical applications to mechanical engineering problems.

3. Influence function in the analysis of the bending curve and reactions of elastic supports of a beam with constant and variable cross-sections

3.1. Example of a general equation of the bending curve of the beam

The differential equation for deflection of an elastic beam in the static case has form (Jaroszewicz and Zoryj, 1997)

$$(fy'')'' = G(x) \tag{3.1}$$

Table 1. Formulas of Cauchy function (Zoryj, 1987)

$L[y(x)]$	$K(x,\alpha)$
$(fy')'$	$\int\limits_{\alpha}^{x} \frac{1}{f(s)}\, ds$
$(fy'')''$	$\int\limits_{\alpha}^{x} \frac{(x-s)(s-\alpha)}{f(s)}\, ds$
$y'' + \frac{1}{x}y'$	$\alpha \ln \frac{x}{\alpha}$
$y'' + \frac{p}{f(x)}y$, $p =$ const	$\sum\limits_{k=0}^{\infty} (-p)^k u_k(x,\alpha) \equiv U(x,\alpha)$ $u_0(x,\alpha) = x - \alpha$ $u_k(x,\alpha) = \int\limits_{\alpha}^{x} \frac{x-s}{f(s)} u_(k-1)(s,\alpha)\, ds, \quad k = 1,2,\dots$
$(f(x)y'')'' + py''$ $p =$ const	$\frac{1}{p}[x - \alpha - U(x,\alpha)] \equiv \int\limits_{\alpha}^{x} \frac{x-s}{f(s)} U(s,\alpha)\, ds$ $\sum\limits_{k=0}^{\infty} (-p)^k u_k(x,\alpha) \equiv U(x,\alpha)$ $u_0(x,\alpha) = x - \alpha$ $u_k(x,\alpha) = \int\limits_{\alpha}^{x} \frac{x-s}{f(s)} u_(k-1)(s,\alpha)\, ds, \quad k = 1,2,\dots$

where $f = EJ(x)$ is the bending rigidity of the beam (Fig. 1), $G(x)$ – transversal loading takin into account all possible spatialy discrete compositions

$$G(x) = \sum_{i=1}^{n} q_i(x)[\theta(x - x_i) - \theta(x - x_i - l_i)] + P_i\delta(x - x_i) + M_i\delta'(x - x_i) \tag{3.2}$$

where $q_i(x)$ is the distributed load acting on the segment $x_i < x < x_i + l_i$, P_i, M_i – discrete force and moment located in section $x = x_i$ and $0 < x_1 < x_2 < \ldots < x_n < l$, l – length of the beam.

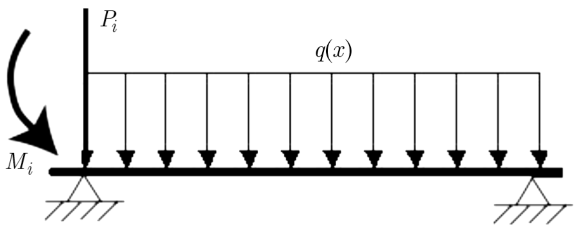

Fig. 1. Bending of elastic beam (Zoryj, 1987)

On the base of (2.3) and (2.6), the particular solution y_* equation (3.1) has the following form

$$\begin{aligned} y_*(x) &= \sum_{i=1}^{n}[\theta(x - x_i)F_i(x, x_i) - \theta(x - x_i - l_i)F_i(x, x_i + l_i)] \\ &+ \sum_{i=1}^{n}\Big[P_i\phi(x, x_i) - M_i\frac{\partial\phi}{\partial\alpha}\Big|_{\alpha=x_i}\Big] \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\phi(x,\alpha) &= \theta(x-\alpha)\int\limits_{\alpha}^{x}\int\limits_{a}^{x}\frac{1}{f(s)}(x-s)(s-\alpha)\;ds \\ F_i(x,\alpha) &= \int\limits_{\alpha}^{x}\phi(x,\tau)q_i(\tau)\;d\tau\end{aligned} \tag{3.4}$$

The particular solution $y_*(0)$ for $x=0$ satisfies the boundary condition

$$y_*(0) = y_*'(0) = y_*''(0) = y_*'''(0) = 0 \tag{3.5}$$

These expressions allow one to describe general solution (3.1) in the following form

$$y(x) = y(0) + y'(0)x - M(0)\int\limits_{0}^{x}\frac{x-s}{f(s)}\;ds + Q(0)\phi(x,0) + y_*(x) \tag{3.6}$$

Expression (3.6) is a general equation of the bending curve of the beam with a variable cross-section. In (3.6), the initial parameters: $y(0)$, $y'(0)$, $M(0)$, $Q(0)$, i.e. deflection, angle of rotation, bending moment and transversal force on the left end of the beam $(x=0)$, respectively, may take arbitrary values.

If $f = EJ =$ const we have

$$\phi(x,\alpha) = \frac{1}{6f}(x-\alpha)^3 \qquad\qquad \frac{\partial\phi(x,\alpha)}{\partial\alpha} = -\frac{1}{2f}(x-\alpha)^2 \tag{3.7}$$

Considering (3.7), we can obtain from (3.6) the well known equation of an elastic beam of constant rigidity. General equation (3.6) gives a new way of the solution of a few problems of static multi-supported beams with rigidity distributed by steps.

3.2. Example of application of the influence function in the analysis of bending curve of the beam

Figure 2 gives a model of an elastically supported beam (Jaroszewicz and Zoryj, 1994) subject to a transverse load G. The rigidity of the support at $x = l_1$ is c. The flexural rigidity is $f = EJ(x)$, where the Young modulus of elasticity $E =$ const, as the material is assumed to be homogeneous and the plane moment of inertia $J(x)$ is variable. The function $1/f(x)$ should be continuous, positive definite and should have a finite value and integral in $[0,l]$.

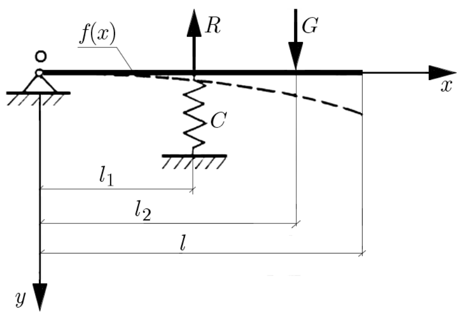

Fig. 2. Model of an elastically supported beam

When we consider the beam shown in Fig. 2, the deflection is defined as follows

$$L[y] = -R\delta(x-l_1) + G\delta(x-l_2) \tag{3.8}$$

where $L[y] = (fy'')''$, δ is Dirac's function.

The boundary conditions have the following form

$$\begin{aligned} &y(0) = 0 && fy''(0) = 0 \\ &fy''(l) = 0 && cy(l_1) = R \end{aligned} \tag{3.9}$$

The following solution to equation (3.8) can be proposed (Jaroszewicz and Zoryj, 2000)

$$y = C_0 + C_1(x-\alpha) + C_2K_{x0} + C_3K'_{x0} - R\phi_{x1} + G\phi_{x2} \tag{3.10}$$

where $K = K(x,\alpha)$ is the Cauchy function for $L[y(x)] = 0$ which can be defined

$$K(x,\alpha) = \int\limits_{\alpha}^{x} \frac{1}{f(s)}(x-s)U(s,\alpha)\;ds \qquad\qquad U(s,\alpha) = s - \alpha \tag{3.11}$$

C_i arre arbitrary constants and

$$K_{x0} = K(x,0) \qquad\qquad K'_{x0} = \frac{\partial K}{\partial \alpha}\Big|_{\alpha=0} \qquad\qquad \phi_{x1} = \phi(x,x_1) = K(x,l_1)\theta(x-l_1)$$

The fundamental solution to equation (3.8) has form (2.3).

After calculation of constants of integration C_i, we have a particular solution for the deflection of the beam at the end $x = l$ in the following form

$$y(l) = \frac{G}{l_1-\alpha}[-(l-\alpha)(D_0 + K_l1) + (l-l_1)K_{l\alpha} \tag{3.12}$$

where

$$\begin{aligned} &K_{l1}(l,l_1) = \int\limits_{l_1}^{l} \frac{1}{f(s)}(x-s)(s-l_1)\;ds && K_{l\alpha}(l,\alpha) = \int\limits_{\alpha}^{l} \frac{1}{f(s)}(x-s)(s-\alpha)\;ds \\ &D_0 = \frac{1}{l_1-\alpha}\Big[(l-l_1)K_{1\alpha} - \frac{l-\alpha}{c}\Big] && K_{1\alpha} = (l_1,\alpha) \end{aligned} \tag{3.13}$$

When we assume the constant cross-section $f(x) = f_0 =$ const

$$K(x,\alpha) = \frac{1}{6f_0}(x-\alpha)^3 \tag{3.14}$$

we have the expression for deflection as follows

$$x(l) = G\Big[\frac{l^2}{cl_1^2} + \frac{1}{3f_0}l(l-l_1)^2\Big] \qquad\qquad \alpha = 0 \tag{3.15}$$

In the case of the rigid support of this beam $c \to \infty$, it has been obtained

$$y(l) \to \frac{1}{3f_0}Gl(l-l_1)^2 \tag{3.16}$$

3.3. Influence function in the determination ofe elastic support reactions of the multi-supported beam

We consider an elastic beam with free ends supported on elastic supports. we examine the displacement in points with coordinates a_j $(j = 1,\dots,n)$, so $0 < a_1 < a_2 < \ldots < a_n < l$. We apply the principle of release from constrains, and considering Hook's law, we have

$$y(a_j) = c_j^{-1}R_j \tag{3.17}$$

where R_j is the modulus of j-th support reactions, c_j – j-th support rigidity, $y(a_j)$ – deflection of the beam in j-th support. Boundary conditions for the considered beam are defined by equaling to zero the bending moments and forces at the beam end

$$\begin{aligned} (fy'')\Big|_{x=0} &= M(0) = 0 \qquad & (fy'')'\Big|_{x=0} &= Q(0) = 0 \\ (fy'')\Big|_{x=l} &= M(l) = 0 \qquad & (fy'')'\Big|_{x=l} &= Q(l) = 0 \end{aligned} \tag{3.18}$$

From (3.6), by taking into consideration $(3.18)_1$, we have

$$y(x) = y(0) + y'(0)x + \widetilde{y}_*(x) \tag{3.19}$$

We put into formula (3.3) the support reaction R_j

$$\widetilde{y}_*(x) = y_*(x) - \sum_{j=1}^{n} R_j\varphi(x, a_j) \tag{3.20}$$

Substituting (3.19) to conditions $(3.18)_2$, we have received the following equations

$$\begin{aligned} &\sum_{j=1}^{n} R_j(l - a_j) = \sum_{i=1}^{\tau} \Big[P_i(l - x_i) + \int\limits_{x_i}^{x_i+l_i} (l - \tau)q_i(\tau)\, d\tau + M_i\Big] \\ &\sum_{j=1}^{n} R_j = \sum_{i=1}^{\tau} \Big(P_i + \int\limits_{x_i}^{x_i+l_i} q_i(\tau)\, d\tau \end{aligned} \tag{3.21}$$

Equation $(3.21)_1$ presents the condition of equilibrium of moments at the point $x = l$ (left end of the beam), the second condition – equilibrium of transverse.

Substituting (3.19) to (3.17), after transformation, we receive the follows equations

$$\begin{aligned} &y_0 + y_0'a_1 - \frac{R_1}{c_1} = -y_*(a_1) \\ &y_0 + y_0'a_2 - R_1(a_2, a_1) - \frac{R_2}{c_2} = -y_*(a_2) \\ &y_0 + y_0'a_n - \sum_{j=1}^{n-1} R_n(a_n, a_j) - \frac{R_n}{c_n} = -y_*(a_n) \end{aligned} \tag{3.22}$$

This way we determine two constants $y_0 = y(0)$ and $y_0' = y'(0)$ and we look for n reactions R_i of the elastic supports through the system $n + 2$ linear algebraic equations (3.21), (3.22). If any arbitrary support of them is rigid (e.g. first) then in (3.22) $c_1 \to \infty$ we obtain the system considered in this case.

Now we consider the determinants $\varDelta_n$ got from the system of equations $n = 2, 3, 4$, and we find

$$\begin{aligned} &\varDelta_2 = (a_2 - a_1)^2 \qquad \varDelta_3 = \varphi_{21}(a_3 - a_2)^2 + \varphi_{32}(a_2 - a_1)^2 \\ &\varDelta_4 = \varphi_{21}\varphi_{32}(a_4 - a_3)^2 + \varphi_{21}\varphi_{43}(a_3 - a_2)^2 + \varphi_{32}\varphi_{43}(a_2 - a_1)^2 \end{aligned} \tag{3.23}$$

Next, we proove that $\varDelta_n > 0$ for $n = 2, 3, 4$, which implies that the consider problem has only one solution in all cases (rigid, elastic supports).

3.4. Examples of calculation of the reaction $R_i(\beta, c_i)$ from the elastic supports

We take into consideration the beam presented in Fig. 3 which has three elastic supports and constant stiffness $(EJ = \text{const})$. The systems of equations (Zoryj, 1987; Jaroszewicz and Zoryj, 1994) can be expressed as follows $(a_1 \to 0,\ a_2 = l,\ a_3 = 2l)$

$$\begin{aligned} & 2R_1 + R_2 = 2ql \qquad R_1 + R_2 + R_3 = 2ql \\ & y_0 - \frac{R_1}{c_1} = 0 \qquad y_0 + y_0' l - \tfrac{R_1 l^3}{6EJ} - \tfrac{R_2}{c_2} = -\tfrac{ql^4}{24EJ} \\ & y_0 + y_0' 2l - \frac{R_1(2l)^3}{6EJ} - \frac{R_2 l^3}{6EJ} - \frac{R_3}{c_3} = \frac{q(2l)^2}{24EJ} \end{aligned} \tag{3.24}$$

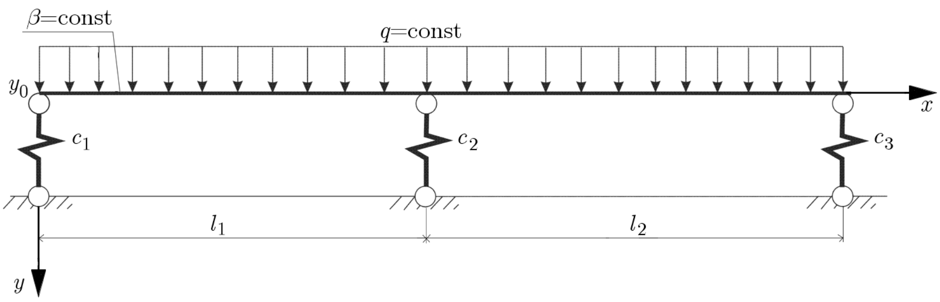

Fig. 3. Beam with three elastic supports (Zoryj, 1987)

From the first equation (3.24) it follows that $R_1 = R_3$. Next, reducing the constants y_0, y_0', we obtain

$$\begin{aligned} & \frac{R_1(\beta, c_i)}{ql} = \frac{R_3(\beta, c_i)}{ql} = \frac{\frac{1}{4} + \frac{4c_2^{-1}}{\beta}}{\frac{c_1^{-1}}{\beta} + \frac{4c_2^{-1}}{\beta} + \frac{c_3^{-1}}{\beta} + \frac{2}{3}} \\ & \frac{R_2(\beta, c_i)}{ql} = 2\frac{\frac{c_1^{-1}}{\beta} + \frac{c_3^{-1}}{\beta} + \frac{5}{12}}{\frac{c_1^{-1}}{\beta} + \frac{4c_2^{-1}}{\beta} + \frac{c_3^{-1}}{\beta} + \frac{2}{3}} \end{aligned} \tag{3.25}$$

where $\beta = l^3(EJ)^{-1}$. Examples of calculations are presented in Table 2.

Table 2. Examples of calculation of reaction forces in the elastic supports of constant stiffness beam

$\beta \to \infty$	Variants			
	I	II	III	IV
c_1	1	100	1000	∞
c_2	2	1	1000	∞
c_3	1	100	1000	∞
$\frac{R_1(\beta,c_i)}{ql} = \frac{R_3(\beta,c_i)}{ql}$	0.375112	0.375375	0.375	0.375
$\frac{R_2(\beta,c_i)}{ql}$	1.249775	1.249251	1.24999	1.25

In particular cases, we find from (3.25) well known values. If all supports are rigid $(c_i \to \infty)$, we have the well known results (Niezgodziński and Niezgodziński, 2012)

$$R_1 = R_3 = \frac{3}{8}ql \qquad R_2 = \frac{5}{4}ql \tag{3.26}$$

For the absolutely rigid bar on elastic supports, in the case $EJ \to \infty$, $\beta = 0$, we receive

$$R_1 = R_3 \frac{4c_2^{-1}ql}{c_1^{-1} + 4c_2^{-1} + c_3^{-1}} \qquad R_2 = 2ql \frac{c_1^{-1} + c_3^{-1}}{c_1^{-1} + 4c_2^{-1} + c_3^{-1}} \tag{3.27}$$

For $EJ \to \infty$ and $c_2 \to \infty$, $R_1 = R_3 \to 0$, $R_2 = 2ql$ and for $EJ \to \infty$, $c_1 = c_3 \to \infty$, we have $R_1 = R_3 \to ql$, $R_2 \to 0$, which can be concluded on the base of equations (3.21) and (3.22). We can also formulate conditions for applying simpler equations, e.g. $c_1 = c_3 = \infty$, $c_2 = c$. We present the corresponding values found from $(3.25)_1$ in Table 3.

Table 3. Results of calculation

$cl^3/(EJ)$	0	0.1	0.5	1	5	1000	∞
$R_1/(ql)$	1	0.99	0.95	0.9	0.55	0.38	0.775
$R_2/(ql)$	0	0.02	0.1	0.2	0.9	1.24	1.25

We can notice that for $0 < cl^3(EJ)^{-1} < 0.1$ the influence of rigidity of the supports on reactions at the end support is very small, and for $cl^3(EJ)^{-1} > 10^3$ we can consider the beam as absolutely rigid and calculate the reactions by means of formulas (3.27).

4. Conclusions

In this work, properties of the influence function applied to solution of mechanical problems of discrete, distributed and discrete-distributed homogeneous components were presented. The advantages of the influence function is the possibility of omission of the boundary of conjugation in the solution to the boundary value problem. The presented method gives possibility of solving the problem of bending curve deflection of a multi-supported beam with a variable cross-section in a closed analytical form. The presented method enables calculation of the force reaction $R_i(\beta, c_i)$ for elastic supports of beams with constant and variable stiffness $\beta(x)$.

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